Notes on 3-manifolds

Jennifer Schultens

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1 Introduction

The study of 3-manifolds lies at the heart of lowdimensional topology. Manifolds of lower dimension have been classified. And there is no more work to be done, at least not for a topologist. The study of 4-manifolds requires, most notably, more analytic and geometric tools. the study of higher dimensional manifolds requires the full force of algebraic topology.

These notes aim to introduce the student to the main topics in the study of 3-manifolds.

Definition 1.1. *A n-manifold is a paracompact Hausdorff space M such that for each point $x \in M$ there is a neighborhood U that is homeomorphic to $\mathbb{R}^n$. The dimension of an n-manifold is n.*

Example 1: The set $S^n = \{x \in \mathbb{R}^{n+1} | \|x\| = 1\}$ is a n-dimensional manifold called the n-sphere. Here S^n is a compact subset of $\mathbb{R}^{n+1}$ and thus necessarily paracompact and Hausdorff. Stereographic projection provides a homeomorphism $h : S^n - \{(0, \dots, 0, 1)\} \to \mathbb{R}^n$. So any point $x \in S^n$ such that $x \neq \{(0, \dots, 0, 1)\}$ has the neighborhood $S^n - \{(0, \dots, 0, 1)\}$ that is homeomorphic to $\mathbb{R}^n$. To exhibit a neighborhood of $(0, \dots, 0, 1)$ that is homeomorphic to $\mathbb{R}^n$ one may turn the idea of stereographic projection on its head to provide a homeomorphism $h' : S^n - \{(0, \dots, 0, -1)\} \to \mathbb{R}^n$. Thus $S^n - \{(0, \dots, 0, -1)\}$ is a neighborhood of $(0, \dots, 0, 1)$ homeomorphic to $\mathbb{R}^n$.

Example 2: The set $T^n = S^1 \times \cdots \times S^1$ (n factors) is called the n-torus. The n-torus may be viewed as a quotient space as follows: In $\mathbb{R}^n$, consider the group G generated by translations of distance 1 along the coordinate axes. Then identify two points $x, y \in \mathbb{R}^n$ if and only if there is a $g \in G$ such that $g(x) = y$. Given this description of the n-torus it follows readily that it is a n-manifold.

Definition 1.2. *A subset of a manifold M that is itself a manifold is called a submanifold of M.*

We will also consider a slightly larger class of objects:

Definition 1.3. *(n-manifold with boundary) A n-manifold with boundary is a paracompact Hausdorff space M such that for each point $x \in M$ there is a neighborhood*

U that is homeomorphic to $\mathbb{R}^n$ *or* $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$*. The dimension of an n-manifold with boundary is n. The boundary of* M *is the set of all points that have a neighborhood homeomorphic to* $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ *but no neighborhood homeomorphic to* $\mathbb{R}^n$*. The boundary of* M *is denoted by* ∂M*.*

Our goal is to understand 3-manifolds. This turns out to be a difficult and perhaps impossible task. In particular, it is far more difficult than classifying 2-manifolds. Recall that 2-manifolds are classified by their Euler characteristic together with the property of being or not being orientable. This information does not come close to classifying 3-manifolds. As it turns out, every closed orientable 3-manifold, of which there are many, in fact has Euler characteristic 0! To study 3-manifolds we must thus develop more effective tools.

Exercise: Show that the boundary of a n-manifold is a (n-1)-manifold without boundary.

2 Fundamental Results in 3-manifolds

Manifolds are studied in a variety of categories. The three main categories in which manifolds are studied are the PL category, the differentiable category and the topological category. In the topological category, TOP, there is no additional structure on a manifold save that in the definition. The maps considered between topological manifolds are merely continuous. Two manifolds are equivalent if they are homeomorphic.

In the PL category we study triangulated manifolds (M, T). Note that there is a natural functor from PL to TOP given by $(M, T) \to M$. In this category we consider maps $h : (M_1, T_1) \to (M_2, T_2)$ that correspond are simplicial maps from T_1 to T_2. Such maps are called PL maps. Unless otherwise noted, a submanifold (M', T') of a triangulated manifold (M, T) is required to have T' be a subcomplex of T. A PL homeomorphism is a PL map that is also a homeomorphism and that has a PL inverse. Two PL manifolds are equivalent if they are PL homeomorphic.

In the DIFF category we study smooth manifolds. A map is smooth if it can be differentiated infinitely many times. A diffeomorphism is a smooth homeomorphism. (The inverse function theorem implies that its inverse will also be smooth.) In a smooth n-manifold M each point $x \in M$ is required to have a neighborhood diffeomorphic to $\mathbb{R}^n$ (or $\{(x_1, \ldots, x_n) \mid x_n \geq 0\}$ in the case of a n-manifold with boundary). In this category we consider smooth maps between smooth manifolds. Two differentiable manifolds are equivalent if they are diffeomorphic.

Interestingly, these three categories coincide for 3-manifolds. For instance, every topological 3-manifold admits a triangulation. (This gives a map from TOP to PL.) Furthermore, given two triangulations of the same 3-manifold there is a PL homeomorphism between the two triangulations. Much of this work is due to R. H. Bing. Similarly every differentiable 3-manifold admits a triangulation. (This gives a map from DIFF to PL.) And there is a PL homeomorphism between any two triangulations arising in this way. Much of this work is due to E. Moise.

It must be mentioned that this equivalence of categories is not true in general. This was first shown by Milnor. He exhibited examples of 7-spheres with distinct differentiable structures. A consequence is that there are also 7-sphres with distinct PL structures. As far as I know it is unknown whether there are smooth structures on S^4 that are not diffeomorphic. A candidate for an "exotic" smooth structure on the 4-sphere was given by Scharlemann. But much later it was proven by Akbulut that this smooth structure is in fact diffeomorphic to the standard smooth structure on the 4-sphere.

The advantage of the equivalence of the categories TOP, PL and DIFF for 3-manifolds is that one may prove a theorem in the category that best suits the theorem. In this chapter we will be working in TOP. In the succeeding chapters we will lay the foundations for the PL category.

2.1 Irreducibility

As will be shown, there is such as thing as prime factorization for 3-manifolds. This prime factorization is almost unique. The obstruction has to do with the more general notion of irreducibility which we here define.

Definition 2.1. *A 3-manifold M is irreducible if every 2-sphere in M bounds a 3-ball.*

The following theorem generalizes the well known Jordan Curve Theorem. The smoothness assumption is necessary. This was shown by J.W. Alexander who exhibited the Alexander horned sphere, a wildly embedded sphere in $\mathbb{R}^3$ that does not bound a 3-ball.

Theorem 2.2. *(The Schönflies Theorem) Any smooth 2-sphere in $\mathbb{R}^3$ bounds a 3-ball.*

A beautiful elementary proof of this theorem was given by Morton Brown in 1960. Students are encouraged to read this proof in the original. The following proof appeared in a lecture series given by Andrew Casson in China in 2002. We include it here because it illustrates some of the techniques employed in the contemporary study of 3-manifolds. This is a DIFF proof. (It could be made into a PL proof, but at the cost of clarity.)

Proof: Let $S \subset \mathbb{R}^3$ be a smooth 2-sphere. Then S may be perturbed so that the "height function" $h : \mathbb{R}^3 \to \mathbb{R}$ given by projection onto the third coordinate has a finite number of maxima, minima and saddle points and no other critical points.

By the Poincaré-Hopf Index Theorem,

$$\#maxima + \#minima - \#saddles = \chi(S^2) = 2.$$

We prove that S bounds a 3-ball by induction on the number n of saddle points.

If $n = 0$, then S has one maximum and one minimum. It is thus isotopic to the standard 2-sphere in $\mathbb{R}^3$ and hence bounds a 3-ball.

If $n = 1$, then there are two minima (one minimum, resp.), one maximum (two maxima, resp.) and one saddle. There are two possibilities: (a "non nested" saddle or a "nested" saddle) In both cases, S bounds a 3-ball.

Suppose now that $n \geq 2$, and suppose that every 2-sphere in $\mathbb{R}^3$ with less than n saddles bounds a 3-ball. We may assume, after a small perturbation, that the saddles do not all occur at the same level. Thus there is a level surface H such that there are saddle points both above and below H. Now $H \cap S$ is a compact 1-manifold (without boundary), so $H \cap S$ is a finite union of disjoint circles in H.

Let C be an innermost component of $H \cap S$ in H. It follows that there is a disk $D \subset H$ that meets S only in C. Here C separates S into two disks, D_1, D_2. Set $S_1 = D \cup D_1$ amd $S_2 = D \cup D_2$. Then both S_1 and S_2 are piecewise smooth 2-spheres in $\mathbb{R}^3$. They may be perturbed slightly to be smooth. There are two cases:

Case 1: If S_1 and S_2 both have saddles, then each of them has less than n saddles. Thus S_1 and S_2 both bound 3-balls in $\mathbb{R}^3$. It follows that S also bounds a 3-ball.

Case 2: If S_1, say, has no saddle, then S_1 bounds a 3-ball. We may thus isotope D_1 to coincide with D and then further isotope D to eliminate the component C of $H \cap S$. Now repeat the process with a new innermost circle. Since H was chosen so that there were saddle points on either side of H, this process eventually leads to Case 1.

□

Corollary 2.3. *$\mathbb{R}^3$, the 3-ball and $\mathbb{S}^3$ are irreducible.*

Here $\mathbb{S}^3 = \mathbb{R}^3 \cup \infty$ and the interior of the 3-ball is $\mathbb{R}^3$.

The same argument as above also gives the following theorem of Alexander.

Theorem 2.4. *If T is a smooth torus in $\mathbb{S}^3$, then one of the components of $\mathbb{S}^3 \backslash T$ has closure homeomorphic to a solid torus $S^1 \times \mathbb{B}^2$.*

Proof: Let $T \subset \mathbb{R}^3$ be a smooth torus. Then T may be perturbed so that the "height function" $h : \mathbb{R}^3 \to \mathbb{R}$ given by projection onto the third coordinate has a finite number of maxima, minima and saddle points and no other critical points.

By the Poincaré-Hopf Index Theorem,

$$\#maxima + \#minima - \#saddles = \chi(torus) = 0.$$

We prove that T bounds a solid by induction on the number n of saddle points. Note that since T is compact it contains at least one maximum and one minimum. The Poincaré-Hopf Index Theorem then implies that T has at least two saddles.

If $n = 2$, then T has one maximum and one minimum. A level curve of $h|_T$ near the maximum bounds a disk in the corresponding level surface of h. At the first saddle, this disk is either

(1) pinched into two disks, or

(2) turns into a pinched annulus.

Between the two saddles a level surface intersects T correspondingly either

(1) in two non nested circles bounding disks or

(2) in two nested circles cobounding an annlus.

Below the second saddle, (near the minimum), a level curve of $h|_T$ again bounds a single disk. Thus at the second saddle, either

(1) the two non nested circles bounding disks are wedged together or

(2) the annulus is pinched.

In both cases, these disks and annuli stack on top of each other to form a solid torus.

Suppose now that $n \geq 3$, and suppose that every torus in $\mathbb{R}^3$ with less than n saddles bounds a solid torus. We may assume, after a small perturbation, that the saddles do not all occur at the same level. Thus there is a level surface H such that there are saddle points both above and below H. Now $H \cap T$ is a compact 1-manifold (without boundary), so $H \cap T$ is a finite union of disjoint circles in H.

Let C be an innermost component of $H \cap T$ in H. It follows that there is a disk $D \subset H$ that meets S only in C. There are two cases:

Case 1: C is separating in T.

Let P_1, P_2 be the surfaces obtained by cutting T along C. Since $\chi(P_i)$ is odd (orientable surface with one boundary component), $\chi(P_i) \leq 2$ and $\chi(P_1) + \chi(P_2) = \chi(T) = 0$, it must be the case that either P_1 or P_2, say P_1, is a disk and that the other component, P_2, is a punctured torus.

Set $S_1 = D \cup P_1$. Then S_1 is a piecewise smooth 2-sphere in $\mathbb{R}^3$ that can be perturbed to be a smooth 2-sphere. It thus bounds a 3-ball. This 3-ball describes an isotopy of P_1 into D. Thus T is isotopic to $(T \backslash P_1) \cup D$. If P_1 contained saddles, then $(T \backslash P_1) \cup D$ is a piecewise smooth torus with fewer than n saddles. It can be perturbed into a smooth torus with fewer than n saddles. Thus we are done by induction. If P_1 contained no saddles, choose a new innermost component of $(H \cap T) \backslash C$.

Case 2: C is not separating in T.

Here $T \backslash C$ is an annulus A. We may color the portion of this annulus that lies above H red and the portion that lies below H blue. Then every point in A is either colored red, or colored blue, or lies on one of the circles in $H \cap T$. Just above C, A is colored red. Just below C, A is colored blue. Thus there must be at least one other component C' of $H \cap T$ that is not separating in T. We may assume that this component is innermost. For if it is not, then we may either proceed as in Case 1 (if it contains an innermost separating component) or choose an innermost such component (if it contains an innermost non separating component).

Now $T \backslash (C \cup C')$ consists of two annuli A_1, A_2. Denote the disk in $H \backslash T$ bounded by C by D and that bounded by C' by D. Set $S_1 = A_1 \cup D \cup D'$ and $S_2 = A_2 \cup D \cup D'$. Both S_1 and S_2 are piecewise smooth 2-spheres. Thus each of them bounds a 3-ball on either side. Two of these 3-balls may be identified along D_1 and D_2 to form a solid torus. □

The following is a notion related to irreducibility for manifolds with nonempty boundary.

Definition 2.5. *A 3-manifold M is boundary irreducible if every curve c embedded in ∂M that bounds a disk in M also bounds a disk in ∂M.*

Exercise 1: Draw a picture of a piecewise smooth closed orientable surface S of genus 2 in $\mathbb{R}^3$ so that neither component of $\mathbb{R}^3 \backslash S$ is a 3-dimensional fattening of two circles joined at a point. (Thus, the Schönflies Conjecture does not generalize to surfaces of genus greater than or equal to 2.)

Exercise 2: Let $\tilde{M}$ be a covering space of a 3-manifold M. Prove that $\tilde{M}$ is irreducible only if M is irreducible. (The converse is true but is more difficult to prove.)

Exercise 3: Use Exercise 2 to deduce that the three torus T^3 is irreducible.

2.2 Primality

We wish to define a notion of primality for 3-manifolds. To do so, we need a few auxiliarly definitions and theorems.

Definition 2.6. *(Isotopy) A continuous function $H : M \times I \to M$ such that $H(,t) : M \to M$ is a homeomorphism for each $t \in I$ is called an isotopy.*

Two submanifolds S_0, S_1 of M are isotopic if there is an isotopy $H : M \times I \to M$ such that $H(,0)|_{S_0}$ is the identity and $H(,1)|_{S_0} : S_0 \to S_1$ is a homeomorphism.

