Lecture 4: Knot surgery

Thursday, July 8, 2021 11:32 AM

Goals for today:

- Define a new constructive operation: Knot surgery
- Understand the effect of Knot surgery on the homeomorphism type of the manifold and on the Seiberg-Witten invariants
- Learn how to calculate the Seiberg-Witten invariants in examples of Knot surgered manifolds.

Knot surgery:

Set-up: Let K be a knot in S^3.

Let $M_K = S^3 - \nu(K)$ denote the complement of a regular neighborhood of K in S^3.

Note: ∂M_K is a 2-torus.

Let X be a smooth 4-manifold containing an embedded 2-torus T of self-intersection 0. (thus a neighborhood of T in X is diffeomorphic to $T^2 \times D^2$)

Knot surgery of X along T is:

$$X_K := (X \setminus (T^2 \times D^2)) \underset{\partial}{\cup} (S^1 \times M_K)$$

Note: each piece has boundary diffeomorphic to T^3!

where the gluing identifies $\{p\} \times \partial D^2$ with $\{0\} \times$ longitude of K.

Remark: This does not uniquely determine the gluing map $T^3 \to T^3$ but [illegible] the same for any two such choices.

<u>Remark</u>: This does not uniquely determine the gluing map $T^2 \longrightarrow T^2$ but for many X the Seiberg-Witten invariants are the same for any two such choices.

Alternate viewpoint :

For a knot $K \subset S^3$, 0-surgery of S^3 along K is the closed 3-manifold

$$S^3_0(K) = \left(S^3 - \nu(K)\right) \cup_{\partial} (S^1 \times D^2)$$

where $\{p\} \times \partial D^2$ is glued to a longitude of K.

Then $S^1 \times S^3_0(K)$ contains an embedded torus $\tilde{T} = S^1 \times C$ where C is $S^1 \times \{0\}$ in $S^1 \times D^2$.

$\tilde{T}$ has self-intersection 0 (since there is a disjoint parallel copy of C nearby in $S^1 \times D^2$)

Then given $T \subset X^4$ as above

X_K is the fiber sum of (X, T) with $(S^1 \times S^3_0(K), \tilde{T})$

<u>Remark</u>: If K is a fibered knot, $S^3_0(K)$ is a surface bundle over S^1 and $S^1 \times S^3_0(K)$ is a surface bundle over T^2. This allows one to define a symplectic structure on $S^1 \times S^3_0(K)$ so that if (X, T) is symplectic, X_K also has a symplectic structure.

Effect of Knot surgery on homeomorphism type

- <u>π_1</u>: Assume that $\pi_1(X) = \pi_1(X \setminus T) = 1$.

  $\pi_1(S^3_0(K))$ is normally generated by a meridian of K.
  The meridian of K is isotopic to C, the core of the 0-surgery.

  $\Rightarrow$ The image of $\pi_1(\tilde{T})$ in $\pi_1(S^1 \times S^3_0(K))$ normally generates $\pi_1(S^1 \times S^3_0(K))$

$$\begin{tikzcd}
 & \pi_1(S^1 \times M_K) \arrow[dr] & \\
\pi_1(T^2) \arrow[ur] \arrow[dr] & & \pi_1(X_K) \\
 & \pi_1(X_0) \arrow[ur] &
\end{tikzcd}$$

Seifert van Kampen $\Rightarrow$ $\pi_1(X_K) = 1$.

Seifert van Kampen $\Rightarrow$ $\pi_1(X_K) = 1$.

$$\begin{tikzcd}[row sep=small] & \cdots \arrow[dr] & \\ \pi_1(T^2) \arrow[ur] \arrow[dr] & & \pi_1(X_K) \\ & \pi_1(X_0) \arrow[ur] & \end{tikzcd}$$

- <u>Homology</u>:

$$H_i(S_0^3(K)) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ \mathbb{Z} & i=3 \end{cases}$$

i.e. $H_i(S_0^3(K)) \cong H_i(S^1 \times S^2)$.

<u>Note</u>: If K is the unknot $S_0^3(K) \cong S^1 \times S^2$.

i.e. the homology of $S_0^3(K)$ does not depend on K.

Correspondingly, the homology of X_K as the fiber sum of X with $S^1 \times S_0^3(K)$ is always the same independent of K (the exact sequences to do the calculation of $H_i(X_K)$ are the same).

