# Free Probability with Left and Right Variables

## (Free Probability for Pairs of Faces)

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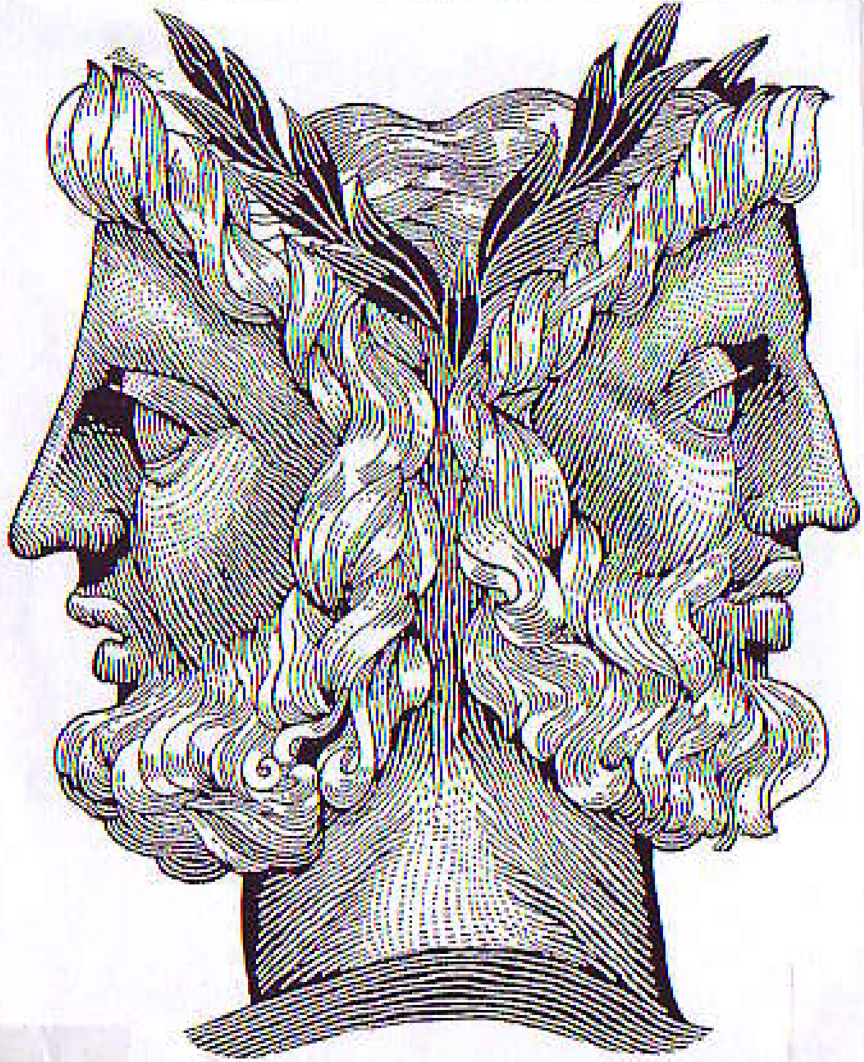

# Janus

2 faces

Past and Future

Transition

[Left Var, Right Var] = 0

Bipartite System

Possible Connections:

- Free Probability of Type B
  (Biane-Goodman-Nica, Belinschi-Shlyakhtenko)

- Second Order Freeness
  (Collins-Mingo-Sniady-Speicher)

- Matricial Freeness
  (Lenczewski)

# Free Product of Vector Spaces with specified State Vectors

$$\mathcal{X}_\iota = \mathring{\mathcal{X}}_\iota \oplus \mathbb{C}\xi_\iota$$

$$\mathcal{X} = \mathbb{C}\xi \oplus \underbrace{\bigoplus_{n\geq 1}\ \bigotimes_{\iota_1\neq\cdots\neq\iota_n} \mathring{\mathcal{X}}_{\iota_1}\otimes\cdots\otimes\mathring{\mathcal{X}}_{\iota_n}}_{\mathring{\mathcal{X}}}$$

$$(\mathcal{X},\mathring{\mathcal{X}},\xi) = \mathop{\ast}_{\iota\in I}(\mathcal{X}_\iota,\mathring{\mathcal{X}}_\iota,\xi_\iota)$$

$$\varphi_\xi : \mathcal{L}(\mathcal{X}) \to \mathbb{C}\ ,\quad T\xi \in \varphi_\xi(T)\xi \oplus \mathring{\mathcal{X}}\ .$$