The following two theorems are due to V.K.A.M. Guggenheim (in the PL category). They are fundamental theorems in the study of 3-manifolds. We will need these theorems in order for Definition 2.9 below to be well defined.

Theorem 2.7. *Every orientation preserving homeomorphism of a n-ball or n-sphere is isotopic to the identity.*

Proof: The proof of this theorem is left as an exercise. □

Theorem 2.8. *If B_1, B_2 are n-balls in the interior of a connected n-manifold M, then there is an isotopy $f : M \times I \to M$ such that $f(,0)|_{B_1}$ is the identity and $f(,1)_{B_1} : B_1 \to B_2$ is a homeomorphism.*

Proof: The proof is left as an exercise. □

Definition 2.9. *(Connected sum of n-manifolds) Given two n-manifolds M_1, M_2, we may delete small open n-balls B_1 from M_1 and B_2 from M_2. We may then identify M_1 and M_2 along the resulting (n-1)-sphere boundary components. The resulting n-manifold is called the connected sum of M_1 and M_2 and denoted by $M_1 \# M_2$.*

Example: The genus 2 surface is the connected sum of two tori.

Definition 2.10. *(Prime n-manifold) A n-manifold M is prime if $M = M_1 \# M_2$ implies that either M_1 or M_2 is the n-sphere.*

Exercise 1: Prove Theorem 2.7.

Exercise 2**: Prove Theorem 2.8.

2.3 3-manifolds that are prime but reducible

In the context of irreducibility and primality, two 3-manifolds stand out. One of these 3-manifolds is $S^2 \times S^1$. The other, a twisted version of this 3-manifold, is defined below.

Definition 2.11. *Consider $S^2 \times I$. Let $a : S^2 \to S^2$ be the antipodal map and let $f : S^2 \times \{0\} \to S^2 \times \{1\}$ be defined by $f(x, 0) = (a(x), 1)$. Then $S^2 \tilde{\times} S^1$ is the quotient space obtained from $S^2 \times I$ by identifying points via f.*

Definition 2.12. *A subset A of a connected set X is separating, if $X \backslash A$ has at least two components.*

The 3-manifolds $S^2 \times S^1$ and $S^2 \tilde{\times} S^1$ contain non separating 2-spheres. Theorem 2.15 below shows that this is a rare property.

The following definition and theorem are fundamental theorems in the study of 3-manifolds. Proofs may be found in Chapter 4 of Rourke and Sanderson's "Introduction to PL topology". (The proof for the analogous theorem in the differentiable category can be found in Chapter 2 of Guillemin and Pollack's "Differential Topology".)

Definition 2.13. *Let M be a n-manifold. Let S be a submanifold of M of dimension m. A regular neighborhood of S is a submanifold $N(S)$ of dimension n such that for each point $x \in S$ there is a neighborhood U of x in S and a neighborhood V of x in $N(S)$ such that $V = U \times \mathbb{B}^{n-m}$.*

Examples

Theorem 2.14. *Let M be a n-manifold. Let S be a submanifold of dimension m. Then there is a regular neighborhood $N(S)$ for S in M. Any two regular neighborhoods of S in M are isotopic. If both M and S are orientable, then $N(S)$ is homeomorphic to $S \times \mathbb{B}^{n-m}$.*

Theorem 2.15. *An irreducible 3-manifold is prime. An orientable prime 3-manifold is either irreducible or $S^2 \times S^1$.*

Remark 2.16. *More generally, a prime 3-manifold is either irreducible or $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$. But we will not prove this fact here.*

Proof: A connected sum of 3-manifolds contains a sphere that does not bound a 3-ball. Hence an irreducible 3-manifold is prime.

Suppose M is prime and let S be a 2-sphere in M. If S is separating, then $M - S$ has two components, N_1, N_2. If neither N_1 nor N_2 is a 3-ball, then $M = N_1 \# N_2$ and M is not prime. Thus either N_1 or N_2 is a 3-ball, i.e., S bounds a 3-ball.

If S is non separating, let $N(S)$ be a regular neighborhood of S in M. Let $\tilde{M}$ be the 3-manifold obtained by removing the interior of $N(S)$ from M. Then $\partial \tilde{M} = \partial N(S)$ consists of two copies of S.

Let a be a point in S. Since S is non separating in M, there is an embedding $\alpha : I \to \tilde{M}$ with enpoints on the two distinct copies of a in $\partial \tilde{M}$. Then $\alpha(I)$ is a

1-dimensional submanifold of $\tilde{M}$. Let $N(\alpha(I))$ be a regular neighborhood of $\alpha(I)$ in $\tilde{M}$.

Let $\bar{M}$ be the 3-manifold obtained by removing teh interior of $N(\alpha(I))$ from $\tilde{M}$. Then $\partial\bar{M}$ is a 2-sphere $\bar{S}$. Note that $\bar{S}$ is also a 2-sphere in M. Moreover, in M, $\bar{S}$ is a separating 2-sphere. To one side of $\bar{S}$ is $N(S) \cup N(\alpha(I))$. Hence, by the argument above, there must be a 3-ball to the other side of $\bar{S}$. Therefore $M = N(S) \cup N(\alpha(I)) \cup 3 - ball = S^2 \tilde{\times} S^1$. □

2.4 Incompressible Surfaces

In the study of surfaces, curves lying on the surface play an important role in the cut and paste techniques that suffice to classify surfaces. Generalizations of these techniques do not quite appear to suffice to classify 3-manifolds. Nevertheless, surfaces lying in 3-manifolds provide some information about the 3-manifold. Not all surfaces are interesting. We here discuss one of the more interesting classes of surfaces.

Definition 2.17. *A submanifold S in a n-manifold M is properly embedded if $\partial S = S \cap \partial M$.*

Definition 2.18. *(Essential curve, essential arc) A curve c in a surface F is essential if it does not bound a disk in F.*

An arc α properly embedded in a surface F is essential if there is no arc β embedded in ∂F such that $\alpha \cup \beta$ is a simple closed curve that bounds a disk in F.

The following definitions generalize these notions.

Definition 2.19. *A surface S embedded in a 3-manifold M is compressible if either*

(1) S is a 2-sphere bounding a 3-ball in M; or

(2) there is a curve c embedded in S that bounds a disk embedded in M but not in S.

A surface that is not compressible is called incompressible.

Example: $T^2 \subset T^3$.

Definition 2.20. *A surface S embedded in a 3-manifold M is boundary compressible if there is an essential arc α properly embedded in S and an arc β properly embedded in ∂M so that $\alpha \cup \beta$ form a simple closed curve that bounds a disk D in M with $D \cap S = \alpha$.*

The following definition honors Wolfgang Haken who pioneered the study of incompressible surfaces and of normal surfaces.

Definition 2.21. *An orientable irreducible 3-manifold that contains a 2-sided incompressible surface is called a Haken 3-manifold.*

Exercise 1: A closed curve in S^3 is necessarily homotopic to a point.

Exercise 2: A torus in S^3 is necessarily compressible.

Exercise 3: A connected incompressible surface $\tilde{S}$ in a 3-ball B is either boundary compressible or it is a disk.

3 Triangulated 3-manifolds

3.1 Simplicial Complexes

One approach to studying 3-manifolds involves the PL category. The following definitions provide the groundwork for this study. This section follows 4.1-4.3 in Singer and Thorpe's "Lecture Notes on Elementary Topology and Geometry" fairly closely. There is a subtle but crucial departure from the traditional definition in Definition 3.6.

Definition 3.1. *(Simplex; simplices) Let V be a vector space over $\mathbb{R}$. Let $\{v_0, \ldots, v_k\}$ be a linearly independent set of vectors. The (convex) set $\{a_0v_0 + \cdots + a_kv_k | a_0 \geq 0, \ldots, a_k \geq 0, \sum_{i=0}^k a_i = 1\}$ is called a k-simplex. It is denoted by $[v_0, \ldots, v_k]$ or simply by $[s]$. The set $\{a_0v_0 + \cdots + a_kv_k | a_0 > 0, \ldots, a_k > 0, \sum_{i=0}^k a_i = 1\}$ is called an open k-simplex and denoted by $(v_0, \ldots, v_k)$ or simply by (s). The dimension of a k-simplex is k.*

Definition 3.2. *(Barycentric coordinates; barycenter) For $v = a_0v_0 + \cdots + a_kv_k$, $\{a_0, \ldots, a_k\}$ are called the barycentric coordinates of v. The point $\frac{1}{k+1}v_0 + \cdots + \frac{1}{k+1}v_k$ is called the barycenter of $[v_0, \ldots, v_k]$ and denoted by $b([v_0, \ldots, v_k])$.*

Remark 3.3. *Note that $b([v_0]) = v_0$.*

Definition 3.4. *(Faces) The faces of $[v_0, \ldots, v_k]$ are the l-simplices of the form $[v_{j_0}, \ldots, v_{j_l}]$, for $0 \leq l \leq k$ and $\{v_{j_0}, \ldots, v_{j_l}\}$ a linearly independent subset of $\{v_0, \ldots, v_k\}$.*

A 0-dimensional face of a simplex is also called a vertex. A 1-dimensional face of a simplex is also called an edge.

Remark 3.5. *Note that $[v_0] = (v_0)$. I.e., a vertex is necessarily open.*

Definition 3.6. *(Simplicial complex) A simplicial complex K is a finite set of open simplices such that*

(1) if $(s) \in K$, then all open faces of $[s]$ are in K;

(2) if $(s_1), (s_2) \in K$ and $(s_1) \cap (s_2) \neq \emptyset$, then $(s_1) = (s_2)$.

The dimension of a simplicial complex K is the dimension of the highest dimensional simplex in K. We denote the set of points in the simplicial complex K by $|K|$.

Example:

Remark 3.7. *A simplex inherits a metric from the vector space in which it lies. The distance between two points on a simplex is realized by a path with that length. Note that a connected simplicial complex is path connected. This induces a natural metric on a connected simplicial complex: the distance between two points on distinct simplices is the minimum length of a path between the two points.*

Note that closed simplices are compact. It follows that a connected simplicial complex is also a compact metric space. A simplicial complex K, and $|K|$ too, is topologized via the metric topology.

Definition 3.8. *A triangulation of a n-manifold M is a simplicial complex K such that K is homeomorphic to M. Given a n-manifold M with triangulation K, we call the pair (M, K) a triangulated n-manifold.*

Remark 3.9. *For (M, K) a triangulated n-manifold, the homeomorphism between M and K provides a natural identification of M and K. We may thus refer to "the simplices in M".*

Example: S^2, S^3

Example: T^2, T^3

Theorem 3.10. *Every compact 1-manifold admits a triangulation. Every compact 2-manifold admits a triangulation.*

The proof of the first part of the above theorem is an easy exercise in understanding manifolds. The proof of the second part of the above theorem can be found for instance in Massey's "A Basic Course in Algebraic Topology". The theorem below is much harder to prove. It was proven simultaneously from two different points of view.

Theorem 3.11. *(R.H. Bing, E. Moise) Every compact 3-manifold admits a triangulation.*

Definition 3.12. *(Subcomplex) A subcomplex of a simplicial complex K is a simplicial complex L such that $(s) \in L$ implies $(s) \in K$.*

Definition 3.13. *(r-skeleton) Let K be a k-simplex. Then for $r \leq k$, the r-skeleton K^r of K is the collection $K^r = [(s) \in K; dims \leq r]$.*

Remark 3.14. *The r-skeleton of a simplicial complex K is a subcomplex of K.*

Definition 3.15. *(General position) Let $v \in \mathbb{R}^n$ and let $A \subset \mathbb{R}^n$. The pair (v, A) is in general position if $v \notin A$ and, for each $a_1, a_2 \in A$ with $a_1 \neq a_2$, $[v, a_1] \cap [v, a_2] = \{v\}$.*

Example:

Definition 3.16. *(Cone) Let (v, A) be in general position. The set*

$$\cup_{a \in A}[v, a]$$

is called the cone of v over A and denoted by $v * A$.

Definition 3.17. *(Subdivision) A subdivision of a simplicial complex K is a simplicial complex K' such that*

(1) $|K| = |K'|$;

(2) if (s) is an open simplex in K', then (s) is a subset of some open simplex of K.

Definition 3.18. *(A partial ordering on simplices) Let K be a simplicial complex. Define a partial ordering on K by $(s_1) \leq (s_2)$ if and only if (s_1) is a face of (s_2). We write $(s_1) < (s_2)$ when $(s_1) \leq (s_2)$ and $(s_1) \neq (s_2)$.*

Theorem 3.19. *(Barycentric subdivision) Let K be a simplicial complex. Set*

$$K^{(1)} = \{(b([s_0]), b([s_1]), \ldots, b([s_l])) \mid [s_0], [s_1], \ldots, [s_l] \in K \ni [s_0] < [s_1] < \cdots < [s_l]\}.$$

Then $K^{(1)}$ is a subdivision of K.

Proof: The proof is left as an exercise. □

Example

Definition 3.20. *The subdivision $K^{(1)}$ of K is called the first barycentric subdivision of K. The subdivision $K^{(n)} = (((K^{(1)})^{(1)})\ldots)^{(1)}$ of K is called the n-th barycentric subdivision of K.*

Definition 3.21. *The mesh of a simplicial complex K is the maximum of the diameters of the simplices in K.*

Theorem 3.22. *(mesh decreases under subdivision) Let K be a simplicial complex of dimension k. Then $meshK^{(1)} \leq \frac{k}{k+1} meshK$.*

Proof: The proof is left as an exercise. □

The upshot of the above two theorems is the following: Given a simplicial complex K, consider the mesh of K. In certain situations, $meshK$ may be larger than required. To remedy this fact, we take barycentric subdivisions until the mesh of the resulting subdivision is as small as required.

Definition 3.23. *Let K and L be simplicial complexes. A map $\phi : K \to L$ is a simplicial map if*

(1) for each vertex v of K, $\phi(v)$ is a vertex of L;

(2) for each simplex $(v_0, \ldots, v_k) \in K$, the vertices $\phi(v_0), \ldots, \phi(v_k)$ all lie in some closed simplex of L; and

(3) for $p = a_0 v_0 + \cdots + a_l v_l \in (s)$, $\phi(p) = a_0\phi(v_0) + \cdots + a_l\phi(v_l)$.

Examples

Definition 3.24. *(Star) The star of v, for v a vertex of a simplicial complex K, is the set*

$$St(v) = \{(s) \in K \mid v \in [s]\}.$$

We denote the set of points in $St(v)$ by $|St(v)|$.

