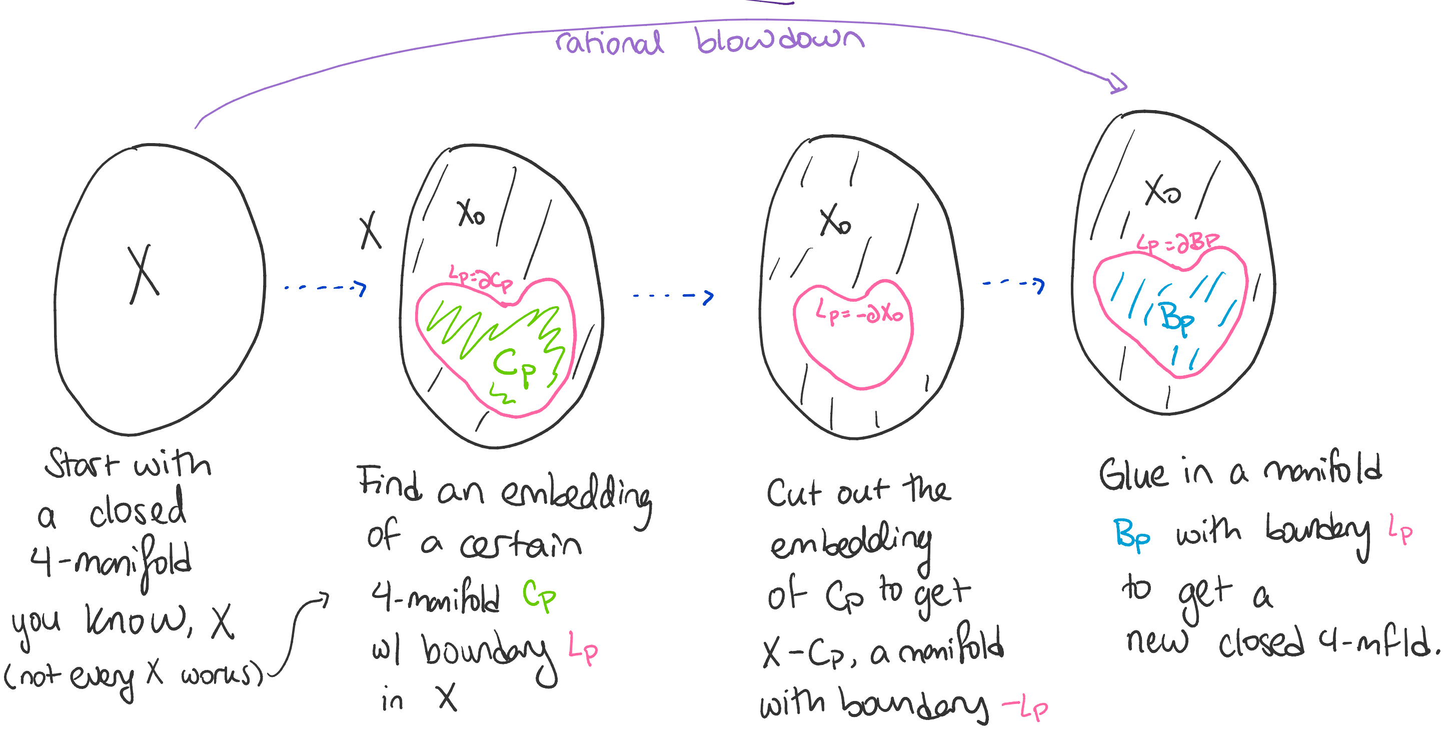
Lecture 3: Rational Blowdown

Thursday, July 1, 2021 11:58 AM

Goals today:

- Define the Fintushel-Stern rational blowdown operations.
- Understand the effect on topological invariants
- Understand the effect on Seiberg-Witten invariants
- Find examples where we can apply the rational blowdown and use it to construct exotic 4-manifolds.

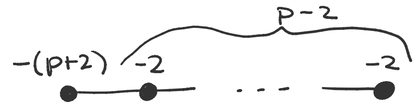
Rational blowdown: big picture



What is C_p? How can we find embeddings of C_p?