# Left and Right Factorizations

$$V_\iota : \mathcal{X}_\iota \otimes \Big(\mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \bigotimes_{\iota\neq\iota_1\neq\cdots\neq\iota_n} \mathring{\mathcal{X}}_{\iota_1} \otimes \cdots \otimes \mathring{\mathcal{X}}_{\iota_n}\Big) \to \mathcal{X}$$

$$W_\iota : \Big(\mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \bigotimes_{\iota_1\neq\cdots\neq\iota_n\neq\iota} \mathring{\mathcal{X}}_{\iota_1} \otimes \cdots \otimes \mathring{\mathcal{X}}_{\iota_n}\Big) \otimes \mathcal{X}_\iota \to \mathcal{X}$$

$$T \in \mathcal{L}(\mathcal{X}_\iota)$$

$$\lambda_\iota(T) = V_\iota (T \otimes I) V_\iota^{-1} \in \mathcal{L}(\mathcal{X})$$

$$\rho_\iota(T) = W_\iota (I \otimes T) W_\iota^{-1} \in \mathcal{L}(\mathcal{X})$$

$$[\lambda_\iota(T), \rho_j(S)] = \delta_{ij} [T, S] \oplus 0 .$$

$(\mathcal{A}, \varphi)$ noncommutative probability space

Pair of Faces in $(\mathcal{A}, \varphi)$

$(\mathcal{B}, \beta)$ left face, right face $(\mathcal{C}, \gamma)$

$$\begin{tikzcd}
 & \mathcal{A} & \\
\mathcal{B} \arrow[ur, "\beta"] & & \mathcal{C} \arrow[ul, "\gamma"']
\end{tikzcd}$$

$\beta, \gamma$ unital homomorphisms

$\mathcal{B}, \mathcal{C}$ unital algebras

Included faces $\mathcal{B} \subset \mathcal{A} \supset \mathcal{C}$.
($\beta$, $\gamma$ are the inclusions)

2-faced family of noncommutative random variables in $(\mathcal{A}, \varphi)$

$$((b_i)_{i \in I}, (c_j)_{j \in J}) \text{ in } \mathcal{A}$$

[Corresponds to

$$\beta : \mathbb{C}\langle X_i \mid i \in I \rangle \longrightarrow \mathcal{A}, \quad \beta(X_i) = b_i$$

$$\gamma : \mathbb{C}\langle Y_j \mid j \in J \rangle \longrightarrow \mathcal{A}, \quad \gamma(Y_j) = c_j]$$

# Bi-freeness of a family of pairs of faces

$((B_\iota, \beta_\iota), (C_\iota, \gamma_\iota))_{\iota \in I}$ in $(A, \varphi)$:

$\exists\ (\mathcal{X}_\iota, \mathring{\mathcal{X}}_\iota, \xi_\iota),\ \iota \in I$, $(\mathcal{X}, \mathring{\mathcal{X}}, \xi) = \underset{\iota \in I}{*} (\mathcal{X}_\iota, \mathring{\mathcal{X}}_\iota, \xi_\iota)$

$l_\iota : B_\iota \longrightarrow \mathcal{L}(\mathcal{X}_\iota)$, $r_\iota : C_\iota \longrightarrow \mathcal{L}(\mathcal{X}_\iota)$

unital homomorphisms, so that

$$\varphi \circ \pi = \varphi_\xi \circ \tilde{\pi}$$

$\pi : \underset{\iota \in I}{*} (B_\iota * C_\iota) \longrightarrow A$, $\pi|B_\iota = \beta_\iota$, $\pi|C_\iota = \gamma_\iota$

$\tilde{\pi} : \underset{\iota \in I}{*} (B_\iota * C_\iota) \longrightarrow \mathcal{L}(\mathcal{X})$, $\pi|_{B_\iota} = \lambda_\iota \circ l_\iota$, $\pi|C_\iota = \rho_\iota \circ r_\iota$.