More generally, the star of a subcomplex K' of K is the set

$$St(K') = \{(s) \in K \mid K' \cap [s] \neq \emptyset\}.$$

We denote the set of points in $St(v)$ by $|St(v)|$.

Definition 3.25. *Let K, L be simplicial complexes and $f : |K| \to |L|$ a continuous map. A simplicial map $\phi : K \to L$ is a simplicial approximation to f if $f(|St(v)|) \subset |St(\phi(v))|$ for each vertex v of K.*

Theorem 3.26. *Let K, L be simplicial complexes and $f : |K| \to |L|$ a continuous map. Let $\{K_n\}$ be a sequence of subdivisions of K such that $lim_{n\to\infty} meshK_n = 0$. Then, for n sufficiently large, f has a simplicial approximation.*

Proof: It is an easy exercise (see Exercises 4 and 5) to show that $\{|St(w)|\}_{w\in L^0}$ is an open covering of $|L|$. It then follows, since f is continuous, that $\{f^{-1}(|St(w)|)\}_{w\in L^0}$ is an open covering for the underlying set of $|K|$. Since the underlying set of K is a compact metric space, the Lebesgue number lemma provides an $\epsilon > 0$ such that every ball of radius ϵ lies in some open set of this covering.

Since $lim_{n\to\infty} meshK_n = 0$, we may choose N so that $meshK_n < \frac{\epsilon}{2}$ for $n \geq N$. Hence for $n \geq N$ and each $v \in K_n^0$, $|St(v)|$ is contained in the ball of radius ϵ based at v. But by our choice of ϵ, this means that $|St(v)| \subset f^{-1}(|St(w)|)$ for some $w \in L^0$. Thus $f(St(v)) \subset St(w)$. Define $\phi : K_n \to L$ by first defining $\phi|_{K_n^0} : K_n^0 \to L^0$ as follows: For $v \in K_n^0$, choose $\phi(v)$ to be any vertex w for which $f(|St(v)| \subset |St(w)|$. There may be more than one choice, but there are at most finitely many such choices. These choices guarantee that $f(|St(v)|) \subset |St(\phi(v))|$ for all $v \in K_n^0$.

Now extend $\phi|_{K_n^0} : K_n^0 \to L^0$ to a simplicial map $\tilde{\phi} : K_n \to L$ via barycentric coordinates. To see that this can be done, we must show that for each simplex $(v_0, \ldots, v_k) \in K_n$, the vertices $\phi(v_0), \ldots, \phi(v_k)$ all lie in some closed simplex of L. Consider a point $p \in (v_0, \ldots, v_k)$. Then $p \in \cap_{i=0}^k |St(v_i)|$. So $f(p) \in \cap_{i=0}^k |St(\phi(v_i))|$. Thus $\cap_{i=0}^k |St(\phi(v_i))| \neq \emptyset$. On the other hand, as each open simplex of L is either contained in or disjoint from $St(w)$ for $w \in L^0$, $\cap_{i=0}^k St(\phi(v_i))$ consists of open simplicies. Thus $p \in (t)$, for some open simplex (t). Since $(t) \in \cap_{i=0}^k St(\phi(v_i))$, $v_0, \ldots, v_k \in [t]$.

Now extending $\phi|_{K_n^0} : K_n^0 \to L^0$ to a simplicial map $\tilde{\phi} : K_n \to L$ linearly via barycentric coordinates yields a simplicial approximation to f. ☐

Definition 3.27. *(Closure) The closure of a simplicial complex or union of simplicies L, denoted by $\bar{L}$, is defined by $\bar{L} = \{[s] \mid (s) \in L\}$.*

Definition 3.28. *Consider a simplicial complex K. For $v \in K^0$, consider the star $St(v; K^{(1)})$ of v in the first barycentric subdivision of K. The link of v, denoted by $link(v)$, is defined by $link(v) = \overline{St(v; K^{(1)})} \backslash St(v; K^{(1)})$.*

Exercise 1: Let $[s] = [v_0, \ldots, v_k]$. Prove that $(v_0, [v_1, \ldots, v_k])$ is in general position and $v_0 * [v_1, \ldots, v_k] = [s]$.

Exercise 2*: Prove that $K^{(1)}$ is a subdivision of K. (Hint: Use induction on the dimension of K and verify the two properties that characterize a simplicial complex.)

Exercise 3: Prove that for K a simplicial complex of dimension k, $meshK^{(1)} \leq \frac{k}{k+1} meshK$.

Exercise 4: Prove that $St(v)$ is open.

Exercise 5: Prove that for a simplicial complex K, the collection $\{St(v)\}_{v\in K^0}$ is a covering of the underlying set of K.

Exercise 6**: Prove that the link of any vertex of a triangulated 3-manifold is homeomorphic to a sphere.

3.2 Normal Surfaces

In the context of triangulated 3-manifolds the natural class of surfaces to consider is that of so called normal surfaces. We shall see that incompressible surfaces can be modified slightly to be normal surfaces. In this context, neither incompressible surfaces nor normal surfaces are thought of being subcomplexes of the triangulations of the 3-manifolds under consideration.

Definition 3.29. *(Normal curve) An arc on a 2-dimensional face $[f]$ of a 3-simplex $[s]$ is normal if its endpoints lie on distinct 1-dimensional faces of $[f]$. A closed curve on the 2-dimensional faces of a 3-simplex $[s]$ is a curve c such that any component of intersection of c with a 2-dimensional face $[f]$ of $[s]$ is a normal arc.*

Definition 3.30. *The length of a normal curve c on the 2-dimensional faces of a 3-simplex $[s]$ is the number of points in $c \cap |[s]^1|$.*

Lemma 3.31. *A normal curve on the 2-dimensional faces of $[s]$ either has length 3 or 4 or it meets some edge more than once.*

Proof: Here $|[s]^2|$ is homeomorphic to a sphere. Thus c is a Jordan curve on the sphere. It thus separates $|[s]^2|$ into an "inside" (disk) and "outside" (disk). Consider $|[s]^0|$. It consists of four points. Up to renaming, there are two possibilities. The "inside" of c contains either one or two points of $|[s]^0|$.

Suppose that the "inside" of c contains only the vertex v. Let e_1, e_2, e_3 be the edges incident to v. Then c must intersect e_1, e_2, e_3. Furthermore, since the other vertices to which e_1, e_2, e_3 are incident lie "outside" of c, c must intersect e_1, e_2, e_3 an odd number of times. Similarly, it must meet the other three edges an even number of times. Thus if c meets no edge more than once, then c has length 3.

Suppose now that the "inside" of c contains the verticies v_1, v_2. Then there are four edges that are met an odd number of times and two edges that are met an even number of times. If the former edges each meet c once and the latter are disjoint from c, then c has length 4. □

Definition 3.32. *(Normal disk) A normal triangle in a 3-simplex $[s]$ is the link of any one of the vertices of $[s]$ in the first barycentric subdivision of $[s]$. A normal quadrilateral in a 3-simplex $[s]$ is the link of any one of the edges of $[s]$ in the first barycentric subdivision of $[s]$. Normal triangles and quadrilaterals are called normal disks.*

Definition 3.33. *(Normal surface) Let (M, K) be a triangulated 3-manifold. A normal surface in M is a surface $S \subset M$ such that for any 3-simplex $[s]$ in K, $S \cap |[s]|$ consists of disjoint normal disks in $[s]$.*

Example 1: A normal sphere in S^3.

Example 2: A normal torus in T^3.

Remark 3.34. *Let M be a n-manifold containing a k-dimensional submanifold K and an l-dimensional submanifold L. We will always assume, unless stated otherwise, that $K \cap L$ is a (possibly empty) (k+l-n)-dimensional submanifold of M. Furthermore,* *if K and L are compact, we will always assume that $K \cap L$ is also compact. These properties are guaranteed by general position. For the formal definition of general position, and for the proof of the fact that general position can always be guaranteed by a small isotopy, see Chapter 5 of Rourke and Sanderson's "Introduction to PL topology". (For the analogous defintion and theorem in the differentiable category, see Chapter 1 of Guillemin and Pollack's "Differential Topology".)*

Definition 3.35. *The weight of a surface S in a triangulated 3-manifold (M, K) is the number of components of $S \cap K^1$. It is denoted by $w(S)$. Similarly, $m(S)$ is the number of components of $S \cap (K^2 \backslash K^1)$.*

Theorem 3.36. *Let M be an irreducible 3-manifold containing an incompressible surface S. Then for any triangulation (M, K) of M there is an isotopy that takes S to a normal surface in (M, K).*

Proof: Let (M, K) be a triangulation of M and let S be an incompressible surface in M. Isotope S so that $(w(S), m(S))$ is minimal (in the dictionary order). The minimality of $w(S)$ implies that for each 3-simplex $[s]$ in K, S meets the 2-dimensional faces of $[s]$ in a finite number of disjoint normal arcs along with simple closed curves entirely contained in the open 2-dimensional faces.

Let (f) be a 2-dimensional face of $[s] \in K$ and suppose that $S \cap |[f]|$ contains a simple closed curve s. Further assume that s is an innermost such curve in $|[f]|$. Then s bounds a disk D in $|[f]|$ that meets S only in its boundary. Since S is incompressible, it follows that s also bounds a disk D' in S. Since D is disjoint from S away from $s = \partial D$, $D \cup D'$ is a 2-sphere. Since M is irreducible, $D \cup D'$ bounds a 3-ball. It follows that D' can be isotoped to coincide with D. A further isotopy then eliminates the component s of $S \cap T^2$. This contradicts the minimality of $(w(S), m(S))$. Thus for each 3-simplex $[s]$ in K, S meets each face (f) of $[s]$ in normal arcs. Hence $S \cap \partial|[s]|$ is a finite number of disjoint normal curves.

Let $[s]$ be a 3-simplex in K. Let c be a normal curve in $S \cap \partial|[s]|$. Let $\tilde{S}$ be the component of $S \cap |[s]|$ such that $c \in \partial\tilde{S}$. Since c bounds a disk E in the 3-ball $|[s]|$, it must in fact bound a disk $\tilde{E}$ in $\tilde{S}$. Here $E \cup \tilde{E}$ is a 2-sphere in an irreducible 3-manifold and hence bounds a 3-ball B. A priori E may not be disjoint from S. But the procedure in the above paragraph shows how to eliminate curves on intersection in $S \cap E$.

If $\tilde{E}$ does not lie entirely in $|[s]|$, then B describes an isotopy lowering $(w(S), m(S))$, a contradiction. Thus $\tilde{S} = \tilde{E}$. In particular, $\tilde{S}$ is a disk.

Suppose that $\partial\tilde{S}$ meets an edge $[e]$ of $[s]$ more than once. Since $\tilde{S}$ is a disk in the 3-ball $|[s]|$, it is isotopic to one of the disks bounded by $\partial\tilde{S}$ in $\partial|[s]|$. In particular, there is a disk E' such that $\partial E' = \alpha \cup \beta$ with $\alpha \subset \tilde{S}$ and $\beta \subset \partial|[s]|$ and such that E' is disjoint from $\tilde{S} \backslash \alpha$. But this contradicts the minimality of $w(S)$.

It now follows from Lemma 3.31 that $\partial\tilde{S}$ has length 3 or 4. Thus $\partial\tilde{S}$ is a normal disk. Hence S has been isotoped to be a normal surface. □

Exercise 1: Choose a triangulation $(\mathbb{S}^3, K)$ of $\mathbb{S}^3$ and give an example of a normal surface in $(\mathbb{S}^3, K)$.

Exercise 2: Consider the following "converse" of Theorem 3.36: A normal surface in a triangulated 3-manifold is incompressible. Is this true or false?

3.3 Diophantine Equations and Inequalities

Normal surfaces are described by vectors whose entries satisfy certain integral equations and inequalities. Specifically, note that in a given 3-simplex there are exactly four types of normal triangles and exactly three types of normal quadrilaterals. Thus to describe a given normal surface, we need only indicate how many of which type of normal triangle and normal quadrilateral occur. Hence if t is the number of 3-simplices in a triangulated 3-manifold, then a normal surface is completely described by a vector with 7t entries.

Given two normal surfaces S_1, S_2 and corresponding vectors $\vec{v}_1, \vec{v}_2$, we wish to define the sum $S_1 + S_2$ corresponding to $\vec{v}_1 + \vec{v}_2$. This is in principle possible, but there is one obstruction. On a face of a 3-simplex, we may see two intersecting normal arcs. (In the 3-simplex, we may be able to isotope these apart, but note that the surface connects up to other normal pieces, so the isotopy may not allow us to make any progress globally.) Changing + to is called a regular switch.

If in a given 3-simplex, two normal triangles intersect, the regular switches on the faces of the 3-simplex extend into the 3-simplex. Similarly, if in a given 3-simplex, a normal triangle and a normal quadrilateral intersect, the regular switches on the faces of the 3-simplex extend into the 3-simplex. If in a given 3-simplex, two normal quadrilaterals intersect, the regular switches on the faces of the 3-simplex extend into the 3-simplex only if the two quadrilaterals are of the same type. If the two quadrilaterals are of different types, the regular switches on the faces of the 3-simplex do not extend into the 3-simplex.

Definition 3.37. *Two normal surfaces S_1, S_2 in the triangulated 3-manifold M are said to satisfy the square restriction if for every 3-simplex σ in M, at most one type of quadrilateral occures in $(S_1 \cup S_2) \cap |\sigma|$.*

It follows that $S_1 + S_2$ is well defined for pairs of surfaces satisfying the square restriction.

Given a normal surface S, the normal triangles and quadrilaterals that constitute S match up along the faces of adjacent 3-simplices. Thus the vector $\vec{v}$ corresponding to S satisfies 2t glueing equations, i.e., $A\vec{v} = 0$ for some integral matrix A. More generally, if we wish to check whether or not the given 3-manifold contains a normal surface, we may do so by checking whether or not $A\vec{v} = 0$ has non negative integral solutions that satisfy the square restriction. This type of system of linear equations is called a Diophantine system of equations.

Theorem 3.38. *Let $Ax = 0$ be a Diophantine system of equations. Then there is a finite set of non negative solutions which generates the full set of all non negative solutions to the Diophantine system of equations.*

Proof: Here A is an m x n-matrix. We consider $\mathbb{Z}^n \subset \mathbb{R}^n$. Set $Y = \{y \in \mathbb{R}^n | y \geq 0 and Ay = 0\}$. Note that each of the rows of A defines a hyperplane through the origin. Thus Y lies in the intersection of these m hyperplanes. The intersection of these hyperplanes is a subspace V of $\mathbb{R}^n$ of dimension d at least n - m. So Y is the intersection of this subspace with the first quadrant of $\mathbb{R}^n$. In particular, Y is convex.

