Lecture 2: Elliptic surfaces

Thursday, June 24, 2021 10:14 AM

Goals this week:

- Learn constructive methods to build new 4-manifolds
- Understand the effect of various cut-and-paste constructions on the Seiberg-Witten invariants
- Build and detect examples of exotic pairs (two 4-manifolds which are homeomorphic but not diffeomorphic)

Goals today:

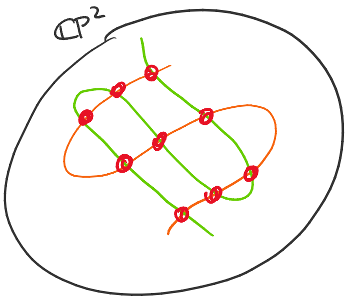
- Understand E(1), the first elliptic fibration
- Learn the fiber sum operation and construct E(n)
- Determine the homotopy & Seiberg-Witten invariants of E(n)
- Learn formulas for Seiberg-Witten invariants under blow-up and connected sum vanishing
- Find our first exotic example

Elliptic fibration -- E(1) :

$\pi : X \to T$ ← cx curve; cx surface ↗ X

$\pi^{-1}(t)$ generically a torus

To build E(1), start in $\mathbb{CP}^2$:



$C_0 = \{ [x:y:z] \in \mathbb{CP}^2 \mid f(x,y,z) = 0 \}$

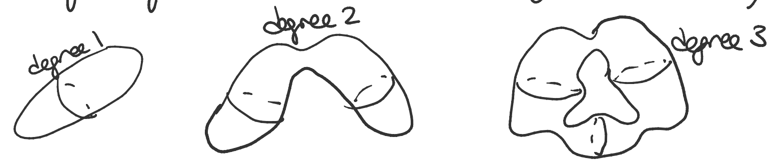
$C_1 = \{ [x:y:z] \in \mathbb{CP}^2 \mid g(x,y,z) = 0 \}$

f and g are generic degree 3 homogeneous polynomials

e.g. $f(x,y,z) = x^3 - y^3 + 5z^3$ $\quad g(x,y,z) = x^2 y + 30 y^2 z - 7z^2 x$

$\boxed{(1)\ C_0 \text{ and } C_1 \text{ are genus } 1}$

(degree genus formula in $\mathbb{CP}^2$: $g = \frac{(d-1)(d-2)}{2}$)



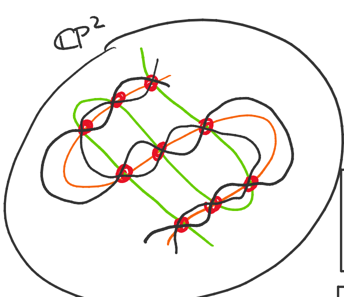
$\boxed{(2)\ C_0 \text{ and } C_1 \text{ intersect transversally in } 9 \text{ points, } p_1, \dots, p_9}$

(Riemann-Roch $3 \cdot 3 = 9$)

Cubic Pencil on $\mathbb{CP}^2$

For $[t:s] \in \mathbb{CP}^1$ let $C_{[t:s]} = \{ [x:y:z] \in \mathbb{CP}^2 \mid t f(x,y,z) + s g(x,y,z) = 0 \}$

$C_{[1:0]} = C_0$, $\quad C_{[0:1]} = C_1$,



(1) $p_1, \dots, p_9 \in C_{[t:s]}$ for all $[t:s] \in \mathbb{CP}^1$
($C_{[t:s]}$ and $C_{[t':s']}$ intersect transversally at $p_1, \dots, p_9$)

(2) Every point in $\mathbb{CP}^2$ is in $C_{[t:s]}$ for some $[t:s]$

(3) For generic $[t:s]$, $C_{[t:s]}$ is a smooth torus

Almost defines elliptic fibration $\pi : \mathbb{CP}^2 \to \mathbb{CP}^1$ but not well defined at $p_1, \dots, p_9$

