Lecture 1: 4-manifolds

Tuesday, June 15, 2021 3:07 PM

4-manifolds -- usually ours will be orientable, closed, and connected.

Basic examples: S^4, $S^2 \times S^2$, $\Sigma_1 \times \Sigma_2$, $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$, $S^1 \times S^3$

Connected sums : $X_1 \# X_2$ (delete a 4-ball from X_1 and a 4-ball from X_2 then glue along common S^3 boundaries)

Algebraic topology invariants of 4-manifolds

- Fundamental group $\pi_1(X)$: Any finitely presented group can be realized as $\pi_1(X)$ for some closed orientable manifold X.

  $\Rightarrow$ classifying all 4-manifolds is unreasonable

  often focus on 4-manifolds where $\pi_1(X) = 1$ (simply connected)

- Homology groups: $H_0(X) \cong \mathbb{Z}$ $H_4(X) \cong \mathbb{Z}$

  If $\pi_1(X) = 1$ then $H_1(X) \cong 0$. $\Rightarrow H_3(X) = 0$

Betti numbers:
$b_i(X) = \text{rank}(H_i(X))$
$b_0 = b_4 = 1$, $b_1 = b_3$
b_2 can be anything.

Most interesting is $H_2(X) \cong H^2(X)$

Poincare duality

has extra structure

Intersection form:

$$Q_X : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to \mathbb{Z}$$

For $\alpha, \beta \in H^2(X;\mathbb{Z})$ their cup product $\alpha \cup \beta \in H^4(X;\mathbb{Z}) \cong \mathbb{Z}$

equiv.

$$Q_X : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}$$

vanishes on torsion elements

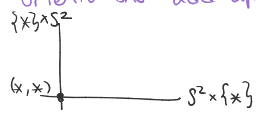
Represent $A, B \in H_2(X;\mathbb{Z})$ by embedded surfaces $\Sigma_A, \Sigma_B \subset X$

$Q_X(A,B) = \Sigma_A \cdot \Sigma_B$ ← signed intersection number (make transverse first by perturbing)

Example: $X = S^2 \times S^2$

$H_2(S^2\times S^2;\mathbb{Z}) \cong \mathbb{Z}^2$ generated by $A=[S^2\times\{*\}]$ and $B=[\{*\}\times S^2]$

$Q_X(A,B) = (S^2\times\{*\})\cdot(\{*\}\times S^2) = 1 \leftarrow$ they intersect transversally at 1 point $(*,*)$ and orientations add up:



$Q_X(A,A) = (S^2\times\{*\})\cdot(S^2\times\{*\})$

$= (S^2\times\{*\})\cdot(S^2\times\{*'\}) = 0$

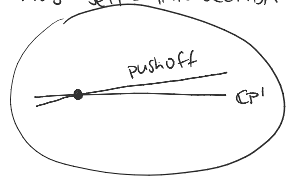
$Q_X(B,B) = 0$

So $Q_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in basis (A,B)

Exercise: $Q_{\mathbb{CP}^2} = [1]$ i.e. $H_2(\mathbb{CP}^2;\mathbb{Z}) \cong \mathbb{Z}$ and the generator $(\mathbb{CP}^1 \subset \mathbb{CP}^2)$ has self-intersection $+1$

If I change the orientation on $\mathbb{CP}^2$, denoted $\overline{\mathbb{CP}^2}$, the sign of intersection is reversed

$Q_{\overline{\mathbb{CP}^2}} = [-1]$



Extracting invariant properties from Q_X: invariant under $\mathbb{Z}$ change of basis

Q_X can be diagonalized over $\mathbb{R}$ (maybe not over $\mathbb{Z}$)

$b_2^+(X) = \#$ positive eigenvalues (counted with multiplicity) (# positive entries on diagonal after diagonalization)

$b_2^-(X) = \#$ negative "

$b_2^0(X) = \#$ zero "

$b_2 = b_2^+ + b_2^- + b_2^0$

The signature of X $\sigma(X) := b_2^+(X) - b_2^-(X)$

Fact: If X is closed (no boundary) $\det Q_X = 1$ (Q_X is unimodular)

(comes from Poincaré duality)

Consequence: $b_2^0 = 0$.