Now consider the hyperplane H defined by

$$x_1 + \cdots + x_n = 1.$$

Then $H \cap Y$ is a convex set that is the convex hull of a finite set of points C.

Let $y \in C$. It is not hard to see (by considering Gauss Jordan elimination), that the entries in y are rational. Thus if we multiply y by the least common multiple of the denominators of its entries, we obtain an integral point x. Let C' be the collection of integral points thus obtained. Note that C' also represents a set of vectors that spans V.

Consider the "parallelogram"

$$P' = \{\sum_{x \in C'} t_x x | t_x \in [0,1]\}.$$

Then P' is compact. Thus the set $L = P' \cap \mathbb{Z}^n$ is a finite number of points. Here L is a finite set of non negative solutions of $Ax = 0$. It remains to show that L generates the full set of non negative integral solutions of $Ax = 0$.

Let $z \in Y$. Then for some $x_1, \ldots, x_l \in C'$ and some $a_1, \ldots, a_l \in \mathbb{R}$, $z = a_1 x_1 + \cdots + a_l x_l$. But then $z - ([a_1]x_1 + \cdots + [a_l]x_l) \in P'$ and all entries of $z - ([a_1]x_1 + \cdots + [a_l]x_l) \in P'$ are differences of integers, hence themselves integers. Thus $z = z - ([a_1]x_1 + \cdots + [a_l]x_l) + ([a_1]x_1 + \cdots + [a_l]x_l)$. I.e., z is generated by C'.

□

Corollary 3.39. *There is an algorithm to detect whether or not a 3-manifold contains an incompressible surface.*

Exercise 1*: Consider the 3-manifold described at the end of Section 3.5. Give a triangulation of this 3-manifold and write down the system of Diophantine linear equations for this 3-manifold.

Exercise 2*: Does the set of fundamental solutions for Exercise 1 contain a 2-sphere?

3.4 2-spheres

Normal surface theory may be used to study 2-spheres in 3-manifolds. In particular, it allows us to lay the foundation for prime decompositions of 3-manifolds. The crucial step is a result due to H. Kneser, proven in 1929.

(A quick google search reveals: Hellmuth Kneser, born 1898, Ph.D. 1921 on Quantum Field Theory, with David Hilbert, at the Georg-August-Universität Göttingen. Instructor at Georg-August-Universität Göttingen 1921-1925, chair at Greifswald 1925-1937, chair at Tübingen from 1937 (?!) until his death in 1973.)

Definition 3.40. *A punctured 3-sphere is a 3-manifold homeomorphic to $S^3 \backslash$(finite union of 3-balls.*

Let $S = S_1 \sqcup \cdots \sqcup S_k$ be a disjoint union of 2-spheres in a 3-manifold M. We say that S is an independent set of 2-spheres if no component of $M \backslash S$ is a punctured 3-sphere.

Example: If $M = M_1 \# \ldots \# M_k$ and $M_i \neq S^3$ for all i, then M contains an independent set of $k-1$ 2-spheres.

Theorem 3.41. *(Kneser's Theorem) Suppose (M, K) is an orientable triangulated 3-manifold. Suppose further that K contains t 3-simplices. If M contains an independent set of k 2-spheres, each of which is separating, then $k < 6t$.*

More generally, Kneser proved that if (M, K) is a triangulated 3-manifold with t 3-simplices, then an independent set of k 2-spheres in M must have $k < 6t + 2dimH_2(M; \mathbb{Z}_2)$.

We first prove a lemma:

Lemma 3.42. *If M contains an independent set of k 2-spheres, each of which is separating, then for any triangulation (M, K) of M, (M, K) contains an independent set of k normal 2-spheres.*

The proof of this lemma is very similar to the proof of Theorem 3.36. The important differences are that (1) M may be reducible and (2) 2-spheres cannot be incompressible.

Proof: Let (M, K) be a triangulation of M. Choose a set $S = S_1 \sqcup \cdots \sqcup S_k$ of k independent 2-spheres such that $(w(S), m(S))$ is minimal.

The minimality of $w(S)$ guarantees that for each 3-simplex $[s] \in K$ and each face (f) of $[s]$, $S \cap |(f)|$ consists of normal arcs and simple closed curves. Suppose that $S \cap |(f)|$ contains simple closed curves and let c be an innermost such curve. Then c bounds a disk D that meets S only in its boundary. Let S_i be the component of S containing c. Furthermore, c cuts S_i into two disks, D' and D''. Set $S_i' = D \cup D'$ and $S_i'' = D \cup D''$. Set $S' = (S \backslash S_i) \sqcup S_i'$ and $S'' = (S \backslash S_i) \sqcup S_i''$.

Claim: Either S' or S'' is independent.

Suppose that both S' and S'' are not independent. Then S_i' meets a punctured 3-sphere B' and S_i'' meet a punctured 3-sphere B''. Suppose that S_i is contained, say, in B'. Then the 2-sphere S_i cuts the punctured 3-sphere B' into two punctured 3-spheres, B_1, B_2. One of these punctured 3-spheres, say B_1, has $S_i' \subset \partial B_1$. The other, B_2, does not meet S_i' and hence is a component of $M \backslash S$. (This is where we use the hypothesis that S_i is separating.) But this contradicts the independence of S.

Thus here B' and B'' meet along D. Hence $B' \cup_D B''$ forms a punctured 3-sphere B in $M \backslash S$. But this also contradicts the fact that S is independent. Thus either S' or S'' is independent.

Suppose S' is independent. Then, after a small isotopy near D that eliminates the component c of $S \cap |(f)|$, $(w(S'), m(S')) < (w(S), m(S))$. But this contradicts the minimality of $(w(S), m(S))$. Thus for each 3-simplex $[s] \in K$ and each face (f) of $[s]$ the intersection $S \cap |(f)|$ consists of normal arcs.

Now suppose $[s] \in K$ is a 3-simplex and $\tilde{S}$ a component of $S \cap |[s]|$. We wish to show that $\tilde{S}$ is a disk. Let $\tilde{c}$ be a component of $\partial \tilde{S}$. Then $\tilde{c}$ bounds a disk $\tilde{D}$ in $|[s]|$. The paragraphs above show how to eliminate components of $S \cap \tilde{D}$. Thus we may assume that $\tilde{D}$ meets S only in $\tilde{c}$. Let S_i be the component of S containing $\tilde{c}$. As above, $\tilde{c}$ cuts S_i into two disks. Here $\tilde{S}$ is contained in one of these disks. The paragaphs above show that $\tilde{S}$ cannot be properly contained in that disk, for that would contradict the minimality of $(w(S), m(S))$. Thus $\tilde{S}$ is a disk. The final two paragraphs of the proof of Theorem 3.36 now show that S is normal.

□

Proof: (Of Kneser's Theorem) Suppose M contains a set of k independent 2-spheres with $k \geq 6t$. By Lemma 3.42 M contains a set $S = S_1 \sqcup \cdots \sqcup S_k$ of k independent normal 2-spheres.

Let $[s]$ be a 3-simplex in K. A component of $\partial|[s]|\backslash S$ is "good" if it is an annulus that contains no point of $|[s]^0|$. At most six components of $\partial|[s]|\backslash S$ are "bad". (Here $S \cap |[s]|$ contains at most one type of normal quadrilateral. If, in addition, it contains all types of normal triangles, there will be six "bad" components. Otherwise, there will be fewer "bad" components.)

A component X of $M\backslash S$ is "good" if every component of $X \cap \partial|[s]|$ is good, for every 3-simplex $[s] \in K$. At most $6t$ components of $M\backslash S$ are "bad".

If $k \geq 6t$, then $M\backslash S$ has at least $6t + 1$ components. Thus there are is at least one "good" component. A "good" component is made up of regions homeomorphic to (triangle) x I and (quadrilateral) x I. It is a nontrivial fact that then $X =$ 2-sphere x I. (This is true because S is an orientable surface in an orientable 3-manifold. It is related to the fact that for an orientable surface in an orientable 3-manifold a regular neighborhood is homeomorphic to a product.) But this is a contradiction. Hence $k < 6t$. □

The above proof also proves the following theorem:

Theorem 3.43. *(Haken's Theorem) Suppose (M, K) is an orientable triangulated 3-manifold. Suppose further that K contains t 3-simplices. If M contains a set of surfaces $F = F_1 \sqcup \cdots \sqcup F_k$, such that no component of $M\backslash F$ is homeomorphic to $(surface) \times I$, then $k < 6t$.*

A similar result holds if M is not orientable. Actually, Haken used the bound 61t. Many arguments for this theorem have been given. The bound has improved over time. Combined the two theorems above are known as Kneser-Haken finiteness.

Exercise 1: Calculate the fundamental group of the three torus.

Exercise 2: List five subgroups of the fundamental group of the three torus corresponding to distinct isotopy classes of incompressible surfaces in the three torus.

Exercise 3: Try to generalize Haken finiteness to the case of incompressible surfaces with boundary properly embedded in 3-manifolds with boundary. Caution: There is a hypothesis that must be added in order to make the more general statement true.

3.5 Prime Decompositions

We here show that every 3-manifold has a prime decomposition that is unique up to a reordering of its factors. The existence of such a decomposition follows from Kneser's Theorem:

Theorem 3.44. *Each compact orientable 3-manifold can be expressed as a connected sum of a finite number of prime factors.*

Proof: Let M be a compact 3-manifold and let (M, K) be a triangulation of M. If M contains a non separating 2-sphere, then by the proof of Theorem 2.15 this non separating 2-sphere in a compact 3-manifold M gives rise to a separating 2-sphere. This separating 2-sphere either bounds a 3-ball or splits off a summand R that is either $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$. Thus $M = M^1 \# R$. If there are non separating 2-spheres in M^1, we may repeat this process. But note that if we repeat this process k times, then $M = M^k \# R_1 \# \dots \# R_k$ with $M^k, R_i \neq S^3$. Thus M contains a independent set of k 2-spheres. Hence by Kneser's Theorem for t the number of 3-simplices in K, $k < 6t$. Thus M has only finitely many summands homeomorphic to $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$.

Similarly, if M^k contains a separating 2-sphere splits M^k into a connected sum. If the summands contain separating 2-spheres, then these 2-spheres split the summands into connected sums. Thus $M = M_1 \# \dots M_l \# R_1 \# \dots \# R_k$ with $M_j, R_i \neq S^3$. Therefore M contains a set of $k + l$ independent 2-spheres. It follows that $k + l < 6t$. □

The analogous theorem holds for non orientable 3-manifolds. We write prime decompositions as $M = M_1 \# \dots M_n$. Consider the case of a connected sum of four 3-manifolds $M_1, \dots, M_4$. We may take the connected sum by removing three small 3-balls from M_1, one small 3-ball from M_2, M_3, M_4 and identifying the resulting boundary components of M_1 with those of M_2, M_3, M_4. However, the "linear" notation would not make sense. Consider the 2-spheres S_1, S_2, S_3 in M which result from the (pairwise identified) boundary components of M_1 and M_2, M_3, M_4. We may isotope them in M_1 to lie inside another 2-sphere $\tilde{S}_1$ that cobounds, together with $S_1 \sqcup S_2 \sqcup S_3$ a four times punctured 3-sphere. Inside this four times punctured 3-sphere we consider another 2-sphere $\tilde{S}_2$ that separates $\tilde{S}_1$ and S_1 from S_2 and S_3. Replace S_1 by $\tilde{S}_1$ and S_2 by $\tilde{S}_2$. Then $\tilde{S}_1 \sqcup \tilde{S}_2 \sqcup S_3$ cuts M into a once punctured copy of M_1 together with a twice punctured copy of M_2 along with a twice punctured copy of M_3 and a once punctured copy of M_4. Furthermore, the once punctured copy of M_1 meets the twice punctured copy of M_2. In addition, the twice punctured copy of M_2 meets the twice punctured copy of M_3. Which, in addition, meets the once punctured copy of M_4. Thus the connected sum can be described as $M = M_1 \# M_2 \# M_3 \# M_4$.

Theorem 3.45. *Let M be a compact orientable 3-manifold. If $M = M_1 \# \dots \# M_k = N_1 \# \dots N_l$, then $k = l$ and, after reordering, $M_i = N_i$.*

We first prove a lemma:

Lemma 3.46. *Let Σ be a non separating and S a separating 2-sphere in a 3-manifold M. Let c be an innermost component of $\Sigma \cap S$ in S. Let D be the disk bounded by c in S that meets Σ only in c and let D', D'' be the two disks into which c separates Σ. Set $\Sigma' = D \cup D'$ and $\Sigma'' = D \cup D''$. Then either Σ' or Σ'' is non separating.*

Proof: Since Σ is non separating, there is a simple closed curve α in M that meets Σ exactly once. After an isotopy, if necessary, we can ensure that α does not meet D. Then the number of points in $(\alpha \cap \Sigma') \cup (\alpha \cap \Sigma'')$ is exactly one. We may assume that this point is in $\alpha \cap \Sigma'$. Now the existence of the simple closed curve α that meets Σ' in exactly one point shows that Σ' is non separating. □

Proof: (Of the Uniqueness Theorem) Suppose first that M contains no non separating 2-sphere. It then follows that each M_i and each N_j is irreducible. Let $S = S_1 \sqcup \cdots \sqcup S_{k-1}$ be a set of 2-spheres such that $M \backslash S = M_1 * \sqcup \cdots \sqcup M_k *$, where $M_i *$ is a punctured copy of M_i for $i = 1, \ldots, k$, Furthermore, let Σ be a 2-sphere such that $M \backslash \Sigma = N_1 * \sqcup M'$, for $N_1 *$ a once punctured version of N_1.

We may assume that both Σ and S have been chosen so that $\#|\Sigma \cap S|$ is minimal. Suppose that $\Sigma \cap S \neq \emptyset$. Let c be an innermost component of $\Sigma \cap S \neq \emptyset$ in Σ. Then c cuts out a disk D in Σ that meets S only in its boundary c. In particular, $D \subset M_j *$ for some j. Let S_i be the component of S containing c.

Here c cuts S_i into two disks, D', D''. Set $S' = D \cup D'$, $S'' = D \cup D''$. Recall that we are assuming that M_j is irreducible. Thus S', S'' bound 3-balls B', B'' in M_j. Since M_j is not S^3, it must be the case that either $B' \subset B''$ or $B'' \subset B'$, say $B' \subset B''$. Then we may set $S_i * = S''$ and $S_j * = S_j$ for $j \neq i$. Then $S* = S_1 * \sqcup \cdots \sqcup S_{k-1} *$ also has the property that $M \backslash S*$ is the disjoint union of punctured copies of $M_1, \ldots, M_k$ and $\#|\Sigma \cap S * | < \#|\Sigma \cap S|$. But this contradicts our choice of Σ and S. Thus $\Sigma \cap S = \emptyset$.