For $p \geq 2$,

C_p is a plumbing of spheres with graph:



a 4-dimensional thickening of a collection of 2-spheres which intersect transversally

vertices $\updownarrow$ 2 spheres

numbers $\updownarrow$ self-intersection numbers of 2-spheres

edges $\updownarrow$ transverse intersections

$b_2^-(C_p) = p-1$, $\chi(C_p) = p$,
$H_2(C_p;\mathbb{Z}) \cong \mathbb{Z}^{p-1}$, $H_0(C_p;\mathbb{Z}) \cong \mathbb{Z}$, $H_i(C_p;\mathbb{Z}) = 0$ for $i \neq 0,2$

To find an embedding of C_p in X, find 2-spheres with self-intersections and pairwise intersections matching the data of the graph and take a small regular neighborhood of the union of 2-spheres.

What is B_p? Why is $\partial B_p \cong \partial C_p$?

Desired properties:

A $\partial B_p \cong \partial C_p$

B $\chi(B_p) = 1$,
$b_1(B_p) = b_2(B_p) = b_3(B_p) = 0$ (rational homology ball $H_i(B_p;\mathbb{Q}) \cong H_i(B^4;\mathbb{Q})$)

Various ways to describe B_p:

(1) Embed $\overline{C_p}$ ← (reversed orientation) into $\#_{p-1}\mathbb{C}P^2$:

$\overline{C_p}$:

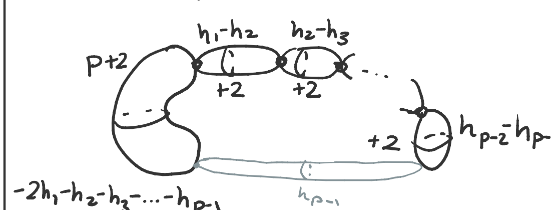
$p=2$: $\mathbb{C}P^2$

$p=3$: $\mathbb{C}P^2 \# \mathbb{C}P^2$

$p=4$: $\#_3 \mathbb{C}P^2$

...

p: $\#_{p-1} \mathbb{C}P^2$



$B_p := \left(\#_{p-1}\mathbb{C}P^2\right) \setminus \overline{C_p}$ $\Rightarrow$ A $\partial B_p \cong -\partial\left(\left(\#_{p-1}\mathbb{C}P^2\right) \setminus \overline{C_p}\right) \cong -\partial \overline{C_p} \cong \partial C_p$

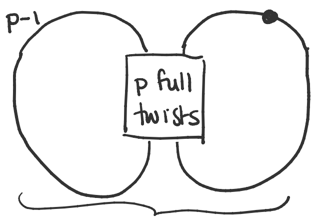
$B_p := \left(\#_{p-1}\mathbb{C}P^2\right) \setminus \bar{C}_p$ $\Rightarrow$ $\boxed{A}$ $\partial B_p \cong -\partial\left(\left(\#_{p-1}\mathbb{C}P^2\right)\setminus \bar{C}_p\right) \cong -\partial\bar{C}_p \cong \partial C_p$

$\boxed{B}$ $H_i(B_p;\mathbb{Q}) \cong H_i(B^4;\mathbb{Q})$ $\leftarrow$ Mayer-Vietoris + $\partial B_p \cong \partial C_p \cong L(p^2, p-1)$

lens space

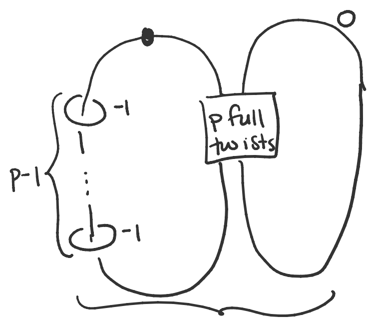
Other ways to define B_p (if you don't know what this means, stick with the 1st way)

Kirby Handle Diagram



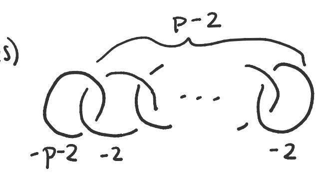
B_p

same boundary as :



C_p

(Kirby calculus) $\cong$



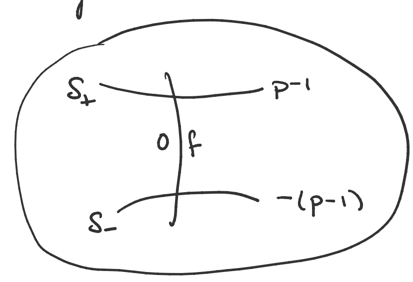
C_p

One 1-handle, one 2-handle which passes through the 1-handle p times
$\rightsquigarrow$ rational homology ball

In Hirzebruch Surface

either diffeomorphic to $S^2\times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

Take H_{p-1}, the S^2 bundle over S^2 with a section S_- of self-intersection $-(p-1)$ and a section S_+ of self-intersection $p-1$.