<u>Remarks</u>: 1° If $((B_\iota, \beta_\iota), (C_\iota, \gamma_\iota))_{\iota \in I}$ bi-free in $(\mathcal{A}, \varphi)$, joint distribution $\varphi \circ \tilde{\pi}$ obtained also as $\varphi_{\xi'} \circ \tilde{\pi}'$ for any other $(\mathcal{X}'_\iota, \mathring{\mathcal{X}}'_\iota, \xi'_\iota)$, $\ell'_\iota, r'_\iota$ so that

$$\varphi \circ \pi_\iota = \varphi_{\xi'_\iota} \circ \pi'_\iota$$

$$\pi_\iota : B_\iota * C_\iota \to \mathcal{A}, \ \pi_\iota|_{B_\iota} = \beta_\iota, \ \pi_\iota|_{C_\iota} = \gamma_\iota$$

$$\pi'_\iota : B_\iota * C_\iota \to \mathcal{L}(\mathcal{X}'_\iota), \ \pi'_\iota|_{B_\iota} = \ell'_\iota, \ \pi'_\iota|_{C_\iota} = r'_\iota$$

2°. $\left((B_\iota, \beta_\iota), (C_\iota, \gamma_\iota)\right)_{\iota \in I}$ bi-free in $(\mathcal{A}, \varphi)$

then $(\beta_\iota(B_\iota))_{\iota \in I}$ free in $(\mathcal{A}, \varphi)$

$\iota \neq j \Rightarrow \beta_\iota(B_\iota), \gamma_j(C_j)$ classically independent in $(\mathcal{A}, \varphi)$.

3°. bi-freeness has the necessary properties to be used as an independence relation in a noncommutative probability theory with left- and right variables i.e. two-faced families.

4°. $C^*$-bi-freeness, $W^*$-bi-freeness
bi-free products of states etc.
bi-free convolution operations
(additive, multiplicative)

$\mu \boxplus\boxplus \nu$ , $\mu \boxtimes\boxtimes \nu$

# Bi-freeness Examples

I. Groups $(G_\iota)_{\iota \in I}$ , $G = \underset{\iota \in I}{*} G_\iota$ .

$L_\iota : \mathbb{C}[G_\iota] \to \mathcal{L}(\mathbb{C}[G])$

$R_\iota : \mathbb{C}[G_\iota]^{op} \to \mathcal{L}(\mathbb{C}[G])$

restrictions of left and right regular representations

$((\mathbb{C}[G_\iota], L_\iota), (\mathbb{C}[G_\iota]^{op}, R_\iota))_{\iota \in I}$

bi-free family of faces in $(\mathbb{C}[G], \tau)$.

v. Neumann trace

## II. Left and right creation and annihilation operators on the full Fock space.

$\mathcal{H}$ complex Hilbert sp. $(e_\iota)_{\iota\in I}$ ONB

$\mathcal{T}(\mathcal{H}) = \mathbb{C}1 \oplus \bigoplus_{n\geq 1} \mathcal{H}^{\otimes n}$

$\ell_\iota \zeta = e_\iota \otimes \zeta$, $r_\iota \zeta = \zeta \otimes e_\iota$, $\zeta \in \mathcal{T}(\mathcal{H})$.

$\omega(T) = \langle T1, 1\rangle$ on $\mathcal{B}(\mathcal{T}(\mathcal{H}))$

$\left( (\ell_\iota, \ell_\iota^*), (r_\iota, r_\iota^*) \right)_{\iota\in I}$

bi-free in $\left(\mathcal{B}(\mathcal{T}(\mathcal{H})), \omega\right)$.

