Lecture 5: Generalizations

Friday, July 9, 2021 12:58 PM

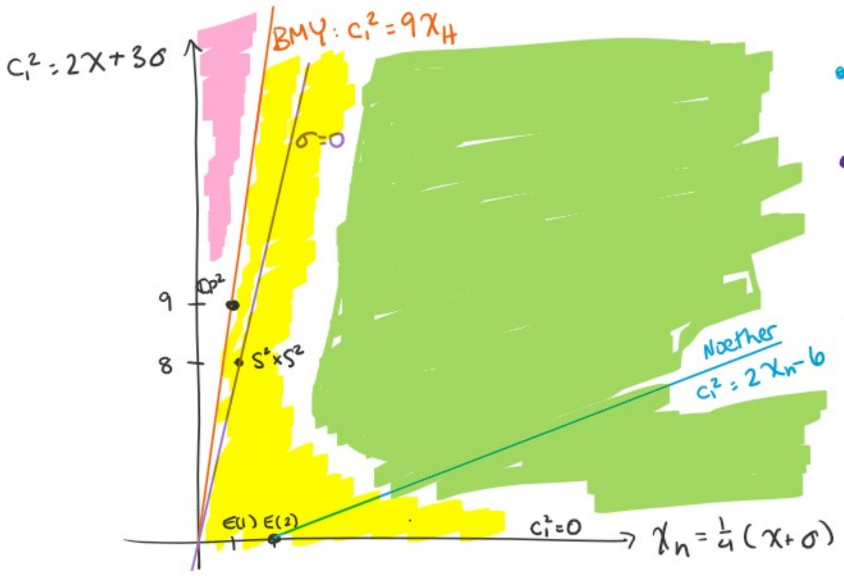
Goals for today:

- Big picture of what kinds of 4-manifolds have been found and what has been harder to find.
- Generalizations of rational blowdown
- Benefits of combining knot surgery with rational blowdown
- Detecting exotic constructions in the $b_2^+ = 1$ setting.

Geography: Two invariants from homeomorphism type:

$$c_1^2 = 2\underset{\substack{\uparrow\\ \text{euler characteristic}}}{\chi} + 3\underset{\substack{\nwarrow\\ \text{signature}}}{\sigma} \qquad \text{and} \qquad \chi_h = \frac{1}{4}(\chi + \sigma)$$

Question: Which values of (c_1^2, χ_h) are realized by smooth, simply connected, minimal, 4-manifolds with well-defined, nonvanishing Seiberg-Witten invariants?



- Complex surfaces of general type are between Noether and BMY
- There are constructions of (symplectic) manifolds for most values between Noether and $c_1^2 = 0$.
- There are some empty spots on/near the BMY line and near the origin (small 4-mflds)
- It is unknown whether there are such (symplectic) 4-manifolds above the BMY line

are such (symplectic) 4-manifolds above the BMY line

Botany Question: For a fixed (c_1^2, χ_h), how many nondiffeomorphic (smooth, simply-connected, minimal, 4-manifolds with well defined nonvanishing SW) are there?

For many values of (c_1^2, χ_h) there are known to be infinitely many.

It is unknown whether there exists any exotic $S^2 \times S^2$, $\mathbb{CP}^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

Fiber sum, rational blow-down, and knot surgery produced many new examples of constructions of interesting 4-manifolds, but to find the trickier examples:

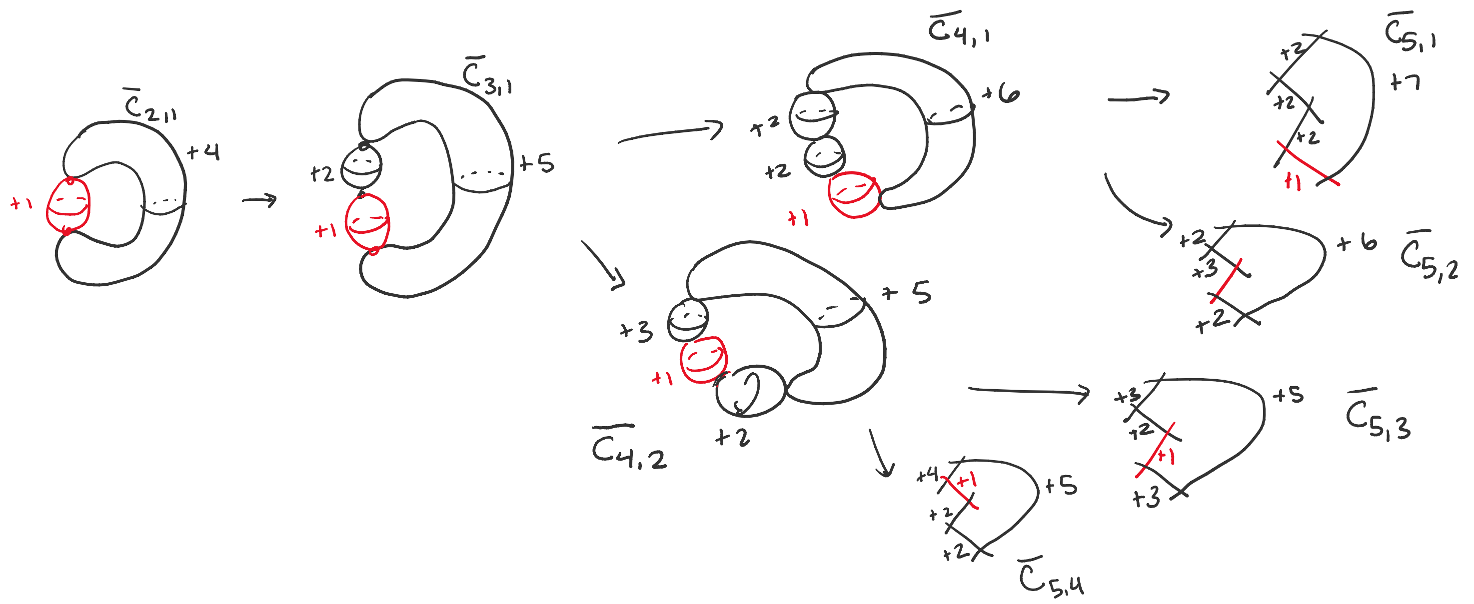
- * combined these operations in creative ways
- * generalized these operations + found some others

Generalizations of rational blowdown

$\boxed{1}$ Jongil Park showed that $\overset{-(p+2)}{\bullet}$ — $\overset{-2}{\bullet}$ — $\cdots$ — $\overset{-2}{\bullet}$ is not the only linear plumbing which can be replaced by a rational homology ball:

There is a more general family $C_{p,q}$ with boundary $L(p^2, 1-pq)$

$B_{p,q}$ can be found in the complement of $\overline{C_{p,q}}$ in $\#_{p-1}\, \mathbb{CP}^2$



$\boxed{2}$ Stipsicz-Szabo-Wahl found families of nonlinear plumbings which can be replaced by rational homology balls (symplectically)

Michalogiorgaki extended the Seiberg-Witten formula for this generalization.

$\boxed{3}$ More generally, can look for plumbings which can be (symplectically) replaced by smaller fillings (star surgery [Karakurt-S.])

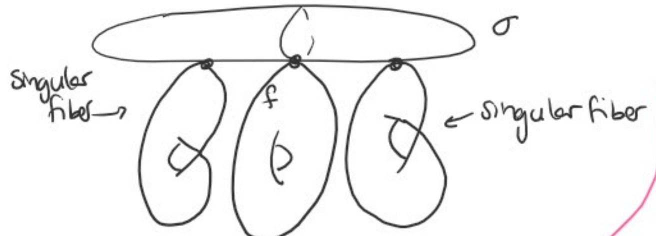
Combining knot surgery + rational blowdown

Sometimes doing a knot surgery first can make it possible to do a rational blowdown that wasn't previously there:

Fintushel-Stern's double node neighborhood:

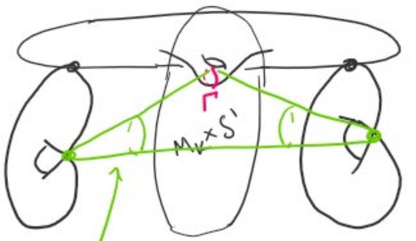
Elliptic surface:

Elliptic surface:



Perform knot surgery using

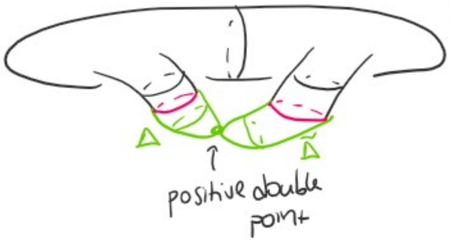
Note: K has a Seifert surface of genus 1



σ becomes genus 1 (meridian of fiber is filled by Seifert surface)