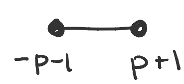


H_{p-1}

Let f be a fiber (self-intersection 0).

Smooth the intersection between S_+ and f to merge them into a single S^2 of self-intersection $p+1$.

This yields an embedding of the plumbing: D_p :



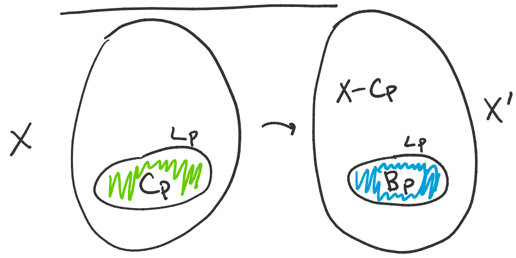
The boundary of this plumbing is $\overline{L(p^2, p-1)}$

$B_p := H_{p-1} \setminus D_p$. This satisfies properties $\boxed{A}$ + $\boxed{B}$ as in the first defn.

Effect of rational blow-down on algebraic topology

- **Effect on π_1:**

Suppose $\pi_1(X) = 1$. We want to know when $\pi_1(X') = 1$.

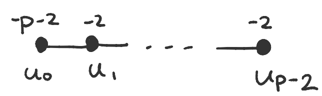


$\pi_1(C_p) = 1$ but $\pi_1(B_p) \cong \mathbb{Z}_p$

$\pi_1(L_p) \cong \mathbb{Z}_{p^2}$

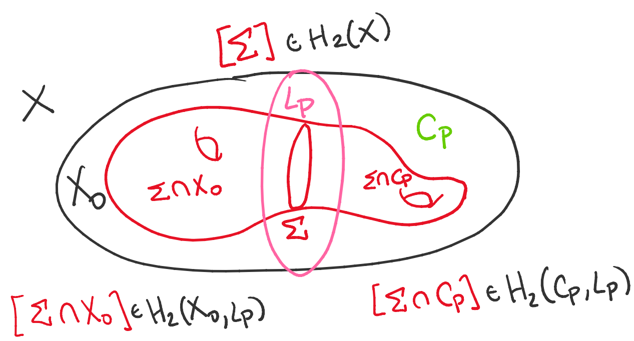
and $i_* : \pi_1(L_p) \to \pi_1(B_p)$ is surjective.

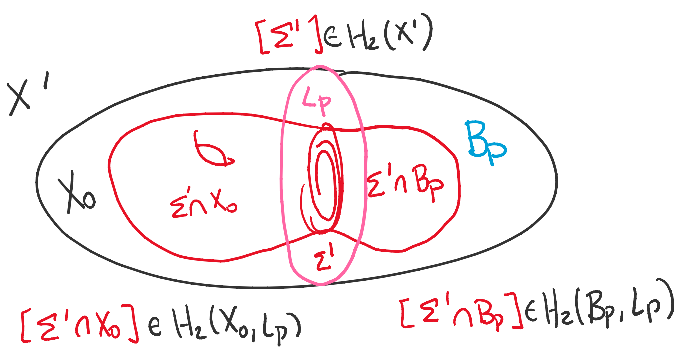
$\boxed{\text{If } \pi_1(X - C_p) = 1 \text{ then } \pi_1(X') = 1}$ ← any generator of $\pi_1(B_p)$ is represented by a loop in $L_p \subset X - C_p$ so it is homotopically trivial in $X - C_p \subset X'$



$\pi_1(X - C_p)$ is generated by a meridian of u_0 ← show this meridian is trivial in $X - C_p$.