# Bi-free Cumulants

$z = ((z_i)_{i \in I}, (z_j)_{j \in J})$ 2-faced family of r.v. in $(A, \varphi)$

Moments $\varphi(z_{\alpha(1)} \cdots z_{\alpha(n)})$, $\alpha : \{1, \ldots, n\} \to I \amalg J$

$R_\alpha$ polynomial in commuting variables $X_{\alpha(k_1) \cdots \alpha(k_r)}$, $1 \le k_1 < \cdots < k_r \le n$.

homogeneous $\deg = n$, $\deg X_{\alpha(k_1) \cdots \alpha(k_r)} = r$.

$R_\alpha(z) = R_\alpha(\varphi(z_{\alpha(k_1)} \cdots z_{\alpha(k_n)}) \mid 1 \le k_1 < \cdots < k_n \le n\}$

$R_\alpha$ bi-free cumulant, exists & unique so that:

1°. coefficient of $X_{\alpha(1)\ldots\alpha(n)} = 1$

2°. $z', z''$ bi-free in $(A, \varphi)$, then

$$R_\alpha(z') + R_\alpha(z'') = R_\alpha(z' + z'').$$

$$\alpha\Pi_m = \{(\alpha(k_1), \dots \alpha(k_r)) \mid 1 \le k_1 < \dots < k_r \le m,\ 1 \le r \le n\}$$

$$M_{z,\alpha} = \left(\varphi(z_{\alpha(1)} \cdots z_{\alpha(k_r)})\right)_{(\alpha(k_1), \dots \alpha(k_r)) \in \alpha\Pi_m}$$

$$(M_{z',\alpha}, M_{z'',\alpha}) \longrightarrow M_{z'+z'',\alpha}$$

polynomial abelian group law on $\mathbb{C}^{\alpha\Pi_m}$

$$\mathbb{C}^{\alpha\Pi_m} \xrightarrow{\exp} \mathbb{C}^{\alpha\Pi_m} \quad \text{isomorphism}$$

(Lie algebra, +) $\qquad\qquad$ $\boxplus$ $\boxplus_n$ law

$\log = (\exp)^{-1}$ yields cumulants

# Two-Bands $\mathcal{R}$-transform

$$\mathcal{R}_{(a,b)}(z,w) = \sum_{\substack{m\geq 0,\, n\geq 0\\ m+n>0}} \mathcal{R}_{m,n}(a,b)\, z^m w^n$$

$$G_a(z) = \varphi\left((z-a)^{-1}\right),\quad K_a(z) = z^{-1} + \mathcal{R}_a(z)$$

$$G_a(K_a(z)) = z$$

$$G_{(a,b)}(z,w) = \varphi\left((z-a)^{-1}(w-b)^{-1}\right)$$

$$\mathcal{R}_{(a,b)}(z,w) = 1 - \frac{z\,w}{G_{(a,b)}(K_a(z), K_b(w))} + z\,\mathcal{R}_a(z) + w\,\mathcal{R}_b(w)$$

# Bi-free Central Limit

$z$ two-faced family in $(A, \varphi)$

has bi-free central limit distribution

(aka bi-free Gaussian)

if $n \neq 2 \Longrightarrow R_{\alpha(1)\cdots\alpha(n)}(z) = 0$

$n = 1 \quad R_a(z) = \varphi(z_a)$

$n = 2 \quad R_{ab}(z) = \varphi(z_a z_b) - \varphi(z_a)\varphi(z_b).$

bi-free central limit distribution

$\gamma_C : \mathbb{C}\langle Z_k \mid k \in I \amalg J \rangle \to \mathbb{C}$

determined by covariance matrix

$C = (C_{k,\ell})_{k,\ell \in I \amalg J}$ .

$\gamma_C(Z_k Z_\ell) = C_{k\ell}$

(equivalently $C_{k\ell} = R_{k\ell}(Z)$ ).

# Realization on full Fock space

$\mathcal{T}(\mathcal{H})$ full Fock space, $T \longrightarrow \langle T1, 1 \rangle$
vacuum expectation

$\ell(h), \ell^*(h), r(h), r^*(h)$
left and right creation and annihilation

$h, h^*: I \amalg J \longrightarrow \mathcal{H}$ maps

$z_i = \ell(h(i)) + \ell^*(h^*(i)) \quad i \in I$

$z_j = r(h(j)) + r^*(h^*(j)) \quad j \in J$

$z = ((z_i)_{i \in I}, (z_j)_{j \in J})$ bi-free Gaussian

covariance $C_{ab} = \langle h(b), h^*(a) \rangle$.