Say X is positive definite if $b_2^+ = b_2$, negative definite if $b_2^- = b_2$

Say Q_X is even if $Q(A,A)$ is even $\forall A \in H_2(X)$ otherwise Q_X is odd

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PAUSE

Can consider 4-manifolds up to homeomorphism ← topological category
or diffeomorphism ← smooth category

In 4-manifold topology these are very different as you will learn.

This summer school will focus on the smooth category (diffeomorphism)
but what is the story in the topological category?

Freedman's Theorem :

Existence • For every unimodular symmetric bilinear form Q, there is a closed, simply connected, topological 4-manifold X with $Q_X \cong Q$.

Uniqueness
- If Q is even X is unique up to homeomorphism.
- If Q is odd there are exactly 2 homeomorphism classes of such X
  (at most one can have a smooth structure).

Contrast to the smooth category :

Existence

There are lots of non isomorphic (over $\mathbb{Z}$) negative definite unimodular forms Q

but

Donaldson's diagonalization Theorem : Any smooth 4-manifold with negative-definite intersection form has $Q_X \cong \bigoplus_n [-1]$ (diagonal matrix) (see this in 1 week)

Uniqueness

Exotic examples : There are (infinitely many) manifolds $\{X_k\}$ such that X_k and $X_{k'}$ are homeomorphic but not diffeomorphic.

(we will use Seiberg-Witten invariants to find exotic examples next week)

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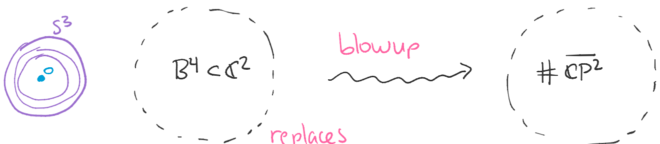
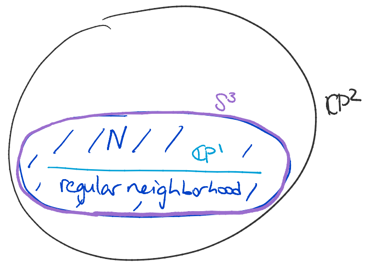
Last topic for this lecture: (Complex) blow-up.

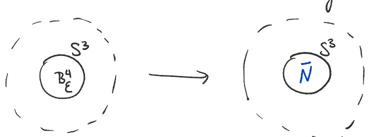
Every 4-manifold locally looks like $\mathbb{C}^2$ w/ complex coordinates (z_1, z_2)



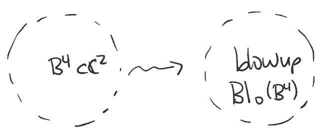
replaces a point $(0,0) \in B^4$ with a $\mathbb{CP}^1$ (a 2-sphere)



Key fact: A neighborhood N of $\mathbb{CP}^1$ in $\mathbb{CP}^2$ has boundary diffeomorphic to S^3
(the complement of the neighborhood is $\cong B^4$)



Another way to understand blow-up:



In coordinates $(z_1, z_2) \in \mathbb{C}^2$

$$Bl_0(B^4) = \{((z_1,z_2), [w_1 : w_2]) \in B^4 \times \mathbb{CP}^1 \mid z_1 w_2 - z_2 w_1 = 0\}$$

$\pi: Bl_0(B^4) \to B^4$ (project $((z_1,z_2),[w_1:w_2]) \mapsto (z_1,z_2)$)

If $(z_1, z_2) \neq (0,0)$ can solve $z_1 w_2 = z_2 w_1$ to find unique $[w_1 : w_2]$
$(\pi^{-1}(z_1,z_2)$ is one point$)$ $([w_1:w_2] = [\lambda w_1 : \lambda w_2])$

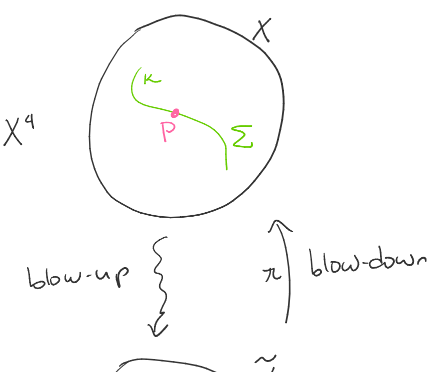
If $(z_1,z_2) = (0,0)$ every $[w_1 : w_2] \in \mathbb{CP}^1$ solves $z_1 w_2 - z_2 w_1 = 0$.