- **Effect on H_2:** Reduces b_2^- by $p-1$





$H_2(X)$ has more generators than $H_2(X')$:

1. $H_2(C_p, L_p)$ has more generators than $H_2(B_p, L_p)$
2. Every class in $H_2(X_0, L_p)$ can be completed by a class in $H_2(C_p, L_p)$ to get a closed class in $H_2(X)$. This is not true in X'.

Every class $A' \in H_2(X')$ has a (nonunique) "lift" $A \in H_2(X)$

A' and A agree on X_0 $\left(PD(A')|_{X_0} = PD(A)|_{X_0}\right)$

but not every $B \in H_2(X)$ has a matching $B' \in H_2(X')$

Formally, we have exact sequences of pairs:

$$0 \to H_2(C_p) \to H_2(C_p, L_p) \to H_1(L_p) \to H_1(C_p) \to \cdots$$

This is surjective (every class in $H_1(L_p)$ bounds a surface in C_p), but not injective (there are many choices of surface in

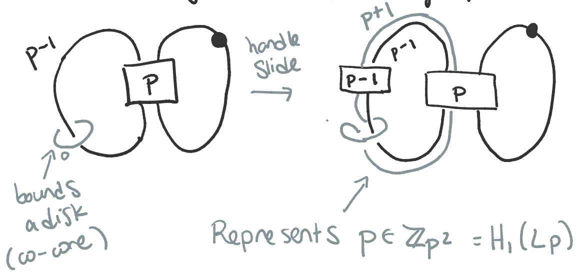
$$\begin{array}{ccccccccc} 0 \to & H_2(C_p) & \to & H_2(C_p, L_p) & \xrightarrow{\partial} & H_1(L_p) & \to & H_1(C_p) & \to \cdots \\ & \wr\| & & \wr\| & & \wr\| & & \| & \\ & \mathbb{Z}^{p-1} & & \mathbb{Z}^{p-1} & & \mathbb{Z}_{p^2} & & 0 & \end{array}$$

but not injective (there are many choices of surface in C_p with the same boundary).

$$\begin{array}{ccccccccc} 0 \to & H_2(B_p) & \to & H_2(B_p, L_p) & \xrightarrow{\cdot p} & H_1(L_p) & \xrightarrow{\text{mod } p} & H_1(B_p) & \to \cdots \\ & \| & & \wr\| & & \wr\| & & \wr\| & \\ & 0 & & \mathbb{Z}_p & & \mathbb{Z}_{p^2} & & \mathbb{Z}_p & \end{array}$$

This is not surjective (the image is $\{mp\} \subset \mathbb{Z}_{p^2}$) so only classes in $H_1(L_p)$ which are multiples of p extend to bound a surface in B_p.
Injectivity $\Rightarrow$ the relative homology class of the extending surface is unique.

Kirby calculus picture of B_p:



Only $mp \in \mathbb{Z}_{p^2}$ for some $m \in \mathbb{Z}$ bounds in B_p.

$p \in H_1(L_p) \cong \mathbb{Z}_{p^2}$ bounds a disk in B_p of relative self-intersection $p+1$.

$mp \in H_1(L_p)$ bounds a disk in B_p of relative self-intersection $m(p+1)$.

Effect on Seiberg-Witten invariants

$$X = X_0 \cup C_p \qquad X' = X_0 \cup B_p$$

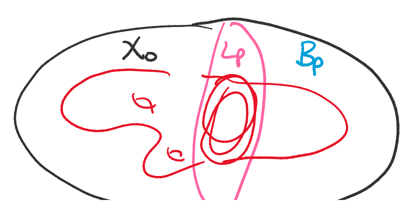
Theorem [Fintushel-Stern] Suppose $k' \in H^2(X';\mathbb{Z})$ is a characteristic class on X' and $k \in H^2(X;\mathbb{Z})$ is a characteristic lift on X $(k|_{X_0} = k'|_{X_0})$.

then $$\underline{SW_{X'}(k') = SW_X(k).}$$

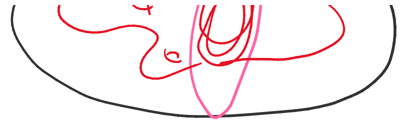
Characteristic classes on X': (identifying $H^2(X') \cong H_2(X')$ with Poincaré duality)