If K is the unknot, $X_K \cong X$ (exercise).

<u>Conclude</u>: The homology of X_K is the same as the homology of X and the intersection pairings are the same

By Freedman's theorem, (assuming $\pi_1(X) = \pi_1(X \setminus T) = 1$) X is homeomorphic to X_K.

Effect on Seiberg-Witten invariants

<u>Theorem</u> [Fintushel-Stern] Suppose $b_2^+(X) > 1$, $\pi_1(X) = \pi_1(X \setminus T) = 1$, $[T] \neq 0 \in H_2(X;\mathbb{Z})$, and $[T]^2 = 0$. Then with X_K defined as above, the Seiberg-Witten polynomial of X_K is:

$$SW_{X_K} = SW_X \cdot \Delta_K(t_T^2)$$

where Δ_K is the symmetrized Alexander polynomial of K.

Alexander polynomial of a knot is a classical knot invariant.

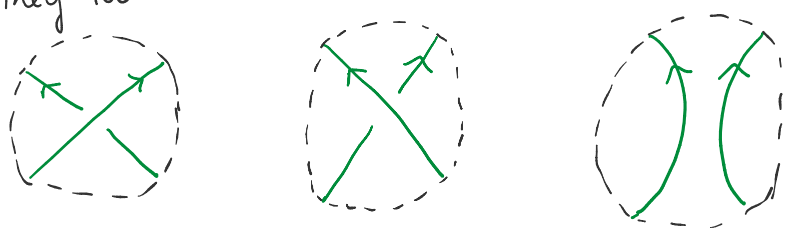
Alexander polynomial of a knot is a classical knot invariant.

There are intrinsic ways of defining it through invariants of the infinite cyclic cover of the knot complement but the easiest definition for computational purposes comes from the Skein relations.

Alexander polynomial via Skein relations

Rule 1: If u is the unknot $\boxed{\Delta_u(t) = 1.}$

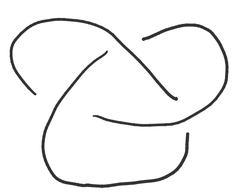
Rule 2: Suppose we have three diagrams of oriented knots/links which agree outside of a specified disk, and inside of the disk they look like this:

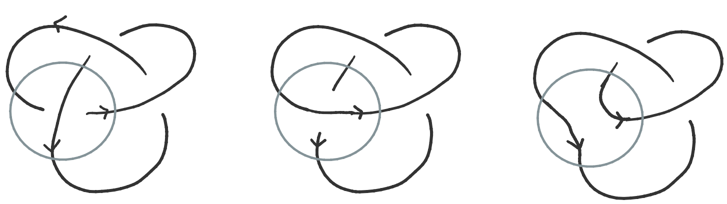


L_+ $\quad$ L_- $\quad$ L_0

$$\boxed{\Delta_{L_+}(t) = \Delta_{L_-}(t) + (t^{1/2} - t^{-1/2})\,\Delta_{L_0}(t)}$$

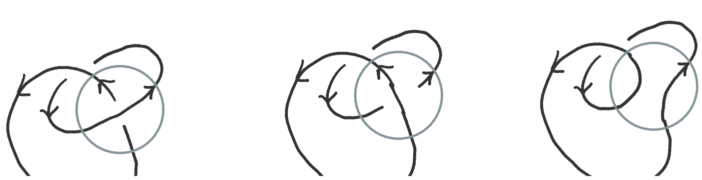
Example:

$K =$ 

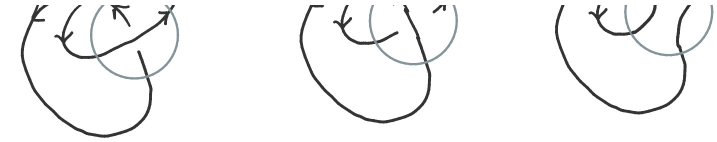


$K = L_+$ $\quad$ L_- $\quad$ $L_0 = H$

$$\Delta_K(t) = \underbrace{\Delta_{L_-}(t)}_{1} + (t^{1/2} - t^{-1/2})\,\Delta_{L_0}(t)$$



