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Unstable minimal surfaces and Heegaard splittings

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§ 1. Introduction and preliminaries

In recent years minimal surfaces have played an important role in the study of three-dimensional manifolds. Least area surfaces, which minimize area in their isotopy [15], homotopy [23] or homology [10] classes, have been used to make progress towards the understanding of three-manifolds which contain such surfaces. Such surfaces can be shown to exist and to have strong regularity properties under the assumption that the homotopy, isotopy or homology class in which they minimize area is non-trivial. However these techniques tend not to be applicable to the study of surfaces which are not essential in one of the above classes, or to the study of manifolds which contain only such non-essential surfaces.

Recently developed techniques open the way for the use of a new class of surfaces in three-dimensional manifolds, namely the class of unstable embedded minimal surfaces. Pitts and Rubinstein have established the existence of unstable minimal submanifolds which arise from a Heegaard splitting in such a way that their genus is less than or equal to that of the Heegaard splitting [19]. In this paper we extend their result to obtain greater topological control of the resulting minimal surface, and apply the resulting theorem to purely topological problems relating to Heegaard splittings of a three-manifold.

A compact manifold is a *handlebody* if it is homeomorphic to a regular neighborhood of a graph in R^3. A *Heegaard splitting* of a three-manifold M is a triple (F, H_1, H_2) where F is an embedded connected orientable surface in M, H_1 and H_2 are handlebodies embedded in M, $M = H_1 \cup H_2$ and $H_1 \cap H_2 = F = \partial H_1 = \partial H_2$. F is called a *Heegaard surface*. We say that two Heegaard splittings of M, (F, H_1, H_2) and (F', H'_1, H'_2), are *equivalent* if there is a homeomorphism $h: M \to M$ such that $f(F) = F'$, $f(H_1) = H'_1$ and $f(H_2) = H'_2$. By considering neighborhoods of triangulations, one can show that any closed orientable three-manifold admits a Heegaard splitting. Of particular interest are the splittings with Heegaard surface of smallest possible genus. The genus of such a splitting is called the *Heegaard genus* of the three-manifold.

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Heegaard splittings of three-manifolds are generally quite difficult to classify. For the three-sphere, it is a result of Waldhausen that there is a unique Heegaard splitting of each genus [26]. This was extended to genus one splittings of Lens spaces by Bonahon [3] and to higher genus splittings of Lens spaces by Bonahon-Otal [4]. More recent work of Boileau-Collins-Zieschang and Moriah has classified genus two splittings of a class of Seifert Fiber spaces [5, 17]. All these results use completely different techniques from those of this paper. This paper's contribution to the theory of Heegaard decompositions is to show that there is a unique genus 3 Heegaard splitting of the three-torus up to isotopy. In [14, conjecture 34] it is conjectured that any two Heegaard splittings of the three-torus having the same genus are equivalent up to homeomorphism. Theorem 4.2 resolves this question when the Heegaard splitting has genus 3, the smallest possible genus. Recent work in crystallography indicates that this question may have some physical significance [11]. We also apply these methods to produce the first example of a triply periodic minimal surface in R^3 whose quotient has genus four in the three-torus.

The paper is organized into five sections. Section 1 is the introduction you are currently reading, which also includes the necessary definitions and a statement of a version of the co-area formula, a well known result in differential geometry. In Sect. 2 we discuss the minimax process for obtaining a minimal surface, as developed by Pitts and Rubinstein, and an improvement, Theorem 2.4, which gives additional topological control on the resulting surface. In Sect. 3 we state some topological results concerning the compressibility of the boundary of a three-manifold and the effect on compressibility of removing an arc properly embedded in the three-manifold. These results, extensions of an unknotting lemma proved in [7], are of independent interest. They are applied in Sect. 4, where we give the proof of the uniqueness of genus 3 Heegaard splittings of the three-torus. In Sect. 5 we use Theorem 2.4 and some topological constructions to give the first example of a genus 4 minimal surface embedded in a flat three-torus.

We will work in the category of smooth compact manifolds and maps unless otherwise stated. An embedded surface F in a three-manifold M, other than a two-sphere, is *incompressible* if whenever D is a two-disk embedded in M such that $D \cap F = \partial D$, then ∂D bounds a two-disk in F. If a surface F meets an embedded disk D along ∂D, then we say that F' is obtained from F by *surgery along* D if $D \times I$ is embedded in M with $(D \times I) \cap F = \partial D \times I$ and if F' is a smooth surface isotopic to $F - (\partial D \times I) \cup (D \times \partial I)$. A regular neighborhood N of an embedded surface F in M is an I-bundle over F, with boundary either a connected double cover of F or two copies of F. We say that F is obtained from the boundary of N by *collapsing the boundary of an I-bundle to its core.* A surface is *two-sided* if its normal bundle is trivial, else it is *one-sided.* Two-sided embedded surfaces other than S^2 are incompressible if and only if their inclusion is injective on fundamental groups. One-sided embedded incompressible surfaces do not necessarily induce an injection of fundamental groups.