$PD(k') = K' \in H_2(X')$ characteristic $\iff$ $K' \cdot A' \equiv A' \cdot A' \mod 2$

Recall that $k'|_{L_p} = mp \in \mathbb{Z}_{p^2} \cong H^2(L_p)$



Claim: k' characteristic $\iff$ $k'|_{L_p} = mp \in \mathbb{Z}_{p^2}$



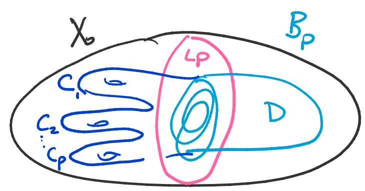
Claim: k' characteristic $\iff$ $k'|_{L_p} = mp \in \mathbb{Z}_{p^2}$ for some odd value of $m \in \mathbb{Z}$

Consequence: Characteristic classes $k \in H^2(X)$ with $k|_{L_p} = mp \in \mathbb{Z}_{p^2}$ for some odd $m \in \mathbb{Z}$ are lifts of characteristic classes on X'.

Proof ("$\Rightarrow$"): If p is odd then if $k'|_{L_p} = mp$ for m even then $mp \equiv (m-p)p \mod p^2$ and $m-p$ is odd.

So the main claim is for p even:

Consider a homology class $A' \in H_2(X')$ represented by



$$A' = [C_1 \sqcup \cdots \sqcup C_p \cup D]$$

p copies of a surface in X_0 which bounds the generator of $\mathbb{Z}_{p^2}$.

disk in B_p with boundary in class p times the generator of $\mathbb{Z}_{p^2}$.

$$A' \cdot A' = p(C_i \cdot C_i) + D \cdot D = pn + p + 1.$$

If $k'|_{L_p} = mp$ then $K' = PD(k')$ can be represented by

$$K' = [F \cup \sqcup_m D] \quad \text{for some } F \subseteq X_0$$

$$K' \cdot A' = p\, C_i \cdot F + m\, D \cdot D = p\ell + m(p+1)$$

k' characteristic $\Rightarrow$ $pn + p + 1 \equiv p\ell + m(p+1) \mod 2$

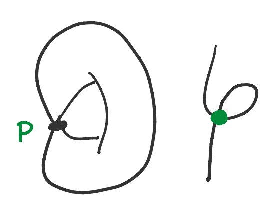
when p is even this reduces to: $1 \equiv m \mod 2$

i.e. m is odd ∎

Finding plumbings C_p to rationally blow down:

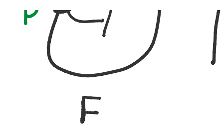
Example: Consider an elliptic surface E_n.

Recall that some of its fibers are singular:



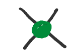
Because it is a fiber, this (immersed) surface

Recall that some of the pieces of a fiber

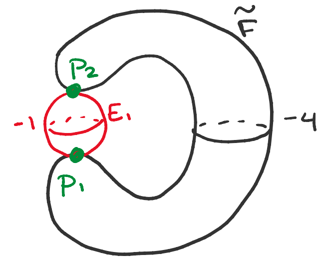


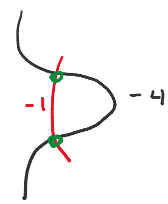
Because it is a fiber, this (immersed) surface represents a homology class with $[F]\cdot[F]=0$.

At p, the singular point: two smooth pieces of surface intersect transversally at a point



Blow up at p.



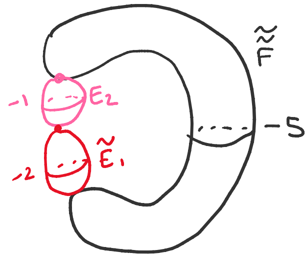


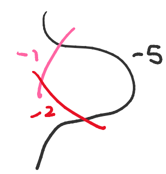
The two pieces of F that intersected transversally at p get separated.
A new exceptional sphere E_1 appears intersecting the proper transform $\tilde{F}$ at two points P_1, P_2

$$\tilde{F} = F - 2E_1 \implies \tilde{F}\cdot\tilde{F} = 0 + 4(-1) = -4$$

This gives an embedding of C_2 in $E(n)\#\overline{\mathbb{CP}^2}$ as a neighborhood of $\tilde{F}$.