Suppose now that a component of S, say S_l, lies in $N_1 *$. Since N_1 is irreducible, S_l bounds a 3-ball in N_1. On the other hand, since S_l is essential in M, it does not bound a 3-ball in M and hence does not bound a 3-ball in $N_1 *$. It follows that S_l bounds the once punctured 3-ball in $N_1 *$ containing $\partial N_1 * = \Sigma$. But this means that S_l is isotopic to Σ. Hence a component of $M \backslash S$ homeomorphic to $N_1 *$. Thus after reordering, $M_1 = N_1$.

If no component of S lies in $N_1 *$, then $N_1 * \subset M_l *$ for some l. In particular, $\Sigma \subset M_l *$. Then similarly, since M_l is irreducible, Σ bounds a 3-ball in M_l. On the other hand, since Σ is essential in M, it does not bound a 3-ball in M and hence does not bound a 3-ball in $M_l *$. It follows that S_l bounds a punctured 3-ball in $M_l *$. Since no component of S lies in $N_1 *$, S_l in fact bounds a punctured 3-ball whose punctures are bounded by the components of S that meet $M_l *$. But this means that M_l is homeomorphic to N_1. Thus after reordering, $M_1 = N_1$.

Now suppose that M contains a non separating 2-sphere $\tilde{S}$. Let S be as above. We may assume that $\tilde{S}$ is chosen so that $\#|\tilde{S} \cap S|$ is minimal.

We proceed as in the argument above. Suppose that $\tilde{S} \cap S \neq \emptyset$. Let c be an innermost component of $\tilde{S} \cap S$ in S and let D be the disk bounded by c in S that meets $\tilde{S}$ only in its boundary c. Here c separates $\tilde{S}$ into two disks D', D''. Set $S' = D \cup D'$, $S'' = D \cup D''$. By Lemma 3.46 either S' or S'',say S', is non separating. Furthermore $\#|S' \cap S| < \#|\tilde{S} \cap S|$. But this contradicts the assumed minimality. Thus $\tilde{S} \cap S = \emptyset$.

It follows that for some i, $M_i = S^2 \times S^1$. Similarly, we may show that for some j, $N_j = S^2 \times S^1$. Thus after reordering, $M_i = N_i$.

□

For non orientable 3-manifolds prime factorizations are not unique. More specifically, if M is non orientable and $M = M_1 \#(S^2 \times S^1)$ then it is also the case that $M = M_1 \#(S^2 \tilde{\times} S^1)$ and vice versa. However, this prime factorization becomes unique if we decree that reducible prime summands for non orientable 3-manifolds always be $S^2 \tilde{\times} S^1$.

To understand the lack of orientability of a 3-manifold in purely topological terms note that a 3-manifold is non orientable if and only if it contains a submanifold homeomorphic to (Möbius band)$\times I$.

The Möbius band can be thought of as a twisted I-bundle over the circle. In particular, he circle is orientable but the Möbius band is not. Twisted I-bundles are interesting when considering the question of orientation. The projective plane is not orientable, but the twisted I-bundle over the projective plane is orientable. Its boundary is a 2-sphere.

3.6 Fundamental 2-spheres and projective planes

In this section we consider algorithmic questions. In particular, we are interested in recognizing certain features of a 3-manifold in a finite number of steps. Any feature of a 3-manifold that can be translated into the existence of some type of fundamental normal surface can be recognized in this sense.

Definition 3.47. *A 2-sphere in a 3-manifold is essential if it does not bound a 3-ball. A surface F in a 3-manifold M is boundary parallel if it cuts off a 3-manifold homeomorphic to $F \times I$ from M. A disk in a 3-manifold is essential if it is not boundary parallel. A surface F not equal to a 2-sphere or disk in a 3-manifold M is essential if it is incompressible and not boundary parallel.*

So far we have always considered embedded surfaces. But occasionally we also wish to consider immersed surfaces. An immersion is locally an embedding, but not globally. I.e., an immersed surface is a surface with self intersections. By isotoping an immersed surface into general position, we arrange that the self intersections consist of double curves and isolated triple points. Deep results on general position guarantee that this is possible.

Lemma 3.48. *Suppose that F is a connected normal surface in the 3-manifold M. Further suppose that $F = G + H$, for G, H normal surfaces in M and that G, H are chosen so that the number $\#|G \cap H|$ is minimal. Then G and H are connected.*

Proof: Suppose that $H = H_1 \sqcup H_2$. Set $G' = G + H_1$. Then $F = G + H = G + H_1 + H_2 = G' + H_2$. Here the components of $G \cap H$ consist of the components of $G \cap H_1$ together with the components of $G \cap H_2$. Since F is connected, neither of the latter two sets are empty. Thus $\#|G' \cap H_2| < \#|G \cap H|$. But this contradicts minimality. Thus G, H are connected. □

Theorem 3.49. *If the 3-manifold M contains an essential 2-sphere or projective plane, then it contains a fundamental 2-sphere or projective plane.*

Proof: Let F be the 2-sphere or projective plane. By proceeding as in Lemma 3.42 we may arrange for F to be normal. We may also assume that $w(F)$ is minimal among all normal 2-spheres and projective planes in M.

Suppose F is not fundamental. By Lemma 3.48 there are connected normal surfaces G, H such that $F = G + H$, $w(G) > 0, w(H) > 0$, $w(F) = w(G) + w(H)$, and $\chi(F) = \chi(G) + \chi(H)$.

Case 1: $\chi(F) = 2$, i.e., F is a 2-sphere.

Up to renaming of G, H, there are only two possibilities: 1) $\chi(G) = 2$ and $\chi(H) = 0$; or 2) $\chi(G) = 1$ and $\chi(H) = 1$. Thus G is either a 2-sphere or projective plane and $w(G) < w(F)$, contradicting minimality.

Case 2: $\chi(F) = 1$, i.e., F is a projective plane.

Up to renaming of G, H, there are only two possibilities: 1) $\chi(G) = 2$ and $\chi(H) = -1$; or 2) $\chi(G) = 1$ and $\chi(H) = 0$. Thus G is either a 2-sphere or projective plane and $w(G) < w(F)$, contradicting minimality.

Thus M contains a fundamental 2-sphere or projective plane. □

Corollary 3.50. *There is an algorithm to decide whether or not a 3-manifold contains an essential 2-sphere or projective plane.*

In a similar vein, though necessarily though much harder work, Haken proved the following theorem:

Theorem 3.51. *(Haken) If the 3-manifold M contains and essential disk and contains no projective planes, then it contains a fundamental disk.*

The application he had in mind was:

Corollary 3.52. *There is an algorithm to decide whether or not a knot $K \subset \mathbb{S}^3$ is the unknot.*

Proof: Let $\eta(K)$ be an open regular neighborhood of K and set $C(K) = \mathbb{S}^3 \backslash \eta(K)$. We need only show that a knot is the unknot if and only if $C(K)$ contains an essential disk.

If K is the unknot, then K bounds a disk in $\mathbb{S}^3$. This disk restricts to an essential disk in $C(K)$. Conversely, suppose there is an essential disk D in K. Then ∂D is a torus knot on $\partial C(K)$. Let $N(K)$ be the closure of $\eta(K)$. Consider the union of $N(K)$ with the regular neighborhood $N(D)$ of D. Then the boundary of $N(K) \cup N(D)$ is a 2-sphere in $\mathbb{S}^3$. Thus by the Schönflies Theorem, it bounds a 3-ball. Since it does not bound a 3-ball on the $N(K) \cup N(D)$, it bounds a 3-ball on the other side. We may use this description to calculate the fundamental group of $\mathbb{S}^3$. Use the Seifert-Van Kampen Theorem. We obtain $\pi_1(\mathbb{S}^3) =< x | x^q >$, where q is the number of times that ∂D wraps around K on $\partial N(K)$. But this tells us that $q = 1$. Thus the disk D extends, along an annulus in $N(K)$ to a disk bounded by K in $\mathbb{S}^3$. □

4 Haken 3-manifolds

This chapter is an overview of the work that has been accomplished following Haken's general program. Recall that the classification of surfaces is accomplished by cutting a surface along essential arcs into simpler and simpler pieces. Haken's general program proceeds along the same lines. We here discuss the key ingredients of this program along with its main result and limitations.

4.1 The Loop Theorem

The precursor of what is now known as the Loop Theorem was first claimed by Max Dehn in 1910 and is therefore known as Dehn's Lemma.

Theorem 4.1. *(Dehn's Lemma) Suppose M is a 3-manifold and $f : D \to M$ a map from the disk into M. If for some neighborhood U of ∂D, $f|_U$ is an embedding then $f|_{\partial D}$ extends to an embedding.*

This innocuous statement turns out to be very hard to prove. Dehn's original argument was found to be incomplete, as was pointed out by Kneser in 1927. (Dehn never got around to giving a complete proof of this theorem. He was fired from his position at Frankfurt in 1933 and took on a teaching position at Black Mountain College in North Carolina that left him little time for research. Dehn found life at Black Mountain College, a progressive institution, even by today's standards, extremely gratifying and turned down offers from more research oriented institutions.)

A proof of this theorem was given by Christos Papakyriakopoulos in 1957. In this proof, Papakyriakopoulos employed what came to be known as a "tower construction". In this construction, Papakyriakopoulos considered a sequence of 2-fold covers of a regular neighborhood of an immersed disk. Very roughly, this sequence allowed a removal of triple points in the immersed disk. At the top of the tower he found an immersed disk with now triple points. Standard cut and paste operations allowed him to obtain an embedded disk at the top of the tower which could then successively be projected down one step at a time.

The following quintuplet is due to John Milnor:

"The perfidious Lemma of Dehn,

put many a man to shame.

But Christos Pap-

akyriakop-

oulos did it without any pain."

The tower construction also allowed Papakyriakopoulos to prove two other theorems that he called the Loop Theorem and the Sphere Theorem. Below is a generalization of the Loop Theorem formulated by John Stallings:

Theorem 4.2. *(The Loop Theorem) Let M be an 3-manifold and F a connected surface in ∂M. If N is a normal subgroup of $\pi_1(F)$ and if $ker(\pi_1(F) \to \pi_1(M)) - N \neq \emptyset$, then there is a proper embedding $g : (D, \partial D) \to (M, F)$ such that $[g|_{\partial D}]$ is not in N.*

The case of most interest in our further discussion is the case in which $N =< 1 >$.

Theorem 4.3. *(The Sphere Theorem) Let M is an orientable 3-manifold and N a $\pi_1(M)$-invariant subgroup of $\pi_2(M)$. If $\pi_2(M) - N \neq \emptyset$, then there is an embedding $g : \mathbb{S}^2 \to M$ such that $[g]$ is not in N.*

Exercise 1: The contrapositive of the Loop Theorem with $N =< 1 >$ states that if a surface F in a 3-manifold M is incompressible, then $\pi_1(F) \to \pi_1(M)$ is injective. Is the converse true?

Exercise 2: Let F be a compact surface without boundary. List all incompressible, boundary incompressible surfaces (those with and those without boundary) in $M = F \times I$.

4.2 Hierarchies

A hierarchy provides a way of cutting certain 3-manifolds into simpler pieces.

Definition 4.4. *Let M be a compact 3-manifold. A hierarchy for M is a finite sequence of pairs $(M_1, F_1), \ldots, (M_n, F_n)$ such that*

1) F_i is a 2-sided incompressible surface in M_i;

2) M_{i+1} is obtained from M_i by cutting along F_i;

3) $M_1 = M$;

4) each component of M_{n+1} is a 3-ball.

E.g., a hierarchy for the three torus (three torus, torus), (torus x I, annulus), (annulus x I, disk)

The main theorem concerning hierarchies is the following:

Theorem 4.5. *Let M be a compact orientable irreducible 3-manifold that contains no projective planes. Then M is Haken if and only if M has a hierarchy.*

A key ingredient in proving this theorem is the following lemma:

Lemma 4.6. *Suppose M is a compact orientable 3-manifold such that ∂M contains a surface of positive genus. Then M contains a properly embedded, 2-sided, incompressible surface F such that $0 \neq [\partial F] \in \pi_1(\partial M)$.*

The proof of the lemma is not hard but does require some homology theory. It may be found in Jaco's or Hempel's book.

It follows from this theorem that the class of 3-manifolds which lends itself to Haken's general program of classifying 3-manifolds is exactly the class of 3-manifolds now known as Haken 3-manifolds. The main result of this program is due to Friedhelm Waldhausen.

Theorem 4.7. *(Waldhausen's Theorem) Homotopy equivalent 3-manifolds are homeomorphic.*

It is unknown whether or not this theorem holds for more general 3-manifolds. One of the most famous open problems in the theory of 3-manifolds is the following:

Conjecture 4.8. *(The Poincaré Conjecture) If M is homotopy equivalent to $\mathbb{S}^3$, then M is homeomorphic to $\mathbb{S}^3$.*

Exercise $1^{*\cdots*}$: Prove or disprove the Poincaré Conjecture.

4.3 Seifert Fibered Spaces

Seifert fibered spaces constitute on of the larger classes of 3-manifolds. Not all of them are Haken, but many are. We discuss them here to provide more examples of 3-manifolds in general and of Haken manifolds in particular. One of the reasons for continued interest in Seifert fibered spaces is the fact that they project in a natural way onto a 2-dimensional "base space". This fact makes them particularly amenable to computations.

Definition 4.9. *A fibered solid torus of type (l, m) is a solid torus with a foliation by circles obtained as follows: Consider the cylinder $\mathbb{D}^2 \times [-1, 1]$. It is foliated by intervals of the form $\{point\} \times [-1, 1]$. Identify $\mathbb{D}^2 \times \{-1\}$ to $\mathbb{D}^2 \times \{1\}$ by setting $((r, \theta), 1)$ equal to $((r, \theta + \frac{2\pi m}{l}), -1)$. A circle formed by intervals of the form $\{point\} \times [-1, 1]$ is called a fiber of the fibered solid torus.*

The core of a fibered solid T is the fiber arising from $\{0\} \times I$. If $l > 1$, then we call a fibered solid torus T of type (l, m) an exceptionally fibered solid torus. In this case the core of T is called an exceptional fiber and all other fibers of T are called regular fibers. If $l = 1$, then we call a fibered solid torus T of type (l, m) a regularly fibered solid torus. In this case all of the fibers of T are called regular fibers.

Suppose T_1 and T_2 are fibered solid tori. A fiber preserving homeomorphism between T_1 and T_2 is a homeomorphism $h : T_1 \to T_2$ that takes fibers to fibers.

The following observations are left as exercises.