An immersed surface is *minimal* if its mean curvature is zero. A minimal surface is *stable* if the second variation of area for all families of surfaces passing through it is greater or equal to zero, and otherwise it is *unstable.* When we consider the three-torus as a Riemannian manifold, we will assume that it has the cubical flat metric unless otherwise stated, this being the metric obtained by taking the unit cube in R^3 and identifying opposite faces by translation. A two-dimensional *varifold* is a Radon measure on $G_2(M)$, the Grassmann bundle over M whose fiber at any point m is the space of unoriented two-dimensional linear subspaces of TM_m. Roughly speaking a varifold is a measurable choice of two-dimensional tangent planes. To each surface F embedded in M with finite two-dimensional Hausdorff measure, we can associate a varifold $v(F)$ defined by:

$$v(F)(A)=\mathscr{H}^2[F\cap\{x:(x, T_x F)\in A\}], \qquad \text{for } A\subset G_2(M)$$

where $\mathscr{H}^2$ denotes two-dimensional Hausdorff measure.

We can define a metric called the *F-metric* on the set of varifolds. It is defined by $F(V, W)=\sup\{V(f)-W(f)\,|\,f$ is a Lipschitz function with Lipschitz constant <1 and $|f|<1\}$ for two varifolds V and W. Roughly speaking, two surfaces represent varifolds which are nearby in the F-metric if they are nearby in the C^1 topology except for subsurfaces of small area. See [16] for an introduction to the above concepts of geometric measure theory, [1] and [6] for a more detailed treatment.

For completeness we next state a result relating the growth as r increases of the area of a surface F in a radius r neighborhood $N(G, r)$ of a second surface G to the length of the curve of intersection of F and the boundary of the r-neighborhood. This result is a consequence of the co-area formula and it does not depend on the surface being minimal. A proof of more general results of this type can be found in [6, Sect. 3.2].

Let M be a Riemannian manifold, let G be an embedded surface in M and let $N(G, r)$ be the tubular neighborhood of radius r about G in M. Let F be a surface meeting $N(G, r)$, let $A(r)$ be the area of $F\cap N(G, r)$, and when F meets $\partial N(G, r)$ transversely, let $L(r)$ be the length of $F\cap\partial N(G, r)$.

Lemma 1.1. *If ε is sufficiently small so that the $N(G, \varepsilon)$ describes a diffeomorphism of the r-normal bundle of G into M and if $R<\varepsilon$ then*

$$\partial A(R)/\partial r \geqq L(R)$$

or equivalently

$$A(R)\geqq \int_0^R L(r)\,dr.$$

Proof. For almost all values of r Sard's theorem implies that $F\cap\partial N(G, r)$ consists of a finite number of smooth curves on $\partial N(G, r)$. Let $p\in F\cap\partial N(G, r)$ and let $e_1, e_2, e_3, \ldots e_n$ be an orthonormal basis for TM in some neighborhood U of p such that e_1 is tangent to the curve $F\cap S(r)$, $e_2, \ldots, e_n$ are tangent to $\partial N(G, r)$ and e_n is tangent to F. Let $\omega_1, \omega_2, \ldots, \omega_n$ be the dual basis of one-forms. Then the formulas for the area $A_u(r)$ of $F\cap U\cap N(G, r)$ and length $L_u(r)$ of $F\cap U\cap \partial N(G, r)$ are expressed by:

$$A_u(r)=\int_{F\cap U\cap N(G,r)} \omega_1\wedge\omega_n \quad\text{and}\quad L_u(r)=\int_{F\cap U\cap \partial N(G,r)} \omega_1.$$

Integrating $L_u(r)$ over $r<R$ in U gives

$$\int_{F\cap U\cap N(G,R)} L_u(r)\,dr=\int_{N(G,R)}\Big(\int_{F\cap\partial N(G,r)\cap U}\omega_1\Big)\,dr=\int_{F\cap U\cap N(G,R)}\omega_1\wedge dr.$$

Since ω_1 and dr both have norm one and since $\omega_1\wedge\omega_n$ restricted to F has norm one we have that the pullback to F of $\omega_1\wedge\omega_n$ is greater or equal to the pullback to F of $\omega_1\wedge dr$, and so

$$A_u(R)=\int_{F\cap U\cap N(G,R)}\omega_1\wedge\omega_n\geqq\int_{F\cap U\cap N(G,R)}\omega_1\wedge dr=\int_{F\cap U\cap N(G,R)}L_u(r)\,dr$$

We now take a collection of closed disjoint neighborhoods U which cover $F\cap N(G,R)$ and the result follows.