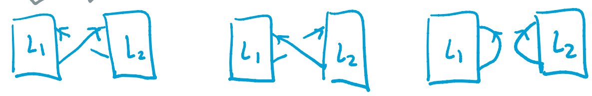
$$\Delta_H(t) = \underbrace{\Delta_{L_-}(t)}_{0} + (t^{1/2} - t^{-1/2})\,\underbrace{\Delta_{L_0}(t)}_{1}$$



$H = L_+$ $\qquad$ L_- $\qquad$ L_0

$$\Delta_K(t) = 1 + (t^{1/2} - t^{-1/2})\Delta_H(t)$$
$$= 1 + (t^{1/2} - t^{-1/2})^2$$
$$= 1 + t - 2 + t^{-1}$$
$$= t - 1 + t^{-1}$$

<u>Lemma</u>: Any split link L has $\Delta_L = 0$.

Pf: 

$L_+ = L_1 \# L_2$ $\qquad$ $L_- = L_1 \# L_2$ $\qquad$ $L_0 = L_1 \sqcup L_2$

$$\Delta_{L_+} = \Delta_{L_-} + (t^{1/2} - t^{-1/2})\Delta_{L_0}$$

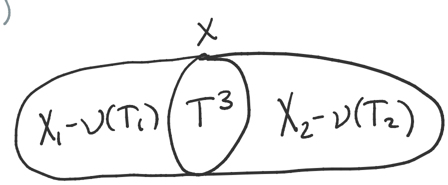
$L_+ = L_- \Rightarrow \quad 0 = (t^{1/2} - t^{-1/2})\Delta_{L_0}$ ☒

A glance behind the scenes for the Seiberg-Witten formulas we've used

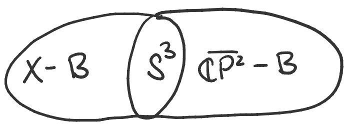
<u>Key ingredient</u>: Neck stretching along a seperating 3-manifold.

* study SW moduli spaces for each side
* study SW moduli space for 3-mfld $\times \mathbb{R}$ ← need this to be relatively simple

(Recall you saw this strategy for the adjunction inequality for Σ with $\Sigma^2 = 0$, stretch along $\Sigma \times S^1$.)

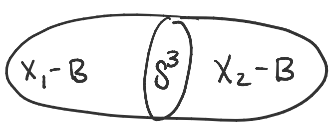
- Fiber sums 

<u>Thm</u> [Taubes]* $\boxed{SW_X = SW_{X_1} \cdot SW_{X_2}}$ ← need a homology condition about $T_1 \subset X_1$, $T_2 \subset X_2$ (can't be null homologous)

- Blow-ups 

$\overline{\mathbb{CP}}^2$ has reducible solutions which contribute

$$\boxed{SW_{X \# \overline{\mathbb{CP}}^2} = SW_X \cdot (t_E + t_E^{-1})}$$

- Connected sums 

$$\boxed{SW_X = 0}$$

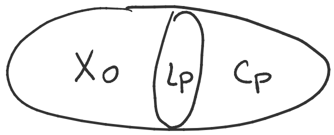
Neck stretching $\rightsquigarrow$ $\mathcal{M}_{X,K} \cong \mathcal{M}_{X_1,K_1} \times \mathcal{M}_{X_2,K_2}$

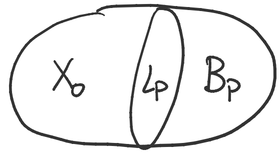
but dim formula says

$0 = \dim \mathcal{M}_{X,K} = \dim \mathcal{M}_{X_1,K_1} + \dim \mathcal{M}_{X_2,K_2} + 1$

$\Rightarrow$ one of $\mathcal{M}_{X_1,K_1}$ or $\mathcal{M}_{X_2,K_2}$ is neg dim $\Rightarrow \emptyset$

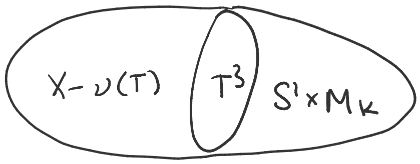
$$\frac{K^2 - 3\sigma(X) - 2\chi(X)}{4} = \frac{K_1^2 + K_2^2 - 3\sigma(X_1) - 3\sigma(X_2) - 2(\chi(X_1) - 1) - 2(\chi(X_2) - 1)}{4}$$

- Rational blowdown 

vs. 

$$\boxed{SW_{X_0 \cup C_p}(\tilde{K}) = SW_{X_0 \cup B_p}(k')}$$

L_p is simple enough
C_p + B_p do not really contribute (negative definite)

- Knot surgery 

Gluing along T^3
but understanding
$SW_{S^1 \times S^3_0(K)}$ is tricky.