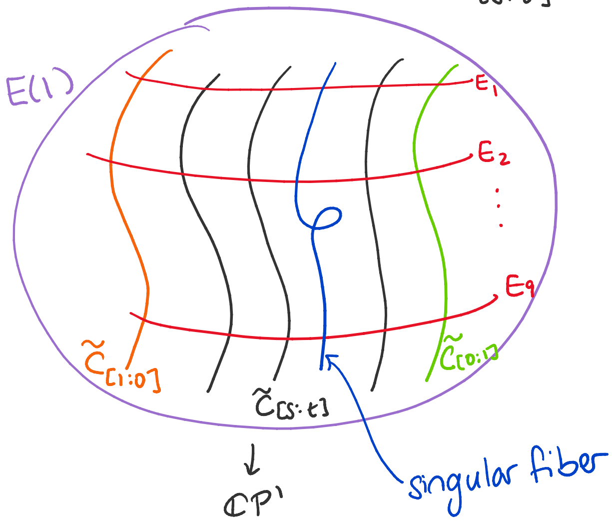
$E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$

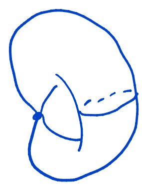
To get from the pencil to an elliptic fibration, blow-up at $p_1, \dots, p_9$

so each $\tilde{C}_{[s:t]}$ gets its own seperate version $p_1^{[s:t]}, \dots, p_9^{[s:t]}$

proper transform

and $\tilde{C}_{[s:t]} \cap \tilde{C}_{[s':t']} = \emptyset$ for $[s:t] \neq [s':t']$





For a generic pencil (generic f, g), all singular fibers will be once pinched tori, and there will be exactly 12 such singular fibers.

Euler char of a nonsingular torus fibration is 0.
Euler Char of $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ is 12.
Each pinched torus contributes 1 to Euler char.

Fiber Sum

Given two elliptic fibrations $\pi_1 : X_1 \to T_1 \ni t_1$ and $\pi_2 : X_2 \to T_2 \ni t_2$

where $F_1 := \pi_1^{-1}(t_1)$, $F_2 := \pi_2^{-1}(t_2)$ are generic fibers

① Cut out a regular neighborhood $N_i \cong T^2 \times D^2$ of F_i in X_i.

② Glue $(X_1 \setminus N_1) \cup_\partial (X_2 \setminus N_2)$ along their boundaries via a

(*) fiber preserving, orientation reversing diffeomorphism of their boundaries.

Result is the fiber sum denoted $X_1 \#_f X_2$

The fiber sum admits an elliptic fibration $\pi : X_1 \#_f X_2 \to T_1 \# T_2$

(*) In general the diffeomorphism type of $X_1 \#_f X_2$ may depend on the choice of gluing map in step ②, but if X_1 or X_2 is $E(1)$, the result is independent of this choice.

or more generally, contains a "nucleus"

$E(n)$:

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We have one elliptic fibration $E(1)$.

We can iteratively fiber sum copies of $E(1)$ to get new elliptic fibrations.

$$E(2) = E(1) \#_f E(1)$$

$$E(n) = E(n-1) \#_f E(1)$$

Basic invariants of $E(n)$: (problem session)

1. $\pi_1(E(n)) = 1$
2. $\chi(E(n)) = 12n$
3. $b_2(E(n)) = 12n-2$, $\quad H_2(E(n)) \cong \mathbb{Z}^{12n-2}$

e.g. $H_2(E(2)) = \mathbb{Z}^{22}$

Surfaces in $E(n)$ generating homology (in order to calculate intersection form)

- Remember: $E(1) \cong \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ $\quad H_2(E(1)) = \mathbb{Z}\langle \underset{\text{class of } \mathbb{C}P^1}{h}, \underbrace{e_1, \dots, e_9}_{\text{classes of exceptional spheres}} \rangle$ $\quad h^2 = 1,\ e_i^2 = -1$

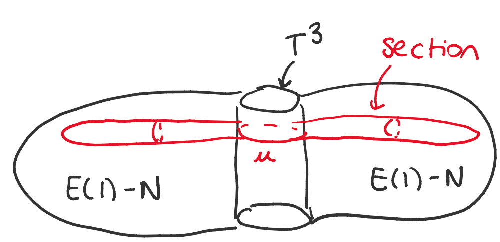
  The generic fiber is the proper transform of a degree 3 curve so its homology class is $f = 3h - e_1 - \dots - e_9$.