Remark 4.10. *We may assume that $0 \leq m < \frac{l}{2}$, for any fibered solid torus is homeomorphic via a fiber preserving homeomorphism to a fibered solid torus of type (l, m) with l, m satisfying these requirements. With this assumption, the existence of a fiber preserving homeomorphism between a fibered solid torus of type (l_1, m_1) and a fibered solid torus of type (l_2, m_2) necessitates $l_1 = l_2$ and $m_1 = m_2$.*

Definition 4.11. *A 3-manifold M is a Seifert fibered space if M is the union of pairwise disjoint simple closed curves called fibers such that each fiber has a closed neighborhood consisting of a union of fibers that is homeomorphic to a fibered solid torus via a fiber preserving homeomorphism. This closed neighborhood of a fiber is called a fibered neighborhood.*

Denote the quotient space of M obtained by identifying each fiber to a point by B and denote this quotient map by $p : M \to B$.

Examples: Lens spaces, prism manifolds, the complement of a torus knot.

Remark 4.12. *Here B is a surface. This can be seen as follows: The quotient of a fibered solid torus is a disk. Let q be a point in B. Then $p^{-1}(q)$ is a fiber f of M. The fibered neighborhood T of f yields a neighborhood $p(T)$ of q that is homeomorphic to a disk. We think of B as a disk with exceptional points. The latter are the images of the exceptional fibers.*

Lemma 4.13. *If T_1 and T_2 are fibered neighborhood of the fiber f, then f is an exceptional fiber of T_1 if and only if f is an exceptional fiber of T_2.*

Proof: This follows directly from Remark 4.10. □

This lemma allows us to refer to the fibers of a Seifert fibered space as exceptional fibers or regular fibers unambiguously.

Remark 4.14. *The interiors of the fibered neighborhoods of fibers form an open cover for a Seifert fibered space. It follows that in a compact Seifert fibered space, there will be only finitely many exceptional fibers.*

Definition 4.15. *A subset Z of X is saturated with respect to $p : X \to Y$ if $Z = p^{-1}(p(Z))$.*

Example: Saturated tori and annuli in Seifert fibered spaces.

We conclude this section with a few well known theorems that provide a brief overview of some of the results on Seifert fibered spaces.

Theorem 4.16. *An orientable Seifert fibered space is either irreducible or homeomorphic to either $S^2 \times S^1$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$.*

Proof: Two cases need to be considered:

Case 1: $\partial M \neq \emptyset$.

In this case $\partial B \neq \emptyset$. The proof is by induction on $(-\chi(B), \#exceptionalpoints)$. Suppose first that B is a disk with at most one exceptional fiber. Then M is a solid torus. Suppose S is a 2-sphere in M. Then a standard innermost disk argument shows that S can be made disjoint from a meridian disk for M. Thus S lies in the 3-ball $M \backslash T$. Hence S bounds a 3-ball by the Schönflies Theorem.

If B there are exceptional points in B, an arc a in B that cuts of a disk containing exactly one exceptional point corresponds to the saturated annulus $A = p^{-1}(a)$. Unless B is a disk with at most one exceptional point, A is incompressible. If B is nonplanar, a non separating arc in B similarly yields a saturated annulus that is incompressible.

Let S be a 2-sphere in M. If S is disjoint from an incompressible saturated annulus A in M, then S lies in the Seifert fibered space $M \backslash A$. By the inductive hypothesis, this Seifert fibered space is irreducible. Hence S bounds a 3-ball in $M \backslash A$. Thus S bounds a 3-ball in M.

Suppose c is an innermost component of $A \cap S$ in S. Then c bounds a disk D in S that is disjoint from A except in its boundary. Since A is incompressible, ∂D is inessential in A and bounds a disk D' in A. As usual, the 3-ball bounded by $D \cup D'$ describes an isotopy reducing the number of components of $A \cap S$. Thus if we isotope S so that $A \cap S$ is minimal, then $A \cap S = \emptyset$.

Case 2: $\partial M = \emptyset$.

If M contains a saturated incompressible torus T, then the argument above can be applied to show that a 2-sphere $S \subset M$ can be isotoped to be disjoint from T. Thus S lies in $M \backslash T$, a Seifert fibered space with boundary. It follows that S bounds a 3-ball in $M \backslash T$ and hence in M.

It remains to consider the cases where B is either a 2-sphere with at most three exceptional fibers or a projective plane with at most one exceptional fiber.

The case of lens spaces ($B = S^2$ with up to two exceptional points) is left as an exercise. Here either M is irreducible or homeomorphic to $S^2 \times S^1$. A theorem of Waldhausen states that prism manifolds ($B = S^2$ with three exceptional points) are irreducible.

So suppose B is a projective plane with at most one exceptional fiber. Let t be a simple closed curve in B that bounds a disk containing the exceptional point if there is one. Let $T = p^{-1}(t)$ be the corresponding saturated torus. Since c is separating, so is T. T cuts M into two components: 1) M', a circle bundle over the Möbius band with the natural fibration and 2) T', a fibered solid torus. Since M, and thus M', is orientable, M' must be the twisted circle bundle over the Möbius band.

Suppose S is a 2-sphere in M. If S is disjoint from T, then it lies in either M' or T' and hence bounds a 3-ball. Suppose that $S \cap T$ is non empty. Let c be an innermost component of $S \cap T$ in S. Then c bounds a disk D in S that is disjoint from T except along its boundary. If ∂D is inessential in T, then, as above, there is an isotopy reducing the number of components of $S \cap T$. So suppose that S has been isotoped so that the number of components of $S \cap T$ is minimal. This assumption implies that there is no component of $S \cap M'$ or $S \cap T'$ that is boundary compressible via a boundary compressing disk that meets more than one component of $S \cap T$.

We make the following observations (the proofs are left to the reader):

1) The minimality assumption on $S \cap T$ implies that there is no component of $S \cap M'$ or $S \cap T'$ that is boundary compressible via a boundary compressing disk that meets more than one component of $S \cap T$;

2) A properly embedded incompressible surface in T' is a disk or an annulus A for which there is a boundary compressing disk that meets more than one component of $A \cap \partial T'$;;

3) $\partial M'$ is incompressible in M';

4) A properly embedded incompressible surface with non empty boundary in M' that is not boundary compressible via a boundary compressing disk that meets more than one component of $S \cap T$ must be an annulus that double covers the Möbius band.

It follows that under the minimality assumption on $S \cap T$, S must consist of an annulus in M' that double covers the Möbius band together with two disks in T'. In this case, S cuts M into two identical pieces, each consisting of one of the two twisted I-bundles over a Möbius band coming from M' and one of the 3-balls coming from T'. Each of these pieces is homeomorphic to $\mathbb{RP}^3$. □

Theorem 4.17. *A compact orientable Seifert fibered space M is either a Haken manifold or a lens space (including $S^2 \times S^1$, S^3) or $\mathbb{RP}^3 \# \mathbb{RP}^3$ or a prism manifold. In the latter case, M is Haken if and only if $H_1(M)$ is infinite.*

Theorem 4.18. *The only Seifert fibered spaces with non unique fiberings are:*

a) lens spaces (including $S^2 \times S^1$, S^3);

b) prism manifolds;

c) the solid torus;

d) the twisted I-bundle over the Klein bottle;

e) the double of the twisted I-bundle over the Klein bottle (this 3-manifold also fibers over S^2 with four exceptional fibers for which $l_1 = l_2 = l_3 = l_4 = 2$.

The following theorem is known as the Seifert "Conjecture".

Theorem 4.19. *Let M be a compact orientable irreducible 3-manifold. Then M is a Seifert fibered space if and only if $\pi_1(M)$ has a normal subgroup isomorphic to $\mathbb{Z}$.*

Exercise 1: Prove Remark 4.10.

Exercise 2: Prove the observations near the end of the proof of Theorem 4.16.

Exercise 3: Let M be a prism manifold obtained as follows: Set $M' =$ (thrice punctured 2-sphere) $\times S^1$ and let T_1, T_2, T_3 be fibered solid tori of type $(l_1, m_1), (l_2, m_2), (l_3, m_3)$ respectively. Let M be the 3-manifold obtained by identifying the first of three boundary components of M' with the boundary component of T_1, the second of the three boundary components of M' with the boundary component of T_2, and the third of the three boundary components of M' with the boundary component of T_3.

Calculate the fundamental group of M

5 Thin position of knots

In this chapter we study knots, bridge position and thin position of knots. These notions are interesting in their own right. So in this short chapter we digress from our main objective. However, in the study of 3-manifolds, knots and links arise naturally in a variety of ways. Often, they arise as the locus of intersection of surfaces. In this context, the notion of thin position, introduced by D. Gabai, has been employed with great success.

Definition 5.1. *A knot in a 3-manifold M is a smooth (or PL) isotopy class of smooth (or PL) embeddings of S^1 into M. More generally, link in a 3-manifold M is a smooth (or PL) isotopy class of smooth (or PL) embeddings of the disjoint sum of some number of copies of S^1 into M.*

The case $M = \mathbb{S}^3$ captures many of the problems arising in the study of knots and links. See for instance the excellent books by Lickorish, Rolfsen, and others.

Definition 5.2. *A function $h : M \to [-1, 1]$ is a height function if it has exactly two critical points (a minimum and a maximum) in interior(M).*

An example is the function $h : \mathbb{S}^3 \to [-1, 1]$ that $\mathbb{S}^3 \subset \mathbb{R}^4$ onto the fourth coordinate. Except for the maximum and minimum, the level surfaces of h are spheres.

Definition 5.3. *Let $h : \mathbb{S}^3 \to [-1, 1]$ be a height function and let $K \subset \mathbb{S}^3$ be knot or link. We say that K is in bridge position if all maxima of K occur above all minima of K. The bridge number of K is the least number of maxima K must have with respect to a height function.*

E.g., the bridge number of the unknot is 1, the bridge number of the trefoil is 2.

Definition 5.4. *Let $h : \mathbb{S}^3 \to [-1,1]$ be a height function and let $K \subset \mathbb{S}^3$ be knot or link.*

Let $\tilde{K} \subset S^3$ be a fixed presentation of a knot or link K (i.e., not an isotopy class) and let $c_1, \ldots, c_n$ be the critical values of $h|_{\tilde{K}}$ listed in increasing order; i.e., so that $h(c_1) < \cdots < h(c_n)$. Choose $r_1, \ldots, r_{n-1}$ so that $c_i < r_i < c_{i+1}$. Set $R_i = h^{-1}(r_i)$. The width of $\tilde{K}$ relative to h, denoted by $w(\tilde{K}, h)$, is $\sum_i |K \cap R_i|$. The width of K, denoted by $w(K)$, is the minimum of this relative width over all height functions and all presentations of K.

We say that K is in thin position if it is presented with respect to a height function realizing its width.

Here R_i is a thin level of K with respect to h if c_i is a maximum value for $h|_K$ and c_{i+1} is a minimum value for $h|_K$; and R_i is a thick level of K with respect to h if c_i is a minimum value for $h|_K$ and c_{i+1} is a maximum value for $h|_K$.

E.g., the width of the trefoil is 2+4+2 = 8.

Little is known about thin position of knots. Here are a few theorems:

Theorem 5.5. *(Thompson) If for a knot $K \subset \mathbb{S}^3$ thin position is knot bridge position, then $\mathbb{S}^3 \backslash \eta(K)$ contains a closed incompressible surface.*

Proof: (Sketch) Let $h : \mathbb{S}^3 \to [-1,1]$ be a height function and let $\tilde{K} \subset \mathbb{S}^3$ be a fixed presentation of K realizing thin position of K. If this presentation is not in bridge position, then there is a thin level R. We may assume that R is the highest such thin level. Set $R^* = R \cap (\mathbb{S}^3 \backslash \eta(K))$. By a theorem of Y.Q. Wu, R^* is incompressible in $\mathbb{S}^3 \backslash \eta(K)$.

To obtain a closed surface in $\mathbb{S}^3 \backslash \eta(K)$ proceed as follows: Let $\mathcal{A}$ be the collection of annuli cut off from $\partial(\mathbb{S}^3 \backslash \eta(K))$ by ∂R^* that lies above R^*. Then a pushoff of $R^* \cup \mathcal{A}$ into $\mathbb{S}^3 \backslash \eta(K)$ is a closed surface. It follows from a (vital) lemma of M. Culler, C. McA. Gordon, J. Luecke and P. Shalen, that this surface is in fact incompressible. □

Definition 5.6. *A knot is small if its complement contains no closed incompressible surface that is not boundary parallel.*

Definition 5.7. *Given two knots $K_1 \subset \mathbb{S}^3$ and $K_2 \subset \mathbb{S}^3$ the connected sum $K_1 \# K_2 \subset \mathbb{S}^3$ is the pairwise connected sum of $(K_1, \mathbb{S}^3)$ and $(K_2, \mathbb{S}^3)$.*

Theorem 5.8. *(Rieck-Sedgwick) For $K_1, K_2 \subset \mathbb{S}^3$ small knots, $w(K_1 \# K_2) = w(K_1) + w(K_2) - 2$.*

Exercise: Find a knot with width bigger than 9.

6 Heegaard splittings

A Heegaard splitting is a splitting of a 3-manifold into two simple pieces. Interestingly enough, every 3-manifold admits such a splitting. It turns out however, that these splittings are not as simple as they appear. In this chapter, we will be interested in applications of these splittings. Later we will discuss structural and classification theorems for Heegaard splittings.

6.1 The basics

Definition 6.1. *A handlebody is a compact 3-manifold that is homeomorphic to a regular neighborhood of a connected graph in* $\mathbb{S}^3$.

E.g.

