MANIFOLDS

(1) What is a manifold of dimension n?

(2) What is an n-dimensional manifold with boundary?

(3) Show that the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a 1-dimensional manifold, by showing that the following four open sets are homeomorphic to open subsets of $\mathbb{R}$ and every point of S^1 is contained in one of these four open sets:

$$U_+ = \{(x, y) \in S^1 \mid x > 0\}, \qquad U_- = \{(x, y) \in S^1 \mid x < 0\}$$
$$V_+ = \{(x, y) \in S^1 \mid y > 0\}, \qquad V_- = \{(x, y) \in S^1 \mid y < 0\}$$

(4) Show that the sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ is a 2-dimensional manifold in a similar manner using six open sets. Try to generalize these two cases to the n-sphere: $S^n = \{(x_1, \cdots, x_{n+1}) \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$.

(5) Show that the semicircle $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \geq 0\}$ is a 1-dimensional manifold with boundary and the hemisphere $D = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \geq 0\}$ is a 2-dimensional manifold with boundary.

(6) Suppose X is an n-dimensional manifolds with boundary. Let ∂X denote the set of points in the boundary of X. Show that ∂X is an $(n-1)$-dimensional manifold.

(7) Explain what an *adjunction space* is (Lee Ch 3 p. 74).

(8) Suppose X and Y are n-dimensional manifolds with boundary and the boundaries are manifolds ∂X and ∂Y respectively. Suppose there is a homeomorphism $f : \partial X \to \partial Y$. Prove that the adjunction space $X \cup_f Y$ is an n-dimensional manifold (no boundary).

(9) Suppose we have a polygonal representation of a surface where each edge appears exactly twice. Show that the identification space is a 2-dimensional manifold.

(10) Describe how we can write $S^3 \cong V_1 \cup_f V_2$ where V_1 and V_2 are solid tori $S^1 \times D^2$ and $f : \partial V_2 \to \partial V_1$ is a homeomorphism of the torus. Describe the homeomorphism f.

(11) Find a way to decompose $S^1 \times S^2$ as the adjunction space of two solid tori, $S^1 \times S^2 \cong V_1 \cup_g V_2$ where $g : \partial V_2 \to \partial V_1$ is a homeomorphism of the torus boundary. What is the map g?