Key ideas behind knot surgery formula

Skein relation on SW_{X_K}

$$\frac{SW_{X_{L_+}}}{SW_X} = \frac{SW_{X_{L_-}}}{SW_X} + (t_T + t_T^{-1})\frac{SW_{X_{L_0}}}{SW_X}$$

$$\frac{SW_{X_u}}{SW_X} = 1$$

Since the Skein relation + value on the unknot uniquely determines the Alexander polynomial,

$$\frac{SW_{X_L}}{SW_X} = \Delta_L(t_T^2)$$

Caveat: In the Skein relation if L_+ is a knot, L_- is a knot,
but L_0 is a 2-component link.

(Similarly if L_+ is a 2-cpt link, L_- is 2-cpt and L_0 is a knot)

If L is a 2-component link, what does X_L mean?

Let $Y_1 := S^3 - \nu(L_1) - \nu(L_2)/_\sim$ where $\sim$ glues $\partial\nu(L_1)$ to $\partial\nu(L_2)$

If L is a 2 component link, what does X_L mean?

Let $Y_L := S^3 - \nu(L_1) - \nu(L_2)/\sim$ where $\sim$ glues $\partial\nu(L_1)$ to $\partial\nu(L_2)$ along $\begin{pmatrix}-1 & 0\\ 2\ell & 1\end{pmatrix}$ where $\ell = lk(L_1, L_2)$

Then $X_L := (X - \nu(T)) \cup_{T^3} (S^1 \times Y_L - \nu(T_L))$ where $T_L = S^1 \times \underset{\substack{\uparrow \\ \text{meridian of } L_1}}{\mu}$

Where does the SW skein relation come from?

Surgery formula on a torus T of self-int 0 (no other homological requirement)

$T \subset X$ with nbhd $T \times D^2$. Fix basis $\{\alpha, \beta, \partial D^2\}$ for $H_1(\partial(T \times D^2))$

$X_{p,q,r} := (X \setminus T \times D^2) \cup_\varphi (T \times D^2)$ where $\varphi_*[p \times \partial D^2] = p\alpha + q\beta + r[\partial D^2]$

Theorem [Morgan-Mrowka-Szabó] For $K \in H_2(X)$,

$$\sum_i SW_{X_{p,q,r}}(K + i[T]) =$$

$$p \sum_i SW_{X_{1,0,0}}(K + i[T]) + q \sum_i SW_{X_{0,1,0}}(K + i[T]) + r \sum_i SW_{\underset{=X}{X_{0,0,1}}}(K + i[T])$$

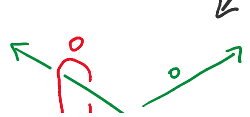
How do we apply this for a Skein relation in knot surgery?

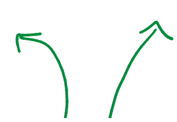
L_+ L_-

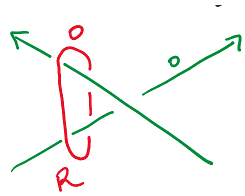
L_+ and L_- become the same after a $+1$ surgery on R

$\rightsquigarrow X_{K_+} = (X_{K_-})_{0,1,1}$ ↙ along the torus $S^1 \times R$

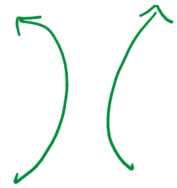
[MMS] Write SW of ↗ in terms of $\underbrace{(X_{K_-})_{(0,1,0)}}$ + $(X_{K_-})_{(0,0,1)} = X_{K_-}$







L_-



L_0

Fiber sum of X with $S^1 \times$ (this 0-surgery on R, L_-) $\xleftrightarrow{\text{[Hoste]}}$ X_{L_0}

See Fintushel - Stern "Knots, links, and 4-manifolds"

or "Six lectures on 4-manifolds" Lecture 3

for more details.