Definition 6.2. *A spine of a handlebody* V *is a graph to which* V *collapses.*

Definition 6.3. *A Heegaard splitting of a closed 3-manifold* M *is a decomposition* $M = V \cup_S W$ *such that 1)* V, W *are handlebodies; and 2)* $S = \partial V = \partial W$. *Here* S *is called the splitting surface of* $M = V \cup_S W$. *Two Heegaard splittings are considered equivalent if their splitting surfaces are isotopic.*

E.g., Genus 0 and genus 1 splitting for $\mathbb{S}^3$, genus 1 splitting for lens spaces, the three torus

Theorem 6.4. *(Moise, Bing) Every closed orientable 3-manifold admits a Heegaard splitting.*

Proof: (Bing's Proof) Let M be a closed 3-manifold. Then M admits a triangulation (M, K). Set $V = N(K^1)$. Then V is a handlebody. It is not too hard to see that the closure W of the complement of V is also a handlebody. □

Definition 6.5. *Let* $M = V \cup_S W$ *be a Heegaard splitting and let* $\mathbb{S}^3 = V' \cup_T W'$ *be the standard genus 1 Heegaard splitting of* $\mathbb{S}^3$. *The pairwise connected sum* $(M, S)\#(\mathbb{S}^3, T)$ *defines a Heegaard splitting* $M = \tilde{V} \cup_{\tilde{S}} \tilde{W}$ *called an elementary stabilization of* $M = V \cup_S W$. *A Heegaard splitting is called a stabilization of* $M = V \cup_S W$ *if it is obtained from* $M = V \cup_S W$ *by performing a finite number of elementary stabilizations.*

Theorem 6.6. *(Reidemeister-Singer) Any two Heegaard splittings of a 3-manifold* M *become equivalent after a finite number of stabilizations.*

Sketch of proof: Let $M = V_1 \cup_{S_1} W_1$ be a Heegaard splitting. Let (M, K) be a triangulation of M. Let X_1 be a spine of V_1. By subdividing K, if necessary, we may assume that each 3-simplex is met by at most one boundary parallel arc. We may then move points of intersection of the spine with the faces of a 3-simplex into the vertices. By subdividing again, if necessary, we avoid collisions. We may the boundary parallel subarcs of the spine in the interior of the 3-simplices entirely into K^1. It then follows that the Heegaard splitting defined as in the proof of Moise and Bing's Theorem is a stabilizatin of $M = V_1 \cup_{S_1} W_1$. Similarly, the Heegaard splitting defined in this way by a subdivision of K is a stabilizatin of $M = V_1 \cup_{S_1} W_1$.

Now consider $M = V_2 \cup_{S_2} W_2$ and proceed analogously. Since any two simplicial complexes of the same with the same underlying space have a common subdivision (exercise), the theorem follows.

Exercise 1: Prove that any two simplicial complexes of the same with the same underlying space have a common subdivision.

6.2 Reducibility Properties

We here discuss two important reducibility properties for Heegaard splittings. These theorems have many consequences. One consequence is that a Heegaard splitting of a connected sum of 3-manifolds factors into Heegaard splittings of the summands.

Definition 6.7. *A Heegaard splitting $M = V \cup_S W$ is reducible if there is an essential circle $c \subset S$ and disks $D \subset V$, $E \subset W$ with $\partial D = \partial E = c$. A Heegaard splitting is irreducible if it is not reducible.*

A Heegaard splitting $M = V \cup_S W$ is weakly reducible if there are disks $D \subset V$, $E \subset W$ such that ∂D, ∂E are essential and $\partial D \cap \partial E = \emptyset$. A Heegaard splitting is strongly irreducible if it is not weakly reducible.

E.g., the genus 3 Heegaard splitting of the three torus is weakly reducible.

Definition 6.8. *A disk D is a handlebody V is essential if ∂D is essential in ∂V.*

Lemma 6.9. *An incompressible boundary incompressible surface is a handlebody is a disk.*

Proof: We prove this by induction on the genus of the handlebody. We leave it as an excercise to prove the assertion for a 3-ball. If V is a handlebody of positive genus then it contains an essential disk D. Denote the incompressible boundary incompressible surface in V by F. Isotope F so that the number of components in $F \cap D$ is minimal. An innermost disk argument then shows that the number of closed curves in this intersection is zero. An outermost arc argument shows that the number of arcs in this intersection is zero. Thus $F \subset (V \backslash \eta(D))$. Since the genus of $V \backslash \eta(D)$ is one less than the genus of V, the inductive hypothesis shows that F is a disk. □

The following theorem is one of the fundamental theorems concerning Heegaard splittings. One consequence of this theorem is that a Heegaard splitting of a connected sum of 3-manifolds can be factored into Heegaard splittings of the summands.

Theorem 6.10. *Suppose M is a reducible 3-manifold and $M = V \cup_S W$ a Heegaard splitting. Then $M = V \cup_S W$ is reducible.*

Proof: Let $\tilde{S}$ be an essential sphere in M. We may assume that $\tilde{S}$ is chosen so that the number of components, $\#|\tilde{S} \cap S|$, of $\tilde{S} \cap S$ is minimal. With this assumption the following holds:

Claim: $\tilde{S} \cap V$ is incompressible in V and $\tilde{S} \cap W$ are incompressible in W.

Suppose that $\tilde{S} \cap V$, say, is compressible in V. Then there is a disk $D \subset V$ with $\partial D \subset \tilde{S}$ and $(D \backslash \partial D) \cap \tilde{S} = \emptyset$. Cut $\tilde{S}$ along ∂D and cap off the resulting boundary components with a copy of D. This creates two 2-spheres $\tilde{S}_1, \tilde{S}_2$ at least one of which is essential. Note that $\#|\tilde{S} \cap S| = \#|\tilde{S}_1 \cap S| + \#|\tilde{S}_2 \cap S|$ and $\#|\tilde{S}_1 \cap S| > 0, \#|\tilde{S}_2 \cap S| > 0$. But this violates our minimality assumption. The same argument holds for $\tilde{S} \cap W$ in W.

It now follows that any component Q of $\tilde{S} \cap V$ (respectively, $\tilde{S} \cap W$) that is not a disk is boundary compressible in V (respectively, W). Isotoping such a component of

$\tilde{S} \cap V$ (respectively, $\tilde{S} \cap W$) across the boundary compressing disk results in one Q' or two components of intersection Q_1, Q_2 such that $\chi(Q') = \chi(Q)+1$ or $\chi(Q_1)+\chi(Q_2) = \chi(Q) + 1$. Thus by locating boundary compressions of $\tilde{S} \cap V$ in V and performing these boundary compressions, we can guarantee that $\tilde{S} \cap V$ consists of disks.

We now assume that $\tilde{S}$ is chosen so that $\#|\tilde{S} \cap S|$ is minimal, subject to the constraint that $\tilde{S} \cap V$ or $\tilde{S}$ consists of disks. Suppose the number of such disks is n. The proof of the claim then still shows that $\tilde{S} \cap W$ is incompressible in W. Again it follows that a component of $\tilde{S} \cap W$ that is not a disk is boundary compressible.

We must distinguish between two types of boundary compressions: Those that produce one component Q' out of a component Q of $\tilde{S} \cap W$ and those that produce two components Q_1, Q_2. Call the former non separating and the latter separating. In case of the former, the portion of the boundary of the boundary reducing disk that meets $\tilde{S}$ has its endpoints on two distinct components of ∂Q. In case of the latter, the portion of the boundary of the boundary reducing disk that meets $\tilde{S}$ has its endpoints on one component of ∂Q. Note that there can be at most $-\chi(Q)$ non parallel essential arcs of the latter type in Q. As we sum over all possible components Q of $\tilde{S} \cap W$, there can be at most $-\chi(\tilde{S} \cap W) = n - 2$ such arcs.

The effect of doing a boundary compression of the former type is to lower the number of components of $\tilde{S} \cap V$ by one and to leave the number of components of $\tilde{S} \cap W$ unchanged. The effect of doing a boundary compression of the latter type is to leave the number of components of $\tilde{S} \cap V$ unchanged and to raise the number of components of $\tilde{S} \cap W$ by one. We now perform boundary compressions on $\tilde{S} \cap W$ to reverse the situation, i.e., to isotope $\tilde{S}$ so that $\tilde{S} \cap V$ is a connected planar surface and $\tilde{S} \cap W$ consists of disks. As we do so, we perform at most $n-2$ boundary compressions of the latter type. It follows that at the end of this procedure, $\tilde{S} \cap W$ consists of disks and that there are at most $n - 1$ components.

We may now play the same game with $\tilde{S} \cap V$. We then end up with $\tilde{S} \cap V$ consisting only disks and of at most $n - 2$ components. A contradiction to the assumed minimality.

□

Exercise: Show that an incompressible boundary incompressible surface in the 3-ball is a disk.

6.3 Weak reducibility and incompressible surfaces

In this section we prove a theorem of A. Casson and C. McA. Gordon. This theorem establishes a connection between a Heegaard splitting being weakly reducible and the manifold of which it is a Heegaard splitting containing an incompressible surface. This idea has come a long way since its inception. In particular, it has led to the concept of a thin manifold decomposition of a 3-manifold pioneered by M. Scharlemann and A. Thompson. This concept in turn gave rise to the notion of a generalized strongly irreducible Heegaard splitting. The concept of a generalized strongly irreducible Heegaard splitting. A structure possessed by every compact 3-manifold.

Theorem 6.11. *Suppose M is a closed orientable 3-manifold and $M = V \cup_S W$ a weakly reducible Heegaard splitting. Then either M contains an incompressible surface* *or $M = V \cup_S W$ is reducible.*

Proof: Let $\mathcal{D}$ be a non empty disjoint union of non parallel essential disks in V and $\mathcal{E}$ be a non empty disjoint union of non parallel essential disks in W such that $\partial\mathcal{D}$ $\cap\partial\mathcal{E} = \emptyset$. Consider the surface S^* obtained by cutting S along $\partial(\mathcal{D} \cup \mathcal{E})$ and capping off the resulting boundary components with copies of the disks in $\mathcal{D} \cup \mathcal{E}$. Note that after a small isotopy, a copy of S^* is embedded in M.

Since $M = V \cup_S W$ is weakly reducible, there are such collection of disks $\mathcal{D} \cup \mathcal{E}$. Thus we may choose $\mathcal{D} \cup \mathcal{E}$ so that $\chi(S^*)$ is maximal.

Case 1: A component, Q, of S^* has positive genus.

In this case it follows from our maximality assumption that Q is incompressible.

Case 2: All components of S^* are 2-spheres.

Let $\mathcal{V}$ be the components of S^* that meet V and let $\mathcal{W}$ be the components of S^* that meet W.

Claim 1: $\mathcal{V} \cap \mathcal{W} \neq \emptyset$.

If $\mathcal{V} \cap \mathcal{W} = \emptyset$, then reversing the cut and paste performed above would connect up components in $\mathcal{V}$ and in $\mathcal{W}$, producing at least two components. Since S was connected, this is impossible.

Let $\tilde{S} \in \mathcal{V} \cap \mathcal{W}$. Then $\tilde{S}$ lies mostly in S. Furthermore, $\tilde{S} \cap V$ and $\tilde{S} \cap W$ are non empty and consist of disks. Let c be a curve in $\tilde{S}$ that separates the components of $\tilde{S} \cap V$ from the components of $\tilde{S} \cap W$. Note that c is a simple closed curve in S.

Claim 2: c is an essential curve in S.

There are essential curves to either side of c, thus c can't bound a disk in S.

Now the disk in $\tilde{\setminus} c$ that meets V can be isotoped slightly to one side of S to lie entirely in V and the other can be similarly isotoped to lie entirely in W. This shows that $M = V \cup_S W$ is reducible. □

In the case in which $M = V \cup_S W$ is reducible, two things way happen: 1) The 2-sphere constructed may be essential. 2) The 2-sphere constructed may be inessential. In the first case, M is reducible. In particular, M contains an incompressible surface, namely the 2-spere. In the second case, it follows from a theorem of Waldhausen that $M = V \cup_S W$ is in fact stabilized.

6.4 Heegaard genus and rank of fundamental group

The description of a 3-manifold via a Heegaard splitting gives a natural way of computing the fundamental group of a 3-manifold. In this section we consider two distinct notions, the Heegaard genus of a 3-manifold and the rank of the fundamental group of a 3-manifold. The insight here translates into an inequality for these invariants.

Definition 6.12. *The Heegaard genus of 3-manifold M, denoted by $g(M)$, is the least possible genus of a splitting surface of a Heegaard splitting for M.*

E.g., $g(\mathbb{S}^3) = 0$, $g(lens\ space) = 1$, $g(prism\ manifold) = 2$.

Definition 6.13. *The rank of a 3-manifold M, denoted by $r(M)$, is the least number of generators required for $\pi_1(M)$.*

Theorem 6.14. $r(M) \leq g(M)$

Proof: Given a Heegaard splitting $M = V \cup_S W$ that realizes $g(M)$, we may compute the fundamental group of M as follows: We consider M to be built from V in $g(M)+1$ steps. At each of the first $g(M)$ steps, we add an open neighborhood of a meridian disk for W. In the final step, we add an open neighborhood of the 3-ball that remains when an appropriate set of $g(M)$ meridian disks are removed from W.

This description translates into a computation of $\pi_1(M)$. Here $\pi_1(V)$ is the free group on $g(M)$ generators. Adding an open neigborhood of a disk (whose fundamental group is trivial) adds a relation at each of the first $g(M)$ steps. In the final step, a 3-ball (also with trivial fundamental group) is added along its boundary 2-sphere. Thus the fundamental group is unchanged.

To summarize: We obtain a ("balanced") presentation

$$\pi_1(M) = < x_1, \ldots, x_{g(M)} | r_1, \ldots, r_{g(M)} > .$$

□

Exercise 1: Show that for M a 3-manifold, $g(M) = 0$ implies $M = \mathbb{S}^3$.

Exercise 2: Design a sufficiently complicated genus 2 Heegaard splitting and calculate the fundamental group of the 3-manifold of which it is a Heegaard splitting.

7 Dehn surgery

The concept of Dehn surgery is one of the fundamental concepts in 3-manifolds being explored today. The most important theorem concerning Dehn surgery is that every 3-manifold can be obtained via Dehn surgery on an appropriate link.

7.1 Coordinates

The idea behind Dehn surgery is simple: Given a knot or link in $K \subset \mathbb{S}^3$, set $C(K) = \mathbb{S}^3 \backslash \eta(K)$. Now create a new 3-manifold by attaching solid tori to the components of $\partial C(K)$. A little more needs to be said concerning the specifics of the reglueing. The goal here is to obtain a new 3-manifold, not to simply reconstruct $\mathbb{S}^3$. For this reason we introduce a coordinate system. To do so, we must first prove a lemma.

Lemma 7.1. *Let $K \subset \mathbb{S}^3$ be a knot. Set $C(K) = \mathbb{S}^3 \backslash \eta(K)$. Let S_1, S_2 be Seifert surfaces for K. Then $S_1 \cap \partial C)K)$ is parallel to $S_2 \cap \partial C(K)$.*

Proof: Recall that Seifert surfaces are oriented. In $S_1 \cap C(K)$ and $S_2 \cap C(K)$ intersect in arcs and simple closed curves. Along an arc of intersection, the right hand rule induces an orientation on this arc. This orientation determines an initial or - endpoint and a terminal or + endpoint of the arc.