§2. Surgeries and unstable surfaces

We begin by recalling briefly some known results on the existence of minimal surfaces in three-manifolds obtained by a minimax process. In [18] Pitts used tools of geometric measure theory to establish that every three-manifold admits an embedded minimal surface, although he obtained no estimate of the topological type of the surface. A technique for controlling the genus of minimal surfaces obtained from such geometric measure theory techniques was introduced in [2] and extended in [25] and [15]. Pitts-Rubinstein further developed this technique and established, among many other results, that a given genus g Heegaard splitting of a three-manifold leads to a minimal submanifold, possibly having several components, of total genus no larger than g, each component being either embedded or a double cover of a one-sided embedded surface [19, Theorem 2]. If the genus of the minimal surface thus obtained is the same as that of the Heegaard surface and the manifold is irreducible, then the Pitts-Rubinstein results show that the two surfaces are isotopic. However, the methods in [19] do not give information about the topological relationship between the resulting minimal surface and the original Heegaard surface if the genus is not the same, beyond controlling the genus of the resulting surface. In this section we study this situation and show that there is a topologically natural way to pass from the Heegaard surface to the minimal submanifold when the genus does drop.

An essential ingredient of our proof will be the following result of Pitts-Rubinstein. If N is a regular neighborhood of an embedded surface F_i with projection $p\colon N\to F_i$, then we say that a path $\tilde\gamma\colon[0,1]\to N$ is a *lift* of a path $\gamma\colon[0,1]\to F_i$ if $p\tilde\gamma=\gamma$. By $v(H_i)$ we denote the varifold associated to the surface H_i. Given a surface S embedded in an orientable Riemannian three-manifold and a positive constant δ which is sufficiently small so that the δ-normal bundle of S is injective, a *δ-neighborhood of S* is the image of the δ-normal bundle of S in M, diffeomorphic to $S\times[-\delta,\delta]$ if S is two-sided and diffeomorphic to a nontrivial I-bundle over S if S is one-sided. In the former case it is the union of surfaces $S_t=S\times\{t\}$, $-\delta<t<\delta$, each homeomorphic to S. In the latter case it is the union of S and surfaces S_t at distance t from S in M, each homeomorphic to a two-fold cover of S. A similar definition applies for a neighborhood of a curve or arc embedded in a surface.

Theorem 2.1. *Let F be a surface in a closed three-manifold M isotopic to the boundary of a handlebody in a Heegaard splitting of M. Then there exist smooth minimal surfaces H_i, a varifold $V=\sum n_i v(H_i)$ and a sequence of surfaces $\{F_j\}$ each isotopic to F, such that $\{v(F_j)\}$ converges to V in the F-metric as $j\to\infty$. Let N be a tubular neighborhood of H_i with projection $p\colon N\to H_i$ and let γ be a simple curve in H_i. Then there is a curve γ_j on H_i and positive functions $\delta(j)$ and $\varepsilon(j)$ such that $\varepsilon(j)\to 0$ as $j\to\infty$, γ_j lies within distance $\varepsilon(j)$ of γ, and each $\gamma_{j,t}$ in a $\delta(j)$-neighborhood of γ_j lifts to a collection of curves in F_j which together project to $\gamma_{j,t}$ with degree n_i.*

Note 2.2. This lifting property of curves on each H_j is used by Pitts and Rubinstein to obtain control of the genus of the H_j. It is similar to the technique used in [25] to show the existence of a minimal two-sphere in the three-sphere endowed with any metric. Pitts-Rubinstein also obtain information about the index of the surfaces H_j but we will not be concerned with that in this paper. Note that it is not asserted that γ itself has the lifting property, but rather an open set of nearby curves.

We next state a rather technical lemma. Our use of the term *length* in this lemma can be taken to mean either the standard definition for rectifiable curves or one dimensional Hausdorff measure.

Lemma 2.3. *Let $\{F_j\}$ be a sequence of surfaces in a Riemannian three-manifold M such that $v(F_j)\to V$ in the F-metric, where V is a varifold in M. Let N be a δ-neighborhood of an embedded surface S in M such that N and support(V) have empty intersection. Then there is a subsequence $\{F_n\}$ of $\{F_j\}$ such that except for t in a measure zero subset of $[-\delta,\delta]$, length$(F_n\cap\{S_t\})\to 0$ as $n\to\infty$.*