∃ Δ disk of self-intersection -1 with boundery Γ

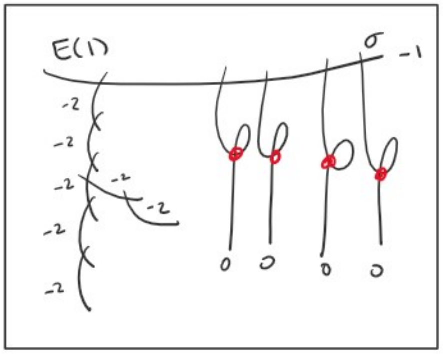
surger σ along Γ using Δ and an oppositely oriented pushoff $\tilde{\Delta}$.



σ becomes an immersed sphere with one positive node

This allows one to find embeddings of C_p with fewer blowups. e.g.:

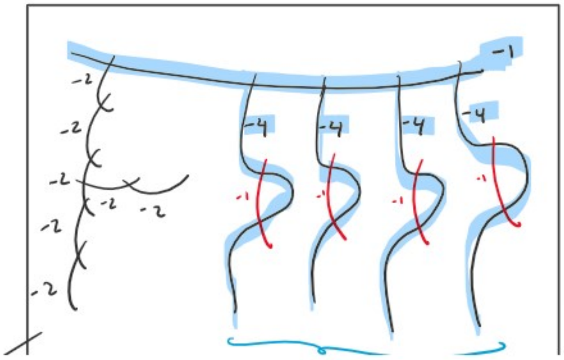
Construction of Jongil Park of the first exotic $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$:

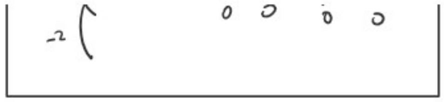


without double node knot surgery

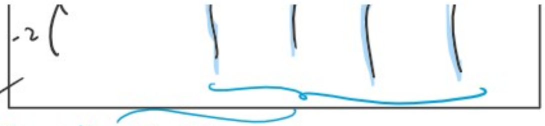
$\longrightarrow$

blow up 4 times (increase b_2^- by 4)

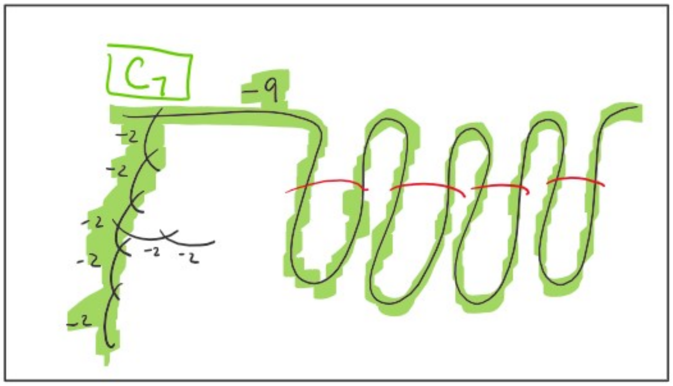




b_2 by 1)



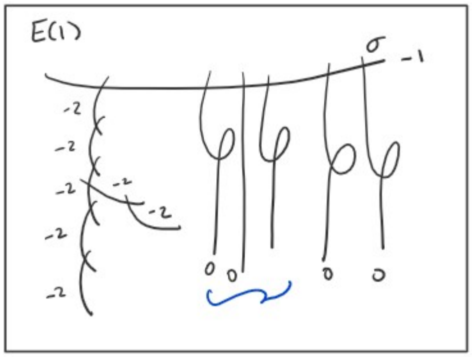
merge these together by smoothing intersections



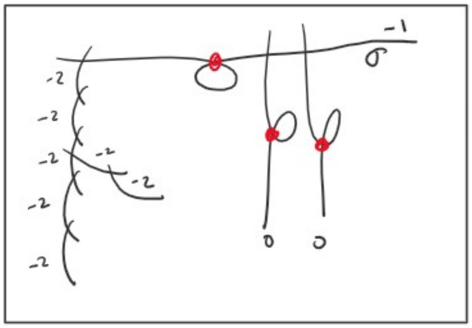
rationally blowdown $\xrightarrow{}$ decreasing b_2^- by 6

Rational blowdown is homeomorphic to $\mathbb{CP}^2 \# \underbrace{(9+4-6)}_{7} \overline{\mathbb{CP}^2}$