# Bi-free Algebraic CLT

bi-free sequence

$$(z^{(n)})_{n\in\mathbb{N}} = \left( ((z_i^{(n)})_{i\in I}, (z_j)_{j\in J}) \right)_{n\in\mathbb{N}} \text{ in } (\mathcal{A}, \varphi)$$

(i) $\varphi(z_k^{(n)}) = 0$, $k \in I \amalg J$

(ii) $\sup\limits_{n\in\mathbb{N}} |\varphi(z_{k_1}^{(n)} \cdots z_{k_m}^{(n)})| = D_{k_1 \cdots k_m} < \infty$

(iii) $\lim\limits_{N\to\infty} N^{-1} \sum\limits_{1\le n\le N} \varphi(z_k^{(n)} z_\ell^{(n)}) = C_{k\ell}$

$$S_N = ((S_{N,i})_{i\in I}, (S_{N,j})_{j\in J})$$

$$S_{N,k} = N^{-1/2} \sum_{1\le n\le N} z_k^{(n)}$$

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$\Rightarrow S_N$ has limit distribution bi-free Gaussian with covariance $(C_{k\ell})_{k,\ell\in I\amalg J}$ as $N \to \infty$

$\mathbb{C} \rightsquigarrow B$ algebra with 1.

# Bi-freeness with amalgamation over $B$

## $B$-$B$ noncommutative probability space

$(\mathcal{A}, p, \varepsilon)$ $\mathcal{A}$ unital algebra over $\mathbb{C}$

$\varepsilon : B \otimes B^{op} \to \mathcal{A}$ unital homomorphism

$\varepsilon|_{B \otimes 1}$, $\varepsilon|_{1 \otimes B^{op}}$ injective

$p : \mathcal{A} \to B$ linear unital

$p(\varepsilon(b_1 \otimes 1)\, a\, \varepsilon(1 \otimes b_2)) = b_1\, p(a)\, b_2$

(in particular $(p \circ \varepsilon)(b_1 \otimes b_2) = b_1 b_2$ .

$(\mathcal{A}, \varphi, \varepsilon)$ $B$-$B$ noncommutative probability space

$\mathcal{A}_r$ commutant in $\mathcal{A}$ of $\varepsilon(1 \otimes B^{op})$

$\mathcal{A}_\ell$ commutant in $\mathcal{A}$ of $\varepsilon(B \otimes 1)$

included pair of $B$-faces in $(\mathcal{A}, \varphi, \varepsilon)$

$(C, D)$ unital subalgebras in $\mathcal{A}$

$$\varepsilon(B \otimes 1) \subset C \subset \mathcal{A}_r$$

$$\varepsilon(1 \otimes B^{op}) \subset D \subset \mathcal{A}_\ell$$

B-B bimodules with specified state vector

$\mathcal{X} = \mathring{\mathcal{X}} \oplus B$ $\qquad$ $\mathcal{X}, \mathring{\mathcal{X}}$ B-B bimodules

Free Product

$$\mathop{\ast_B}_{\iota \in I} (\mathcal{X}_\iota, \mathring{\mathcal{X}}_\iota) = (\mathcal{X}, \mathring{\mathcal{X}})$$

$$\mathring{\mathcal{X}} = \bigoplus_{n \geq 1} \bigoplus_{\iota_1 \neq \cdots \neq \iota_n} \mathcal{X}_{\iota_1} \otimes_B \mathcal{X}_{\iota_2} \otimes_B \cdots \otimes_B \mathcal{X}_{\iota_n}$$

$$\mathcal{X} = \mathring{\mathcal{X}} \oplus B$$

$\mathcal{X} = \mathring{\mathcal{X}} \oplus \mathcal{B}$ $\mathcal{B}$-$\mathcal{B}$ bimodule