$\boxed{A}$ 8 homology classes in each copy of $E(1) - N$:

$\varphi_1 = e_1 - e_2,\ \varphi_2 = e_2 - e_3,\ \dots,\ \varphi_7 = e_7 - e_8,\ \varphi_8 = e_6 + e_7 + e_8 - h$

(they do not intersect the fiber $f = 3h - e_1 - \dots - e_9$)

$\boxed{B}$ The 9^{th} sections (e_9)'s from each copy of $E(1)$ glue together to a section σ of $E(2)$ $\quad \sigma^2 = -n$



$\boxed{C}$ In each T^3 (∂N) where the gluing occurs we have 3 tori:

$f = S^1 \times S^1 \times \{r\}$ $\leftarrow$ the fiber (same in all T^3's if $n > 2$)

$t_1 = \{p\} \times S^1 \times S^1$ $\leftarrow$ one S^1 from fiber $\times$ meridian $\Big\}$ (get one of each for each

3 tori : $f = S^1 \times S^1 \times \{r\}$ ← the fiber (same in all 1°'s if $n > 2$)

$t_1 = \{p\} \times S^1 \times S^1$ ← one S^1 from fiber × meridian

$t_2 = S^1 \times \{q\} \times S^1$ ← other S^1 from fiber × meridian

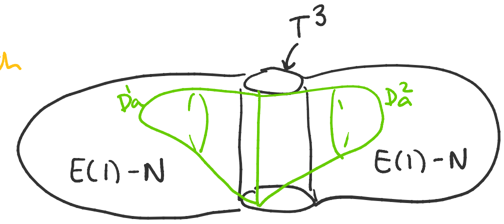
} (get one of each for each fiber sum when $n > 2$)

$\boxed{D}$ The last two surfaces to generate $H_2(E(2))$ are similar to the section σ, except instead of gluing together two disks whose boundary is the meridian ($\mu = \{p\} \times \{q\} \times S^1$), we glue together two disks whose boundary is the other circles in T^3

$S_1 = D_a^1 \underset{S^1 \times \{q\} \times \{r\}}{\cup} D_b^1$

$S_2 = D_a^2 \underset{\{p\} \times S^1 \times \{r\}}{\cup} D_b^2$

} (get one of each for each fiber sum when $n > 2$)



D_a^1, D_b^1 are disks in each of the copies of $E(1) - N$ with boundary on $S^1 \times \{*\} \times \{*\}$ in ∂N. They exist because $E(1) - N$ is simply connected. (Similarly D_a^2, D_b^2)

<u>Lemma:</u> D_a^1, D_a^2 can be chosen to be disjoint from the 8 classes in $\boxed{A}$, disjoint from the section from $\boxed{B}$ and disjoint from each other. (Similarly with D_b^1, D_b^2) $S_1^2 = -2$ and $S_2^2 = -2$

(Problem session: use this basis to calculate the intersection form of $E(2)$)

PAUSE

Seiberg-Witten basic classes

$SW_X : Spin^c(X) \to \mathbb{Z}$

More practically, when X is simply connected

$$c_1 : Spin^c(X) \to H^2(X;\mathbb{Z})$$

is injective (each spin^c structure $\mathfrak{s}$ is uniquely determined by $c_1(\mathfrak{s})$)

and its image in $H^2(X;\mathbb{Z})$ is precisely the set of characteristic elements:

$$Spin^c(X) \cong \{K \in H^2(X;\mathbb{Z}) \mid K(\alpha) \equiv \alpha\cdot\alpha \mod 2 \quad \forall \alpha \in H_2(X;\mathbb{Z})\}$$

From now on, we identify a spin^c structure with the characteristic element in $H^2(X;\mathbb{Z})$

A Seiberg-Witten basic class is a characteristic element K with $SW_X(K) \neq 0$.

Seiberg-Witten polynomial: (equivalent way to encode $SW_X : Spin^c(X) \to \mathbb{Z}$)

$$\mathcal{SW}_X = \sum_{\substack{k \in H^2(X;\mathbb{Z}) \\ \text{characteristic}}} SW_X(k)\, t_k \qquad t_k \text{ is a formal variable}$$

Convention: $t_k^n = t_{nk}$, in particular $t_k^{-1} = t_{-k}$

and $t_0 = 1$

Generalized adjunction inequality:

X smooth 4-mfld, $\Sigma \subset X$ embedded, oriented, connected surface, genus of Σ is g

Hypotheses a: $[\Sigma]^2 \geq 0$, $[\Sigma] \neq 0$,

OR Hypotheses b: X has Seiberg-Witten simple type, $g > 0$

Then for any Seiberg-Witten basic class K (as a char. elt in $H^2(X;\mathbb{Z})$)

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$$2g-2 \geq [\Sigma]^2 + |K([\Sigma])|$$

Basic classes for $E(n)$

Suppose $K \in H^2(E(n);\mathbb{Z})$ is a SW basic class.