Proof. For fixed j and n, Lemma 1.1 gives an upper bound $m_{j,n}=n\cdot\text{Area}(F_k\cap N)$ on the measure of the points $t\in[-\delta,\delta]$ for which $\text{length}(F_j\cap\{S_t\})>1/n$. Since $\{F_j\}$ is converging to V in the F-metric as $j\to\infty$ and support(V) has empty intersection with N we have that $\lim_{j\to\infty} m_{j,n}=0$. After passing to a subsequence $\{F_{j,n}\}$ of $\{F_j\}$ we can assume that $m_{j,n}<1/2^j$, so that $\sum_{j\geqq J} m_{j,n}<1/2^{J-1}$. Thus $\text{length}(F_{j,n}\cap\{S_t\})<1/n$ for all $j\geqq J$ except possibly for t in a set of measure $1/2^{J-1}$. We now repeat the above process with $n+1$ replacing n, finding a subsequence $\{F_{j,n+1}\}$ of $\{F_{j,n}\}$ which satisfies $\text{length}(F_{j,n+1}\cap\{S_t\})<1/n+1$ for all $j\geqq J$ except possibly on a set of measure $1/2^{J-1}$. Taking the diagonal sequence $\{F_n\}=\{F_{n,n}\}$ we have that $\text{length}(F_{n,n}\cap\{S_t\})<1/N$ for all $n\geqq N$ except on a set of measure at most $1/2^{N-1}$. Thus the set of t for which $\text{length}(F_{n,n}\cap\{S_t\})$ does not converge to zero has measure zero, proving the lemma.

We now state our strengthened version of Theorem 2.1. Pitts and Rubinstein, in more recent work, incorporate the added information contained in Theorem 2.4 into a new statement of Theorem 2.1 [20, Theorem 2], [21].

Theorem 2.4. *Let F be an oriented surface in a closed orientable three-manifold M isotopic to the boundary of a handlebody in a Heegaard splitting of M. Let $V=\sum n_j v(H_j)$ be the Pitts-Rubinstein mini-max minimal surface obtained from* *the given Heegaard splitting. Then $\cup H_j$ is isotopic to a surface obtained from F by:*

1. *Successively surgering a series of curves on F.*
2. *Successively collapsing boundaries of I-bundles over a surface.*
3. *Discarding two-spheres which bound balls.*

Proof. We begin by picking a tubular neighborhood N of $\cup H_i$ with components N_i. Let F_j be a sequence of smooth surfaces with $v(F_j)$ converging to $V=\sum n_i v(H_i)$ in the F-metric, as in Theorem 2.1. In N_i, $F_j \cap N_j$ converges in the F-metric to H_i with multiplicity n_i. We can assume, after a small perturbation of the width of N, that each F_j intersects ∂N_i transversely in a finite collection of curves. Noting that V has empty intersection with ∂N_i and applying Lemma 2.3, we can assume that length$(F_j \cap \partial N_i) \to 0$ as $j \to \infty$.

On each component H_i of $\cup H_i$ let $\{S_{k,i}\}$ be a standard set of essential curves for H_i consisting of embedded curves on H_i such that each $s_{k,i}$ intersects at most one distinct $s_{k',i}$ and such that the complement of the set of curves $\{s_{k,i}\}$ in H_i is a disk. Applying Theorem 2.1, we can perturb the $\{s_{k,i}\}$ so that after passing to a subsequence of $\{F_j\}$ which we continue to denote by $\{F_j\}$, each $s_{k,i}$ lifts to a collection of curves in any fixed F_j which together project to $s_{k,i}$ with degree n_i. Any isotopic collection of curves sufficiently nearby to $\{s_{k,i}\}$ has the same property. By picking j sufficiently large, we can assume that curves in $F_j \cap \partial N_i$ have length arbitrarily close to 0, and that their projections to H_i also have length arbitrarily close to 0. In particular, after a small perturbation of the collection $\{s_{k,i}\}$ we can assume that the projections of the curves in $F_j \cap \partial N_i$ to H_i are disjoint from $\{s_{k,i}\}$. By passing to a subsequence we can assume this property holds for all the F_j.

Now let $\{A_{k,i}\}$ denote $p^{-1}(s_{k,i})$ so that each $A_{k,i}$ is either an annulus or a mobius band in N_i. The previous argument implies that $A_{k,i} \cap F_j \cap \partial N_i = \emptyset$. Using the fact that F_j is converging to H_i with multiplicity n_i in the F-metric and applying Lemma 2.3, we again perturb the curves of $\{s_{k,i}\}$ slightly so that each $A_{k,i}$ intersects F_j transversely, the intersection of $A_{k,i}$ with $F_j \cap \partial N$ remains empty, and the intersection of F_j with $A_{k,i}$ consists of a collection of curves which together project to $s_{k,i}$ with degree n_i and some other trivial curves whose total length approaches 0 as $j \to \infty$. The projection of the trivial curves to H_i also has length converging to 0 as $j \to \infty$. These conditions hold for a dense set of curves nearby to $\{s_{k,i}\}$, so we can further assume, by passing to a subsequence of $\{F_j\}$, that the projection of the trivial curves to H_i lies in the complement of the points of intersection $\{s_{k,i} \cap s_{k',i}\}$. It follows that the intersection of any pair of $A_{k,i}$'s, which is either a single arc or empty, does not intersect non-essential curves of $F_j \cap A_{k,i}$.