$p_{\mathcal{X}} : \mathcal{L}(\mathcal{X}) \longrightarrow \mathcal{B}$

$T(0 \oplus 1) \in \mathring{\mathcal{X}} \oplus p_{\mathcal{X}}(T)$

$\varepsilon_{\mathcal{X}} : \mathcal{B} \otimes \mathcal{B}^{op} \longrightarrow \mathcal{L}(\mathcal{X})$ left & right $\mathcal{B}$ multiplication

$(\mathcal{L}(\mathcal{X}), p_{\mathcal{X}}, \varepsilon_{\mathcal{X}})$ $\mathcal{B}$-$\mathcal{B}$ noncommutative probability space

$\mathcal{L}_r(\mathcal{X})$ right $\mathcal{B}$-linear operators

$\mathcal{L}_\ell(\mathcal{X})$ left $\mathcal{B}$-linear operators

$$(\mathcal{X}, \mathring{\mathcal{X}}) = \mathop{\ast_B}_{\iota \in I} (\mathcal{X}_\iota, \mathring{\mathcal{X}}_\iota)$$

$$V_\iota : \mathcal{X}_\iota \otimes_B \Big(B \oplus \bigoplus_{n \geq 1} \bigoplus_{\iota \neq \iota_1 \neq \cdots \neq \iota_n} \mathring{\mathcal{X}}_{\iota_1} \otimes_B \cdots \otimes_B \mathring{\mathcal{X}}_{\iota_n}\Big) \to \mathcal{X}$$

$$W_\iota : \Big(B \oplus \bigoplus_{n \geq 1} \bigoplus_{\iota_1 \neq \cdots \neq \iota_n \neq \iota} \mathring{\mathcal{X}}_{\iota_1} \otimes_B \cdots \otimes_B \mathring{\mathcal{X}}_{\iota_n}\Big) \otimes \mathcal{X}_\iota \to \mathcal{X}$$

$$\lambda_\iota : \mathcal{L}_r(\mathcal{X}_\iota) \to \mathcal{L}_r(\mathcal{X})$$

$$\rho_\iota : \mathcal{L}_\ell(\mathcal{X}_\iota) \to \mathcal{L}_\ell(\mathcal{X})$$

$$\lambda_\iota(T) = V_\iota (T \otimes 1) V_\iota^{-1}$$

$$\rho_\iota(T) = W_\iota (I \otimes T) W_\iota^{-1}$$

$(\mathcal{A}, p, \varepsilon)$ $B$-$B$ noncomm. prob. sp.

<u>family</u> $((C_\iota, D_\iota))_{\iota \in I}$ <u>of pairs of $B$-face</u>

<u>in</u> $(\mathcal{A}, p, \varepsilon)$ <u>is bi-free over</u> $B$ if :

$\exists \quad \mathcal{X}_\iota = \mathring{\mathcal{X}}_\iota \oplus B \quad$ $B$-$B$ bimodules

unital homomorphisms

$\gamma_\iota : C_\iota \longrightarrow \mathcal{L}_r(\mathcal{X}_\iota)$, $\gamma_\iota(\varepsilon(b \otimes 1)) = \varepsilon_{\mathcal{X}_\iota}(b \otimes 1)$

$\delta_\iota : D_\iota \longrightarrow \mathcal{L}_\ell(\mathcal{X}_\iota)$, $\delta_\iota(\varepsilon(1 \otimes b)) = \varepsilon_{\mathcal{X}_\iota}(1 \otimes b)$

so that

if $c_k \in C_{\iota(k)}$, $d_k \in D_{\iota(k)}$, $1 \leq k \leq n$

then

$$p(c_1 d_1 c_2 d_2 \cdots c_n d_n) =$$

$$= p_{\mathcal{X}}\big(\lambda_{i(1)}(\gamma_{\iota(1)}(c_1))\,\rho_{\iota(1)}(\delta_{\iota(1)}(d_1)) \cdots$$

$$\cdots \lambda_{i(n)}(\gamma_{\iota(n)}(c_n))\,\rho_{\iota(n)}(\delta_{\iota(n)}(d_n))\big).$$