Generalized adjunction formula can give restrictions on $|K(x)|$

if we have a surface representing $x \in H_2(E(n);\mathbb{Z})$ of genus g satisfying hypotheses a/b

$\boxed{A}$ $\varphi_1, \dots, \varphi_8$ are represented by spheres ($g=0$), $\varphi_i^2 = -2$

These do not satisfy hypotheses a/b but if we connect sum the spheres

Trick: Can connect sum a sphere with a trivial torus (in homology class 0) to get a torus ($g>0$) representing the same homology class to satisfy hypotheses (b) *

* assuming simple type

Gen adj $\Rightarrow$ $2(1)-2 \geq -2 + |K(x)| \Rightarrow |K(x)| \leq 2$

$\forall$ $x \in H_2(E(n);\mathbb{Z})$ represented by a (-2)-sphere

+ K characteristic $\Rightarrow$ $K(x) \in \{-2,0,2\}$

Lemma (exercise): Suppose $K: \mathbb{Z}\langle\varphi_1,\dots,\varphi_8\rangle \longrightarrow \mathbb{Z}$ is linear and for any class $x \in \mathbb{Z}\langle\varphi_1,\dots,\varphi_8\rangle$ repr by a (-2) sphere $K(x) \in \{-2,0,2\}$. Then K is identically 0 on $\mathbb{Z}\langle\varphi_1,\dots,\varphi_8\rangle$.

Conclusion: Any basic class K on $E(n)$ vanishes on each copy of $\varphi_1,\dots,\varphi_8$.

Next consider $K(t_i)$ and $K(s_i)$:

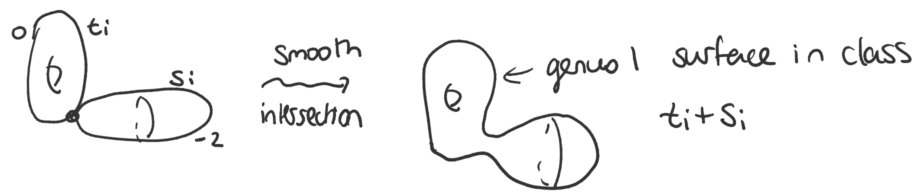
- t_i is represented by a torus ($g=1$) and $t_i^2 = 0$ (hyp a applies)

$$2(1)-2 \geq 0 + |K(t_i)| \Rightarrow |K(t_i)| \leq 0 \Rightarrow K(t_i) = 0$$

$$2(1)-2 \geq 0 + |K(t_i)| \Rightarrow |K(t_i)| \leq 0 \Rightarrow K(t_i) = 0$$

- s_i is repr by a sphere, $s_i^2 = -2$

$s_i \cdot t_i = 1$



$$(t_i + s_i)^2 = t_i^2 + 2t_i \cdot s_i + s_i^2 = 0 + 2(1) + (-2) = 0$$

$$2(1)-2 \geq 0 + |K(t_i+s_i)| \Rightarrow \cdots K(t_i+s_i) = 0$$

<u>Conclusion</u>: A basic class K vanishes on each copy of $\mathbb{Z}\langle t_1, s_1, t_2, s_2 \rangle$.

Similarly $K(f) = 0$.

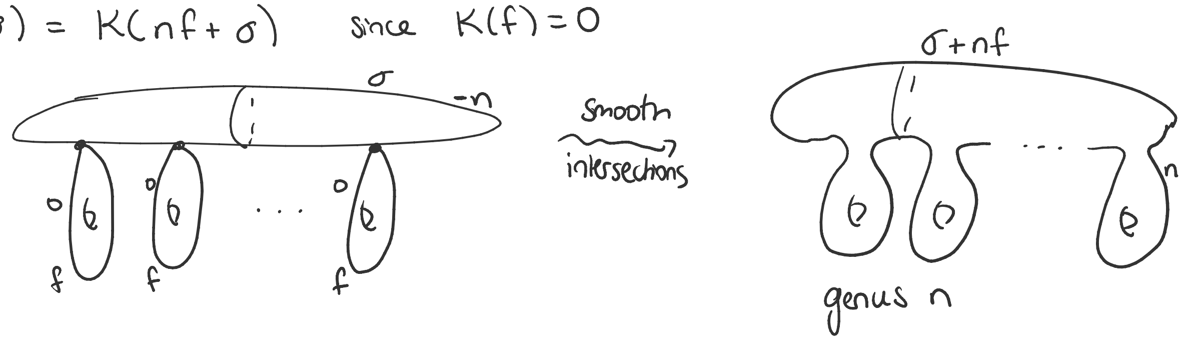
$\Rightarrow$ basic class K has $K(x) = 0$ on all generators except possibly σ.