Now, picking an F_j with j large, we have that the total length of all the curves of $F_j \cap \partial N_i$ is smaller than the injectivity radius of M, so that each component of $F_j \cap \partial N_i$ is homotopically trivial on ∂N_i and bounds a disk on ∂N_i. Such a disk may or may not intersect other components of $F_j \cap \partial N_i$, but we can find an innermost disk whose interior has empty intersection with $F_j \cap \partial N_i$. Let C be a curve of $F_j \cap \partial N_i$ which is the boundary of such an innermost disk D. Then surgering F_j along D yields a surface, not necessarily connected, which has one fewer component of intersection with ∂N_i. Continuing we obtain a surface G_j which is disjoint from ∂N_i. The intersection of G_j with $A_{k,i}$ is the same as the intersection of F_j with $A_{k,i}$ and the intersection of any pair of $A_{k,i}$'s, which is either a single arc or empty, does not intersect trivial curves of $G_j \cap A_{k,i}$. We now surger G_j to successively eliminate innermost trivial curves of $G \cap A_{k,i}$. We obtain finally a surface L whose intersection with N_i consists of a collection of components whose intersection with each $A_{k,i}$ is precisely a collection of simple curves projecting with total degree n_i to $s_{k,i}$. If we cut N_i open along $\cup A_{k,i}$ we obtain a ball whose boundary intersects L in simple closed curves. Performing a series of surgeries on the intersection of L and this ball yields a surface R whose intersection with the ball consists of a collection of disks and some two-spheres bounding balls. If H_i is two-sided then $R \cap N_i$ consists of n_i parallel copies of H_i together with some trivial two-spheres. If H_i is one-sided then $R \cap H_i$ consists of $n_i/2$ parallel copies of ∂N_i, together with some trivial two-spheres. Thus we have shown how to pass by a series of surgeries, throwing away of two-spheres bounding balls and collapsing of I-bundles from F_j to $\cup H_i$. This proves Theorem 2.4.

Note 2.5. The process of discarding two-spheres which bound balls can be replaced by performing an isotopy at an appropriate time. Thus with a little more care one can show that processes 1 and 2 are sufficient in Theorem 2.4. In process 2, either two copies of a surface coalesce or a single surface collapses to double cover a 1-sided surface.

§3. Compressing the boundary of a three-manifold

In this section we consider the effect on the compressibility of the boundary of a three-manifold caused by removing a properly embedded arc from the three-manifold. The following two lemmas extend the unknotting lemma of [7].

Let M be a three-manifold with boundary. There are two standard ways to enlarge M by gluing a copy of $D^2 \times I$ to it. One is to identify $D^2 \times \partial I$ to two disks in ∂M. This is called attaching a *one-handle* to M. The *co-core* of the one-handle is $D^2 \times \{1/2\} \subset D^2 \times I$. The second way to add a copy of $D^2 \times I$ is to identify the annulus $\partial D^2 \times I$ to a regular neighborhood of an embedded curve in ∂M. This is called attaching a *two-handle* to M and the *co-core* of the two-handle is $\{0\} \times I \subset D^2 \times I$. Up to homeomorphism, the effect of adding a handle is canceled by removing an open neighborhood of its co-core.

Lemma 3.1. *Let M be an irreducible orientable three-manifold with incompressible boundary. Let M' be an orientable three-manifold obtained from M by adding a one-handle to M with the two attaching disks of the one-handle going to distinct components of ∂M. Let α be a properly embedded arc in M' such that the manifold M'', obtained from M' by removing a neighborhood of α, has compressible boundary. Then α can be isotoped to miss the co-core of the one-handle.*

Proof. M' is obtained from M'' by adding a two-handle along a curve c. Since M' has compressible boundary, a theorem of Jaco and Przytycki [13, 22, see also 24] implies that $\partial M'' - c$ is compressible. Let D'' be a compressing disk for $\partial M''$ with $\partial D'' \subset \partial M'' - c$. Then D'' gives rise to a compressing disk D' in M' missing α. If this disk is boundary parallel in M' then D' together with

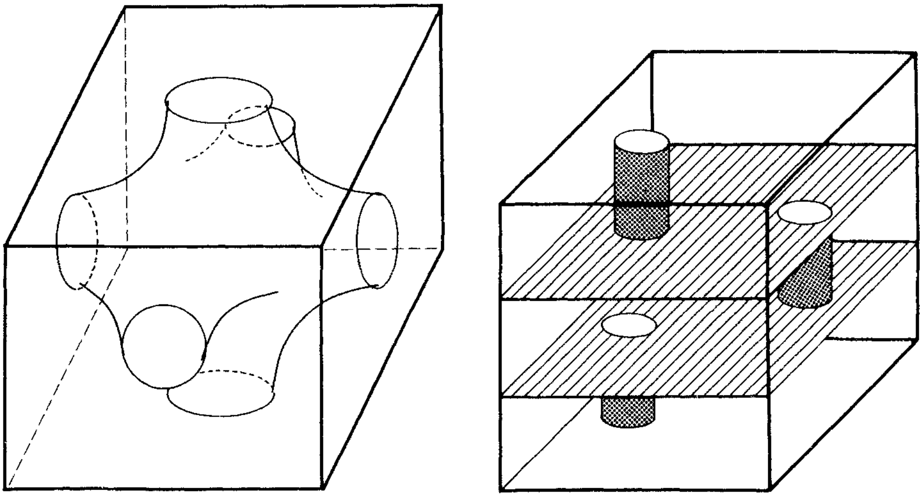

Fig. 4.1. Two views of a standard Heegaard surface of genus three for the three-torus in a cubic fundamental domain