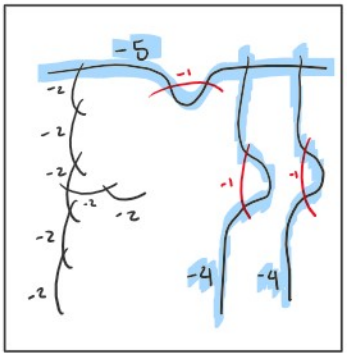
With Fintushel-Stern's double node neighborhood knot surgery:



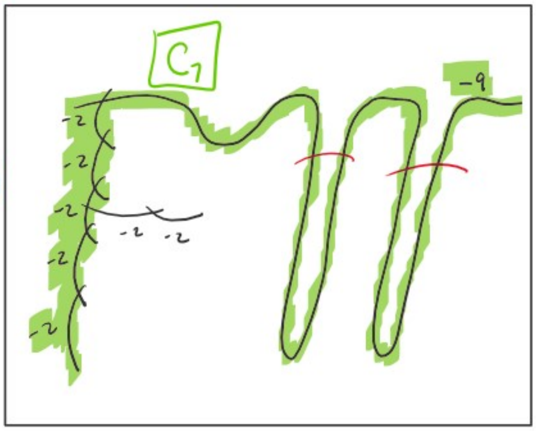
double node nbhd knot surgery



blow-up 3 times (b_2^- goes up by 3)



merge these by smoothing intersections



Rational blowdown $\xrightarrow{}$ (decrease b_2^- by 6)

Exotic $\mathbb{CP}^2 \# (9+3-6)\overline{\mathbb{CP}^2}$ $\shortparallel$ $\mathbb{CP}^2 \# 6 \overline{\mathbb{CP}^2}$

Versions of Seiberg-Witten formulas in $b_2^+ = 1$ case:

Recall that the Seiberg-Witten invariant for a manifold with $b_2^+ = 1$ depends on the choice of metric and perturbation (g, η).

More precisely, it depends on which <u>chamber</u> (g, η) is in (which side of the wall).

We looked at formulas with the general structure:

$$X \xrightarrow[\text{operation}]{\text{cut-paste}} X'$$

$$\Downarrow$$

$$SW_{X'} = \text{something determined by } SW_X$$

To have this make sense, we needed to relate characteristic classes on X to characteristic classes on X'.

When $b_2^+ = 1$, the same formulas hold if we match chambers on X to chambers on X' appropriately, aligning neck stretched metrics.

How to keep track of chambers

- Associate the metric g to a 2-form representing an elt of $H^2(X;\mathbb{R})$

$$g \longleftrightarrow \omega_g \in H^2_+(X;\mathbb{R})$$

unique g-self-dual harmonic 2-form with $\omega_g^2 = 1$ and corresponding to the positive orientation on $H^2_+(X;\mathbb{R})$

(Since $b_2^+ = 1$ the space of g-self dual harmonic 2-forms is 1-dimensional)

- If we have a reducible solution in the Seiberg-Witten moduli space with spin^c structure $\mathcal{S}$ with $c_1(\mathcal{S}) = K \in H^2$ (characteristic)

  we have (A, ψ) where $K = \frac{i}{2\pi}[F_A] \in H^2$

$$\boxed{\begin{array}{l} \text{sw eqns} \begin{cases} D_A \psi = 0 \\ \rho\left(F_A^+ - i\underset{\substack{\uparrow \\ \text{perturbation}}}{\eta}\right) = (\psi\psi^*)_0 \end{cases} \\ \\ \text{and} \quad \psi = 0 \quad \text{(reducible).} \end{array}} \quad \iff \quad \underbrace{F_A^+ - i\eta}_{\nwarrow\ \text{self-dual}} = 0$$

$$2\pi K = i[F_A] = i[F_A^+] + i[F_A^-] = -\eta + i[F_A^-]$$

$$\iff \quad 2\pi K + \eta = i[F_A^-]$$

$$\iff \quad (2\pi K + \eta) \cdot \omega_g = 0$$

Conclusion: The wall for $SW(X, K)$ is defined by the set of (g, η) s.t.

$$(2\pi K + \eta) \cdot \omega_g = 0$$

Define $SW_X^+(K)$ to be the invariant where (g, η) has $(2\pi K + \eta) \cdot \omega_g > 0$

$SW_X^-(K)$ " " " $(2\pi K + \eta) \cdot \omega_g < 0$

Wall crossing formula: $SW_X^+(K)$ and $SW_X^-(K)$ differ by ± 1.