Now consider how the two torus knots $S_1 \cap \partial C(K)$ and $S_2 \cap \partial C(K)$ intersect on the torus $\partial C(K)$. By marking the plus and minus sides of $S_1 \cap \partial C(K)$ and $S_2 \cap \partial C(K)$ on $\partial C(K)$ we can keep track of whether the points of intersection are initial or terminal points. Note, however, that if there are such intersections, then they are either all initial points or all terminal points. Thus there can be no such intersections and the curves in $S_1 \cap \partial C)K)$ and $S_2 \cap \partial C(K)$ are parallel. □

Definition 7.2. *Let $K \subset \mathbb{S}^3$ be a knot. Set $C(K) = \mathbb{S}^3 \backslash \eta(K)$. Denote by m the curve on $\partial C(K)$ that bounds a disk in $\mathbb{S}^3$. We call this curve the meridian. Let S be a Seifert surface for K. Denote by l the curve $S \cap \partial C(K)$. We call this curve the longitude.*

The process of removing $\eta(K)$ from $\mathbb{S}^3$ and attaching a solid torus to the resulting 3-manifold in such a way that a meridian goes to a curve of slope m/l on $\partial C(K)$ is called m/l-Dehn surgery.

noindent Exercise 1: Draw the longitude for some non trivial knot.

noindent Exercise 2: Generalize the notion of Dehn surgery on a knot to Dehn surgery on links.

7.2 Dehn surgery and Dehn twists

We here outline the proof of one of the most impressive results in the study of 3-manifolds. This result states that every closed orientable 3-manifold is obtained via Dehn surgery on a link in $\mathbb{S}^3$. The details of this argument can be found in W.B.R. Lickorish's well written account: A representation of orientable combinatorial 3-manifolds. Ann. of Math. (2) 76 1962 531–540.

Definition 7.3. *Let S be a closed orientable surface. Let c be a simple closed curve in S. A map $f : S \to S$ is called a Dehn twist around c if*

1) $f|_{S \backslash \eta(c)}$ is the identity map; and

2) $f|_{N(c)}$ is a map of the annulus that is parametrized by $f(r, \theta, t) = (r, \theta + 2\pi t, t)$, for each $(r, \theta, t) \in \mathbb{S}^1 \times [0, 1]$.

As it turns out, Dehn twists form the building blocks for all surface homeomorphisms.

Theorem 7.4. *Every surface homeomorphism can be expressed as a composition of Dehn twists.*

The proof of this theorem is rather lengthy. So we omit it here. But the methods employed in the proof are elementary. Students should be able to read through the argument in Lickorish's paper on their own.

We now prove two lemmata.

Lemma 7.5. *Let M be the connected sum of g factors of $\mathbb{S}^2 \times \mathbb{S}^1$. Then M is obtained by $(1/0, \ldots, 1/0)$-Dehn surgery on the g component unlink in $\mathbb{S}^3$.*

Proof: We consider first the case in which $g = 1$. In this case $1/0$-Dehn surgery involves removing a regular neighborhood of the unlink, which creates a solid torus V, and then attaching a solid torus W to the resulting boundary component in such a way that a meridian of W goes to a meridian of V. This yields $\mathbb{S}^2 \times \mathbb{S}^1$.

More generally, consider the g component unlink. Separate the g components by a disjoint collection $\mathcal{S}$ of $g - 1$ 2-spheres in $\mathbb{S}^3$. On each component of the unlink, perform $1/0$-Dehn surgery. Now $\mathcal{S}$ is a set of decomposing spheres that factors the resulting 3-manifold into g factors, each homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. □

Lemma 7.6. *Let S be closed orientable surface of genus g. Let c be a simple closed curve in S. Let f be a Dehn twist around c. Let M_1 be the 3-manifold obtained by identifying two genus g handlebodies along their boundaries via f. Let M_2 be the 3-manifold obtained by identifying S with the splitting surface of the standard genus g Heegaard splitting of the connected sum of g factors of $\mathbb{S}^2 \times \mathbb{S}^1$ and performing $1/1$-Dehn surgery along c. Then M_1 is homeomorphic to M_2.*

Proof: Both 3-manifolds in question have genus g Heegaard splittings. This is more obvious in the case of the former 3-manifold. In the case of the latter 3-manifold, we may isotope c to lie just below the splitting surface and then consider the Dehn surgery to be taking place entirely in one handlebody. Removing and replacing the solid torus as required then still yields a handlebody (as does any $n/1$-surgery).

There is thus a natural homeomorphism between the two pairs of genus g handlebodies. We must show that these homeomorphisms extend across the splitting surface of the Heegaard splittings to yield a homeomorphism of 3-manifolds. Note that in the original Heegaard splitting of the connected sum of g factors of $\mathbb{S}^2 \times \mathbb{S}^1$ the two genus g handlebodies are identified along their boundaries via the identity map.

We consider c to be lying in the splitting surface of the connected sum of g factors of $\mathbb{S}^2 \times \mathbb{S}^1$. Let A_1 and A_2 be the two components of $\partial N(c)\backslash S$. Denote the boundary of the meridian disk of the solid torus attached during Dehn surgery by d. Then we may isotope d so that it is parallel to a meridian of $\partial N(c)$ in A_1. This means that in A_2, d will wind once around the longitude of $\partial N(c)$ as it crosses from one component of ∂A_2 to the other. Collapsing the interior of $N(c)$ we see that we are identifying A_1 to A_2 with $d \cap A_1$ going to $d \cap A_2$. Thus the 3-manifold under consideration is homeomorphic with M_1. □

We now prove the main theorem:

Theorem 7.7. *Every closed orientable 3-manifold can be obtained by Dehn surgery on a link in $\mathbb{S}^3$.*

Proof: Let M be a closed orientable 3-manifold. By Lemma 7.5 it suffices to show that M can be obtained by Dehn surgery on a link in a connected sum of manifolds homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$.

Let $M = V \cup_S W$ be Heegaard splitting of M. Since the two handlebodies V, W are homeomorphic, M may be obtained by taking two copies of the handlebody V and identifying these along their boundaries via a homeomorphism f. Denote the genus of ∂V by g.

By Theorem 7.4, we may factor f into Dehn twists, i.e., $f = f_1 \circ \cdots \circ f_n$, where each f_i is a Dehn twist around a curve c_i. Consider now the 3-manifold that is the connected sum of g factors of $\mathbb{S}^2 \times \mathbb{S}^1$. Further consider n parallel copies $S_1, \ldots, S_n$ of the standard splitting surface S of this 3-manifold. Cutting along each surface S_i and reidentifying via the Dehn twist f_i yields M. By Lemma 7.6, this goal may also be attained via 1/1-Dehn surgeries along the curves c_i.

□

8 Hyperbolic 3-manifolds

One of the most active areas of research into 3-manifolds concerns hyperbolic 3-manifolds. This is another class of 3-manifolds with extra structure. This extra structure adds methods of both differential geometry and algebra to the standard bag of tools used in studying 3-manifolds. We will give only a very superficial glimpse of this rich subject.

8.1 Basic hyperbolic geometry

Hyperbolic space can be realized in n-dimensional space in a variety of ways. We here discuss the upper half space model.

Definition 8.1. *The upper half space model for hyperbolic space is a metric space obtained as follows: Let* $\mathbb{U}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$. *The element of hyperbolic arc length is* $\frac{|dx|}{x_n}$ $(|dx| = \sqrt{dx^2 + dy^2})$ *and the element of hyperbolic volume is given by* $\frac{dx_1 \ldots dx_n}{(x_n)^n}$. *The distance between two points is the minimal hyperbolic arc length of an arc connecting the points. We denote this metric space by* $(\mathbb{U}^n, d_{\mathbb{U}^n})$.

E.g. 1) Calculate the arc length of the horizontal path from $(0, 1)$ to $(1, 1)$.

We parametrize the path by $(t, 1)$. Then,

$$L = \int_0^1 \frac{\sqrt{1^2 + 0}}{1} dt = 1$$

E.g. 2) Calculate the arc length of the vertical path from $(0, 1)$ to $(0, 2)$.

We parametrize the path by $(0, t)$. Then,

$$L = \int_1^2 \frac{\sqrt{0 + 1^2}}{t} dt = [lnt]_1^2 = ln2 - ln1 = ln2$$

More generally, the length from $(0, a)$ to $(0, b)$ will be $ln(b/a)$.

E.g. 3) The subarc of the unit circle from $(0, 1)$ to $(1, 0)$ not including this second endpoint has infinite length. To see this, we parametrize this arc as $(sint, cost)$ and compute the following improper integral

$$L = \int_0^{\frac{pi}{2}} \frac{\sqrt{cos^2 t + sin^2 t}}{cost} dt =$$

$$\int_0^{\frac{pi}{2}} \frac{1}{cost} dt = lim_{a \to \frac{\pi}{2}} [ln(sect + tant)]_0^a \to \infty$$

Definition 8.2. *Given a metric space (X, d) a geodesic in (X, d) is a unit speed path γ that is locally distance minimizing. I.e., for each $x \in X$, there is a neighborhood U such that for any two points in $\gamma \cap U$ the distance between these two points is the length of the subarc of $\gamma \cap U$ connnecting them.*

Theorem 8.3. *The geodesics in $(\mathbb{U}^n, d_{\mathbb{U}^n})$ are rays and half circles that are orthogonal to $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$.*

For a proof of this fact see for instance John G. Radcliffe's book "Foundations of Hyperbolic Manifolds".

Definition 8.4. *A geodesic subspace of $(\mathbb{U}^n, d_{\mathbb{U}^n})$ is a subset $X \subset \mathbb{U}^n$ such that X contains every geodesics that meets X along an open subset.*

Definition 8.5. *A hyperbolic triangle is a subset of a 2-dimensional geodesic subset of $(\mathbb{U}^n, d_{\mathbb{U}^n})$ that is bounded by connected subsets of three geodesics. In the case that it is bounded by three complete geodesics and all "vertices" lie at infinity, the triangle is called an ideal triangle.*

E.g. 4) Calculate the area of the ideal triangle in $(\mathbb{U}^2, d_{\mathbb{U}^2})$ bounded by the geodesic ray with x-coordinate -1, the geodesic ray with x-coordinate 1, and the upper half of the unit circle.

$$A = \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{dydx}{y^2} = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = [arcsinx]_{-1}^{1} = \pi$$

As it turns out, the area of any ideal triangle is π.

One of the first questions one is likely to ask concerning a metric space is that of its symmetries. The following notion generalizes this idea:

Definition 8.6. *An isometry of a metric space is a homeomorphism that preserves distances between points.*

E.g. 1) $(x_1, \ldots, x_n) \to (ax_1, \ldots, ax_n)$

E.g. 2) $(x_1, \ldots, x_n) \to (x_1 + b_1, \ldots, x_{n-1} + b_{n-1}, x_n)$

E.g. 3) $(x_1, \ldots, x_n) \to (\frac{x_1}{x_1^2 + \cdots + x_n^2}, \ldots, \frac{x_n}{x_1^2 + \cdots + x_n^2})$

To verify that these are indeed isometries, consider a parametrized path $(x(t), y(t))$ and plug into the formula for arc length.

Note that the group generated by the three isometries above acts transitively on rays orthogonal to $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ (via isometries of the second type) and on half-circles orthogonal to $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ (via compositions of isometries of the first and second type). Now note that the ray orthogonal to $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ limiting on $(1/2, 0)$ is mapped to a half circle orthogonal to $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ and limiting on $(0, 0)$ and $(2, 0)$. Thus the group generated by the isometries of these three types acts transitively on geodesics. It follows that this is the complete group of isometries. This group is denoted by either $Isom(\mathbb{U}^n, d_{\mathbb{U}^n})$ or $M\ddot{o}(\mathbb{U}^n, d_{\mathbb{U}^n})$.

8.2 Hyperbolic n-manifolds

Definition 8.7. *A hyperbolic n-manifold is a n-manifold that is locally isometric to n-dimensional hyperbolic space (e.g., in the upper half space model).*

Before discussing examples, we wish to state the relation between hyperbolic n-manifolds and subgroups of the isometry group. To do so, we must define two notions.

Definition 8.8. *Let Γ be a group acting on X. Γ acts discontinuously if for any compact subset $K \subset X$, $K \cap gK$ is non empty for only finitely many $g \in G$. Γ acts freely if for every $x \in X$, the stabilizer of x, $\Gamma_x = \{g \in \Gamma \mid gx = x\}$, is trivial.*

Theorem 8.9. *M is a hyperbolic n-manifold if and only if it is the quotient of $\mathbb{U}^n$ by a subgroup Γ of isometries of $\mathbb{U}^n$ that acts freely and discontinuously on $\mathbb{U}^n$.*

The proof of this theorem is not difficult if one accepts the (highly non trivial) fact that a simply connected hyperbolic n-manifold must be isometric with $\mathbb{U}^n$.

E.g. 1, The complement of the figure 8 knot. Later, we will see why this 3-manifold is hyperbolic.

E.g. 2, A closed orientable surfaces of genus at least 2. We consider the closed orientable surface of genus 2. The genus 2 surface may be obtained by identifying opposite sides of an octagon. This octagon may be cut into eight isosceles triangles by adding a vertex v in its center and connecting this vertex to the original eight vertices. We may assume that each of these triangles meets v in an angle of $2\pi/8 = \pi/4$. Consider the two equal angles. In the genus 2 surface the eight corners of the octagon match up. Thus the angles at the eight corners of the octagon must add up to 2π. Each of the two equal angles in an isosceles triangle must thus be $2\pi/(8)(2) = \pi/8$.

We begin by realizing this triangle in the upper half plane model. First consider the ideal triangle bounded by the upper half of the unit circle and the vertical rays limiting on $(-1, 0)$ and $(1, 0)$. The angles in this triangle are all 0. Now consider replacing the vertical rays by very large half circles that limit on $(-1, 0)$ and $(1, 0)$ and intersect the y-axis in $(0, y)$ for y very large. Then the angle between the rays is very small. As these two circles get smaller, i.e., as y gets smaller, the angle between them gets larger. Any value strictly between 0 and π can be obtained. We choose the two half circles C_1, C_2 to obtain an angle of $\pi/4$. Denote the point of intersection of C_1 and C_2 by x.

Next consider expanding the upper half of the unit circle. As we do so, the equal angles in the isosceles triangle get larger. Any value strictly between 0 and $3\pi/4$ may be obtained. (Notice how small triangles are almost Euclidean and thus the sum of the angles in the small triangle is almost π.) We choose the half circle C_3 to obtain angles of $\pi/8$.

Reflect C_1 in C_2 and vice versa. Then further reflecting C_1 and C_2 in the resulting circles. In this manner we obtain an octagon consisting of eight copies of the orginal hyperbolic triangle meeting in x. Note that the reflection of this octagon in C_3 maps the octagon off itself. Let Γ be the group generated by the reflections in the eight copies of C_3. As it turns out, this group acts freely and discontinuously on the upper half plane. Its quotient is a closed orientable genus 2 surface.

Remark 8.10. *An argument similar to that used to construct the hyperbolic triangle with angles $\pi/4, \pi/8, \pi/8$ may be used to show that in fact for any $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma < \pi$ there is a hyperbolic triangle with angles α, β, γ.*