a disk on $\partial M'$ form a two-sphere bounding a ball containing α. α can then be isotoped off the co-core of the one-handle as required. Suppose now that D' is not boundary parallel, so that it is an essential compressing disk in M'. The co-core of the one-handle is the unique essential compressing disk in M' up to isotopy, as can be seen by intersecting it with another compressing disk and simplifying the intersections by doing boundary compressions until the disks are disjoint. So D' is parallel to the co-core, α misses D' and Lemma 3.1 follows.

Note 3.2. The generalization of the proof of Lemma 3.1 fails if more than one one-handle is added to M, since in that case one does not know that a curve missing a compressing disk misses one of the co-cores of the set of one-handles added.

Lemma 3.3. *Let M be an orientable three-manifold obtained by adding a one-handle to $F \times I$, the product of a closed surface F with the interval, so that the two attaching disks of the one-handle go to distinct components. Let α be a properly embedded arc in M such that $M - N(\alpha)$ is homeomorphic to a handlebody H, where $N(\alpha)$ is an open neighborhood of α. Then α is isotopic to the arc $\{b\} \times I$ in $F \times I$, where b is a point in F.*

Proof. By Lemma 3.1 α can be isotoped off the one-handle to lie in $F \times I$. The manifold obtained by removing α from $F \times I$ is a handlebody, as adding a one-handle to it gives rise to H. The unknotting lemma of [7] now implies that α is isotopic in $F \times I$ to $\{pt\} \times I$.

§ 4. Genus 3 Heegaard splittings of the three-torus

In this section we will be working in the three-torus. We will say that a genus 3 Heegaard splitting of the three-torus is *standard* if it is isotopic to the splitting depicted in Fig. 4.1. Alternately, it can be described as follows: Take T^3 to be the cube $[-1, 1] \times [-1, 1] \times [-1, 1] \subset R^3$ with opposite faces identified, and let A be the graph consisting of the intersection of the x, y and z axes of R^3 with the cube. A regular neighborhood of A gives a handlebody of genus 3 whose complement is also a handlebody, and any isotopic Heegaard splitting is called standard. Yet another way to construct this surface of genus three is to start with any pair of parallel incompressible embedded two-tori in the three-torus, and tube them together with one unknotted tube in each of the two complementary $T^2 \times R$ regions.

Theorem 4.2. *Any genus three Heegaard splitting of the three-torus is standard.*

Proof. Given a Heegaard splitting of the three-torus with Heegaard surface F, we apply the Pitts-Rubinstein method to obtain an unstable minimal surface V from the Heegaard splitting. If V has genus 3 then it is isotopic to F and standard by [7]. If V is not genus three, then by Theorem 2.4 there is a collection of embedded minimal submanifolds H_i obtained from F by successively performing surgeries and collapsing of I-bundles. A component H_i can not be a two-sphere as there are no minimal two-spheres in the three-torus, as if there were one could lift to get a minimal two-sphere in R^3 which does not exist by the maximal principle. Similarly it can not be a projective plane. Suppose H_i is a genus two surface. Then since the Gauss map of a minimal surface in R^3 is anti-conformal, the Gauss map of H_i gives a branch covering of the two-sphere. The Gauss-Bonnet theorem tells us that this map has degree 1, so that it has no branch points, but a surface of genus two does not cover the two-sphere, giving a contradiction. Thus there are no genus two minimal surfaces in the three-torus. Similarly, there are no non-orientable surfaces of Euler characteristic equal to -1, as these would lift to genus two orientable minimal surfaces in a double cover. Non-orientable minimal surfaces of Euler characteristic equal to -2 do exist in the three-torus. However, Theorem 2.4 implies that the Pitts-Rubinstein process can lead to such surfaces only by a collapse of the boundary of a one-sided I-bundle and no surgeries. This the Heegaard surface is the boundary of a one-sided I-bundle over a non-orientable surface of Euler characteristic equal to -2, which is impossible since a Heegaard surface bounds a handlebody on each side. So H_i can not be a non-orientable surface of Euler characteristic equal to -2. Minimal tori do exist in the three-torus, and any such is totally geodesic as can be seen by applying the Gauss-Bonnet theorem. If H_i is a torus then Theorem 2.4 implies that there is either a single torus H_i which has multiplicity two or there are two H_i which are tori, each with multiplicity one. A minimal Klein bottle can not be embedded in the three-torus as a double cover would be a totally geodesic torus, and so it too would have to be totally geodesic. But totally geodesic compact surfaces in the three-torus are tori.