A case when the "small perturbation chamber" is well defined

Assumptions: $b_1 = 0$, $b_2^+ = 1$, $b_2^- \leq 9$

For any spin^c structure $\mathcal{S}$ where the moduli space is nonempty

$$d = \frac{c_1(\mathcal{S})^2 - 2\chi(X) - 3\sigma(X)}{4} \geq 0$$

$$d = \frac{c_1(\mathfrak{s})^2 - 2\chi(X) - 3\sigma(X)}{4} \geq 0$$

$$\Rightarrow \quad c_1(\mathfrak{s})^2 \geq 2\chi(X) + 3\sigma(X) = 2(3+b_2^-) + 3(1-b_2^-)$$

$$\Rightarrow \quad c_1(\mathfrak{s})^2 \geq 9 - b_2^- \geq 0$$

With $K = c_1(\mathfrak{s})$, $K^2 \geq 0 \Rightarrow K \cdot \omega_g \neq 0$

and the sign of $K \cdot \omega_g$ is independent of g.

$\Rightarrow$ For suff small η, the sign of $(2\pi K + \eta) \cdot \omega_g$ is independent of g.

For $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, with any N (potentially > 9)

we have a metric of positive scalar curvature $g \longleftrightarrow \omega_g = h \leftarrow$ represented by $\mathbb{CP}^1 \subset \mathbb{CP}^2$

In this chamber (η small) $SW_X^{(g,\eta)}(K) = 0 \quad \forall K$

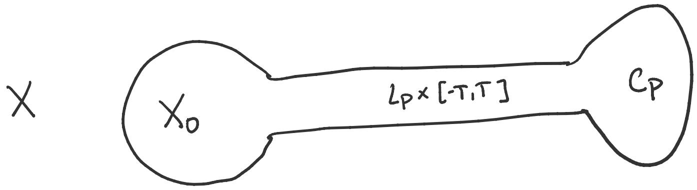
Wall-crossing $\Rightarrow$ $SW_X^{(\tilde{g},\tilde{\eta})}(K) \neq 0$ if $(\tilde{g},\tilde{\eta})$ is in a different chamber.

Suppose X' is obtained from $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$ by rational blowdown

and X' is homeomorphic to $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ with $n \leq 9$.

To calculate $SW_{X'}(K')$ (in the well-defined small perturbation chamber):

- Find a lift K on X of K'.



Use Neck stretching $g \leftrightsquigarrow \omega_g$

The (small perturbation) chamber defined by ω_g contains contains all $\alpha := \omega_{\tilde{g}}$ s.t.

① $\alpha \cdot C = 0 \quad \forall \ C \in H_2(C_p)$

② $\alpha \cdot \alpha \geq 0$.

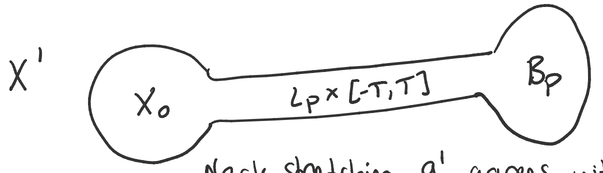
③ $\alpha \cdot h > 0$

(2) $\alpha \cdot \alpha \geq 0$.

(3) $\alpha \cdot h > 0$

① Find such a class α to represent a common chamber where:

$$SW_X^{g,\overset{\text{small}}{\eta}}(K) = SW_{X'}^{g',\overset{\text{small}}{\eta'}}(K')$$



Neck stretching g' agrees with g on X_0 and $L_p \times [-T,T]$

② Determine: on X, with respect to K, does α determine the same or the opposite chamber as h (the positive scalar curvature chamber)?

i.e. is the sign of $K \cdot \alpha$ the same or opposite of the sign of $K \cdot h$?

If they are opposite,

Wall crossing $\Rightarrow$ $SW_X^{g,\overset{\text{small}}{\eta}}(K) = SW_X^{g_{pos},\overset{\text{small}}{\eta}}(K) \pm 1$

$= \pm 1$

0 by pos scalar curvature

$\Rightarrow$ $SW_{X'}^{g,\overset{\text{small}}{\eta}}(K') = \pm 1$ and this is independent of g

$\Rightarrow$ X' cannot be diffeomorphic to $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ $(n \leq 9)$

Since $SW^{g,\overset{\text{small}}{\eta}}_{\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}}(\bar{K}) = 0$ $\forall \bar{K}$.