$\Rightarrow$ $K = PD(kf)$ for some $k \in \mathbb{Z}$ $\quad (K(x) = PD(K) \cdot x)$

$(k = K(\sigma))$

<u>Generalized adjunction with σ:</u>

$K(\sigma) = K(nf + \sigma)$ since $K(f) = 0$



$$(\sigma + nf)^2 = \sigma^2 + 2n\sigma \cdot f + n^2 f^2 = -n + 2n(1) + n^2 \cdot 0 = n$$

<u>Gen adj</u>: $2n - 2 \geq n + |K(nf + \sigma)|$

$\Rightarrow n - 2 \geq |K(nf + \sigma)| = |K(\sigma)| = k$

K characteristic $\Rightarrow K(\sigma) \equiv \sigma^2 = -n \mod 2$

Conclusion: The only possible basic classes on $E(n)$ are

$$\boxed{\{PD(kf) \mid k \equiv n \mod 2 \text{ and } |k| \leq n-2\}}$$

Fintushel-Stern prove all of these really are basic classes ($SW_{E(n)}(kf) \neq 0$ for such k)

Theorem [Fintushel-Stern] The Seiberg-Witten polynomial for $E(n)$ is

$$\boxed{SW_{E(n)} = (t_f - t_f^{-1})^{n-2}.}$$

(Each monomial of the form $a\,t_f^k$ means $SW_{E(n)}(kf) = a$.)

Next: How do Seiberg-Witten invariants change when we change $E(n)$ by particular topological operations?

Blow-up formula:

$$H^2(\underset{\substack{\uparrow \\ \text{N-fold blow-up of } X}}{X \#_N \overline{\mathbb{CP}^2}}) \cong H^2(X) \oplus \mathbb{Z}\langle \underset{\uparrow}{E_1}, \dots, \underset{\uparrow}{E_N} \rangle$$

Poincaré dual to exceptional spheres

If X is simply connected, simple type with basic classes $Bas_X \subset H^2(X;\mathbb{Z})$

$$\boxed{Bas_{X \#_N \overline{\mathbb{CP}^2}} = \{K_i \pm E_1 \pm \dots \pm E_N \mid K_i \in Bas_X\}}$$

Example: $Bas_{E(2)} = \{0\} \implies Bas_{E(2) \# \overline{\mathbb{CP}^2}} = \{\pm E\}$

The values of the Seiberg-Witten invariant on these basic classes:

$$\boxed{SW_{X \#_N \overline{\mathbb{CP}^2}}(K_i \pm E_1 \pm \dots \pm E_N) = SW_X(K_i)}$$

Written as a polynomial: $SW_{X \#_N \overline{\mathbb{CP}^2}} = SW_X\,(t_{E_1} + t_{E_1}^{-1}) \cdots (t_{E_N} + t_{E_N}^{-1})$

Connected Sum Vanishing Theorem:

Suppose $X = X_1 \# X_2$ and $b_2^+(X_i) > 0$ $i = 1, 2$.

Then $SW_X \equiv 0$ (no basic classes).

Corollary: If $X = \#_n \mathbb{CP}^2 \#_m \overline{\mathbb{CP}^2}$ and $n > 1$

$SW_X = 0$.

Example: $\#_3 \mathbb{CP}^2 \#_{20} \overline{\mathbb{CP}^2}$ has no basic classes

$\Rightarrow$ $\#_3 \mathbb{CP}^2 \#_{20} \overline{\mathbb{CP}^2}$ is not diffeomorphic to $E(2) \# \overline{\mathbb{CP}^2}$.

However, you will show they <u>are</u> homeomorphic!