Thus we have proved the theorem except in the case where the H_i consist of two parallel tori or a single torus with multiplicity 2. In either of these cases we know that a series of surgeries on F results in two parallel essential tori T_1 and T_2. We now show that we need to do a sequence of only two surgeries to arrive at T_1 and T_2. If a surgery produces a trivial two-sphere then it can be replaced with an isotopy which moves the surface across the ball bounded by the two-sphere, with the same effect. If a surgery of F occurs along a disk D with ∂D separating and the result of the surgery produces a compressible torus, then there is another surgery along a disk D' which compresses this torus to a two-sphere, which is discarded. We can eliminate the first surgery along D and only perform the second surgery along D'. Following this by an isotopy has the same effect as the original surgery on the two disk and a discard of the trivial two-sphere. The first surgery can not cut off an incompressible torus, as if it did then the complement of F would contain an essential torus on one side or the other of this incompressible torus. Thus we can assume always that the first surgery disk has non-separating boundary on F and surgers it to a surface of genus two. The second surgery disk now must be separating if we are to arrive at two tori after the surgery, so we can assume that exactly two surgeries occur in passing from F to two parallel tori. The first surgery disk D_1 lies in the complement of F and we let F_{D_1} denote the surface obtained by surgering F along D_1. The second surgery disk D_2 may intersect F but is disjoint from F_{D_1}.

F separates T^3 into two handlebodies H_1 and H_2, which we label so that D_1 lies in H_1. Consider the two components E_1 and E_2 of $T^3 - F_{D_1}$. E_1 is the complement of D_1 in H_1 and E_2 consists of H_2 with a two-handle attached along D_1. Thus adding a one-handle to E_1 gives H_1 and removing an arc a_1 from E_2 gives H_2. It follows that E_1 is a handlebody of genus two or a handlebody of genus two disjoint union a handlebody of genus one. $T_1 \cup T_2$ separates T^3 into two components, C_1 and C_2, each homeomorphic to $T^2 \times I$. C_1 is obtained from E_1 by removing a neighborhood of D_2, while C_2 is obtained from E_2 by adding a two-handle along D_2. It follows that E_1 is obtained from C_1 by removing an arc while E_2 is obtained from C_2 by adding a one-handle. Thus E_2 is homeomorphic to $T^2 \times I$ with a one-handle attached. Since removing the arc a_1 from E_2 gives the boundary compressible H_2, Lemma 3.3 implies that the arc a_1 is standard and misses the disk D_2. We thus see the two handlebodies of the Heegaard splitting are constructed from C_1 and C_2 by first removing an open neighborhood of a standard arc from C_1 and then removing an open neighborhood of a standard arc from C_2. This shows that any two Heegaard splittings of the three-torus are equivalent up to homeomorphism of the three-torus. It is straightforward to check that any two standard Heegaard surfaces obtained in this way are in fact isotopic, independent of which pair of parallel essential tori one starts with, by sliding handles to take one to the other.

§5. A genus 4 minimal surface in a three-torus

Schwartz's minimal surface gives a genus three minimal surface in a three-torus, from which it is easy to construct examples of any odd genus by passing to a covering space of appropriate order. For even genus cases a new technique is needed. We show here that there exists an embedded minimal surface of genus four in a three-torus, or equivalently, a triply periodic minimal surface in R^3 which is invariant under a lattice and of genus 4 in the quotient by this lattice. Our construction involves some arbitrary choice of scaling, so that

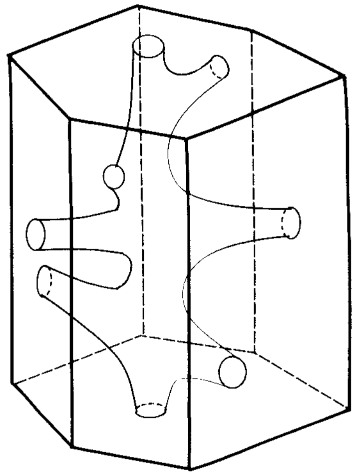

Fig. 5.1. A fundamental domain for the three-torus with a surface of genus four contained in it

we can produce non-isometric families of surfaces of the above type by varying the ratio of lengths of the sides and height of a regular hexagon prism. By passing to coverings, we also produce the first examples of minimal surfaces in the three-torus of genus 10, 16 and in general $4+6m$, $m=1, 2, 3, \ldots$. This genus four minimal surface is roughly depicted in Fig. 5.1. In this figure we construct the three-torus as the product of a hexagon with an interval, with opposite faces identified. The actual minimal surface of genus 4 is isotopic to the surface in Fig. 5.1 and has the same symmetries.