$\Rightarrow$ the point $(0,0)$ is replaced by a $\mathbb{CP}^1$ $(\pi^{-1}(0,0) \cong \mathbb{CP}^1)$

Call the new $\mathbb{CP}^1$ in the blowup the exceptional divisor.

Effect of blow-ups on surfaces in 4-manifolds

If we blow-up X at a point p on a smooth surface Σ: $p \in \Sigma^2 \subset X^4$



In the blow-up, p becomes an exceptional divisor E (a (-1)-sphere) blow-up

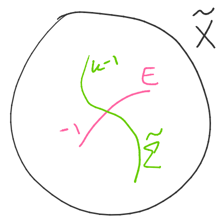
exceptional divisor E (a (-1)-sphere)

$$\underbrace{\pi^{-1}(\Sigma)}_{\text{"total transform of } \Sigma\text{"}} =: \overline{\Sigma} = \tilde{\Sigma} \cup E$$

E ← exceptional divisor

$\tilde{\Sigma}$ ← "proper/strict transform of Σ"

$\tilde{\Sigma} = \overline{\pi^{-1}(\Sigma - p)} \subset \tilde{X}$



$$[\tilde{\Sigma}] = [\Sigma] - [E] \Longrightarrow \boxed{[\tilde{\Sigma}]^2 = [\Sigma]^2 - 1}$$

Extras about indefinite intersection forms and existence of smooth 4-mflds

If a unimodular form is indefinite ($b_2^+, b_2^- > 0$) and odd, there is an integer change of basis showing it is equivalent to $[1]^{\oplus b_2^+} \oplus [-1]^{\oplus b_2^-}$.

This can be realized by smooth 4-manifold: $\#_{b_2^+} \mathbb{CP}^2 \#_{b_2^-} \overline{\mathbb{CP}^2}$.

If an indefinite unimodular form is even and has signature 0 $(\sigma = b_2^+ - b_2^- = 0)$ then it is equivalent ($\mathbb{Z}$ change of basis) to $\bigoplus_{b_2^+} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

This can be realized by smooth 4-manifold $\#_{b_2^+} S^2 \times S^2$.

If a unimodular form is even, then its signature σ is divisible by 8 and an indefinite even form is equivalent ($\mathbb{Z}$-change of basis) to

$$\oplus^{\frac{|\sigma|}{8}} \operatorname{sgn}(\sigma) E_8 \oplus \frac{b_2 - |\sigma|}{2} H$$

where

$$E_8 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \qquad -E_8 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

So even indefinite forms look like $\oplus n E_8 \oplus \ell H$

Thm: [Rohlin] If X^4 is a smooth simply connected closed, or. 4-manifold σ is divisible by 16.

$\Rightarrow n = 2k$ for some $k \in \mathbb{Z}$.

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Partially open: Which values $k, \ell \in \mathbb{Z}$ does there exist a smooth, $\pi_1 = 1$, closed, or 4-manifold with intersection form $\oplus 2k\, E_8 \oplus \ell H$?

There exists such a smooth 4-manifold we will see (K3-surface) with intersection form $\oplus 2\, {-E_8} \oplus 3H$.

Taking connected sums, of this and $S^2 \times S^2$ we get examples with int. form

$$\oplus 2m\, {}^{-}E_8 \oplus (3m+n)\, H \qquad n, m \geq 0$$

Reversing orientations, (+ $\mathbb{Z}$-change of basis) we get $\oplus 2m\, E_8 \oplus (3m+n) H \qquad n, m \geq 0.$

Thm [Furuta] If the intersection form of a smooth, $\pi_1 = 1$, closed, or 4-manifold is $\oplus 2k\, {}^{\pm}E_8 \oplus \ell H$ then $\ell \geq 2k+1$.

Conjecture ("11/8"): $\ell \geq 3k$. (i.e. the connected sums above realize all possible intersection forms for smooth closed or $\pi_1 = 1$ 4-manifolds).