Blow up at P_2





$$\tilde{\tilde{F}} = F - 2E_1 - E_2$$

$$\tilde{\tilde{F}}^2 = 0 + 4(-1) + (-1) = -5$$

$$\tilde{E}_1 = E_1 - E_2$$

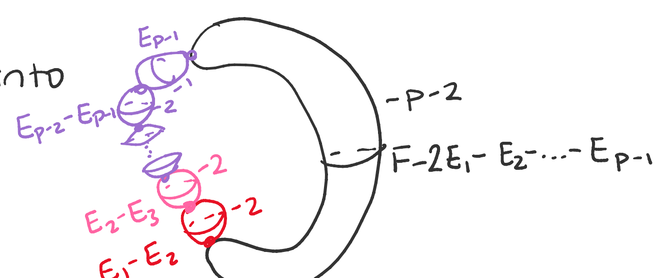
$$\tilde{E}_1^{\,2} = -1 + (-1) = -2$$

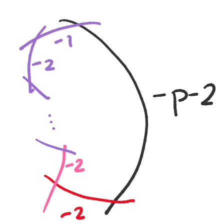
$\rightsquigarrow$ Get an embedding of C_3 in $E(n)\#_2\overline{\mathbb{CP}^2}$ as a nbhd of $\tilde{\tilde{F}}\cup\tilde{E}_1$.

$\vdots$

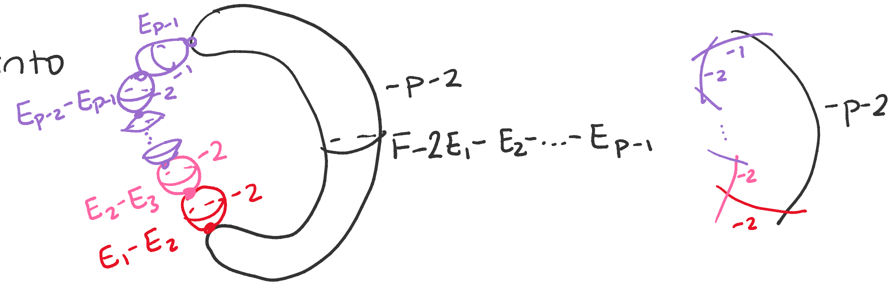
Continue repeatedly blowing up at the intersection of the proper transform of F with the newest exceptional sphere:

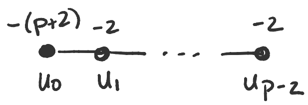
Get an embedding of C_p into $E(n)\#(p-1)\overline{\mathbb{CP}^2}$





Get an embedding of C_p into $E(n) \# (p-1)\overline{\mathbb{CP}^2}$





$u_0 = F - 2E_1 - E_2 - \dots - E_{p-1}, \quad u_i = E_i - E_{i+1} \quad i = 1, \dots, p-2$

Given this embedding, we can rationally blow down, replacing C_p by B_p.

$X = E(n) \# (p-1)\overline{\mathbb{CP}^2}$ $\qquad X_0 = X \setminus C_p$

$X' = X_0 \cup B_p$ (the rational blowdown)

$$SW_{E(n)\#(p-1)\overline{\mathbb{CP}^2}} = (t_f - t_f^{-1})^{n-2}(t_{E_1} + t_{E_1}^{-1}) \cdots (t_{E_{p-1}} + t_{E_{p-1}}^{-1})$$

Lets consider $n=2$: $SW_{E(2)\#(p-1)\overline{\mathbb{CP}^2}} = (t_{E_1} + t_{E_1}^{-1}) \cdots (t_{E_{p-1}} + t_{E_{p-1}}^{-1})$

i.e. basic classes are $\pm E_1 \pm E_2 \cdots \pm E_{p-1}$ & $SW(\pm E_1 \cdots \pm E_{p-1}) = 1$

1. Does each basic class $\pm E_1 \pm E_2 \pm \dots \pm E_{p-1}$ on X desend to a characteristic class on X'?

i.e. is $(\pm E_1 \pm E_2 \pm \dots \pm E_{p-1})\big|_{L_p} = mp \in \mathbb{Z}_{p^2} = H_1(L_p)$ for some odd m?