The genus 4 minimal surface does not arise directly from a minimax process, as its index turns out to be three and minimax methods produce surfaces of index one. However we use Theorem 2.4 to obtain its existence as follows. The three-torus is a three-fold cover of a Seifert Fiber space M with orbit space the two-sphere and with 3 exceptional fibers, each of order 3. This manifold can also be understood as a torus bundle over the circle with monodromy map of order 3. By the methods of [9], it can be shown that M contains no minimal embedded two-spheres, projective planes or Klein bottles, and that any embedded minimal torus is isotopic to a fiber of the fibration of M over S^1. Moreover M has Heegaard genus two. A genus two Heegaard surface can be constructed as the boundary of the regular neighborhood of two exceptional fibers connected by a horizontal arc [5, 17]. A genus two Heegaard surface can not be surgered to yield two parallel torus fibers, nor can it be the boundary of an I-bundle over a non-orientable surface of genus 3, so applying Theorem 2.4 we obtain an embedded minimal surface of genus 2 in M, isotopic to the Heegaard surface. The three-fold cover of this surface lifts to a three-fold cover of M, giving a genus 4 minimal surface embedded in the three-torus.

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References

1. Allard, W.: On the first variation of a varifold. Ann. Math. **95**, 417–491 (1972)
2. Almgren, Jr., F.J., Simon, L.: Existence of embedded solutions of Plateau's Problem. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. **6**, 447–495 (1979)
3. Bonahon, F.: Diffeotopies des espaces lenticulaires. Topology **22**, 305–314 (1983)
4. Bonahon, F., Otal, J.P.: Scindements de Heegaard des espaces lenticulaires. Ann. Sci. Ec. Norm. Super., IV. Ser. **16**, 451–466 (1983)
5. Boileau, M., Collins, D.J., Zieschang, H.: Scindements de Heegaard des petites varietes de Seifert. Math. Ann. (in press)
6. Federer. H.: Geometric measure theory. Berlin-Heidelberg-New York: Springer 1969
7. Frohman, C.: Minimal surfaces and Heegaard splittings of the three-torus. Pac. J. Math. **124**, 119–130 (1986)
8. Hass, J., Scott, G.P.: The existence of least area surfaces in 3-manifolds. Trans. Am. Math. Soc. (in press)
9. Hass, J.: Minimal Surfaces in Seifert Fiber Spaces. Topology Appl. **18**, 145–151 (1984)
10. Hass, J.: Surfaces minimizing area in their homology classes and group actions on 3-manifolds. Math. Z. (in press)
11. Hyde, S.: Triply periodic minimal surfaces and crystallography. Thesis, Canberra University 1986
12. Hempel, J.: Three-manfolds. Ann. Math. Studies **86**, Princeton University Press (1976)
13. Jaco, W.: Adding a two-handle to a three-manifold: An application to property *R*. Proc. Am. Math. Soc. **92**, 288–292 (1984)
14. Meeks, W.: Lectures on Plateau's problem. I.M.P.A., Rio, Brazil (1978)
15. Meeks, W., Simon, L., Yau, S.T.: Embedded minimal surfaces, exotic spheres and manifolds of positive Ricci curvature. Ann. Math. **116**, 621–659 (1982)
16. Morgan, F.: Geometric measure theory: A beginners guide. New York: Academic Press 1988
17. Moriah, Y.: Heegaard splittings of Seifert fibered spaces. Invent. Math. (in press)
18. Pitts, J.T.: Existence and regularity of minimal surfaces on Riemannian manifolds. Princeton, J: University Press 1981
19. Pitts, J.T., Rubinstein, J.H.: Existence of minimal surfaces of bounded topological type in three-manifolds. Proc. Cent. Math. Anal. Aust. Natl. Univ. Canberra, **10**, 163–176 (1986)
20. Pitts, J.T., Rubinstein, J.H.: Applications of minimax to minimal surfaces and the topology of three-manifolds. Proc. Cent. Math. Anal. Aust. Natl. Univ. Canberra, **12**, 137–170 (1987)
21. Pitts, J.T., Rubinstein, J.H.: Equivariant minimax and minimal surfaces in geometric 3-manifolds, Bull. A.M.S. **19**, 303–309 (1988)
22. Przytycki, J.: Incompressibility of surfaces after Dehn surgery. Mich. Math. J. **30**, 289–308 (1983)
23. Scott, G.P.: There are no fake Seifert fiber spaces. Ann. Math. **117**, 35–70 (1983)
24. Scharlemann, M.: Outermost forks and a theorem of Jaco. Proceedings of the Rochester conference, 1982
25. Simon, L., Smith, F.: On the existence of embedded minimal two-spheres in the 3-sphere endowed with an arbitrary metric. Preprint, University of Melbourne
26. Waldhausen, F.: Heegaard-Zerlugen der 3-sphäre. Topology **7**, 195–203 (1968)

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