<u>Exercise</u>: $E_i\big|_{L_p} = -p \in \mathbb{Z}_{p^2} = H_1(L_p)$ for $i = 1, \dots, p-1$

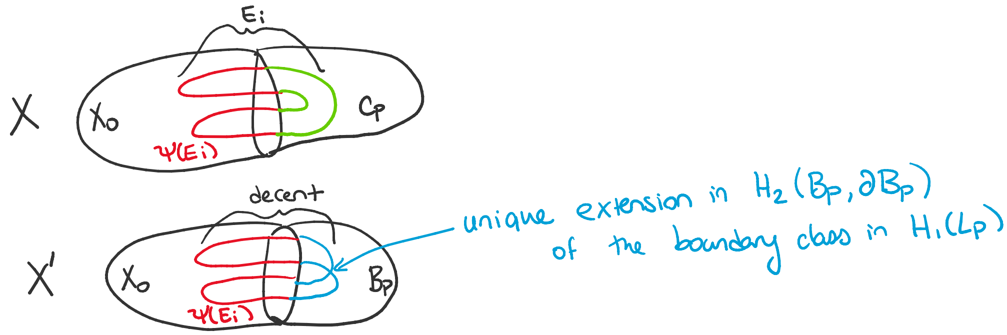
<u>Consequence</u>: $(\pm E_1 \pm E_2 \pm \dots \pm E_{p-1})\big|_{L_p} = mp \in \mathbb{Z}_{p^2}$ for some odd m ✓

2. What sort of class in $H_2(X')$ does $\pm E_1 \pm E_2 \pm \dots \pm E_{p-1}$ descend to?

As above, each E_i descends to some class in $H_2(X')$

As above, each E_i descends to some class in $H_2(X')$

Claim: All E_i descend to the same class in $H_2(X')$:



Exact seq of pair (X, C_p):

$$\begin{array}{ccccccc}
 & & H_2(E(2)) \oplus \mathbb{Z}\langle E_1, \dots, E_{p-1}\rangle & & & & \\
 & & \Vert & & & & \\
H_2(C_p) & \xrightarrow{\Phi} & H_2(X) & \twoheadrightarrow & H_2(X, C_p) & \longrightarrow & H_1(C_p) = 0 \\
\Vert & & & \searrow^{\Psi} & \wr\Vert \text{ excision} & & \\
\mathbb{Z}\langle u_0, u_1, \dots, u_{p-2}\rangle & & & & H_2(X_0, \partial X_0) & &
\end{array}$$

$$\Psi(H_2(X)) \cong H_2(X)/\mathrm{im}\,\Phi \cong H_2(E(2)) \oplus \mathbb{Z}\langle E_1, \dots, E_{p-1}\rangle \Big/ \left\{\begin{array}{l} F - 2E_1 - E_2 - \dots - E_{p-1} = 0 \\ E_1 - E_2 = 0 \\ \vdots \\ E_{p-2} - E_{p-1} = 0 \end{array}\right\}$$

In here $\Psi(E_1) = \dots = \Psi(E_{p-1}) =: E$

and $\Psi(F) = pE$

E has a unique extension by a class in $H_2(B_p, L_p)$ to give a well defined class $E' \in H_2(X')$ such that each E_i is a lift of E' to $H_2(X)$.

Conclude: $1 = SW_X(\pm E_1 \pm \dots \pm E_{p-1}) = SW_{X'}(\underbrace{\pm E \pm \dots \pm E}_{p-1})$

So the basic classes of X' are $\{nE \mid -(p-1) \le n \le p-1,\ n \equiv p-1 \bmod 2\}$

and for each such n, $SW_{X'}(nE) = 1$.

Where E is a class in $H_2(X')$ with $pE = F$.

In fact X' is homeomorphic to $E(2)$, but its SW invariants are

In fact X' is homeomorphic to $E(2)$, but its SW invariants are different so X' is an exotic copy of $E(2)$.

(Different choices of p give an infinite family of exotic $E(2)$'s).

Through Kirby calculus, one can also see that these $p-1$ blowups followed by this rational blowdown are equivalent (diffeomorphic) to performing a <u>log transform</u> on a regular fiber in $E(2)$.