MSRI SUMMER SCHOOL: 4-MANIFOLDS PROBLEM SESSION 1

(1) In this problem you will calculate the intersection form for $\mathbb{CP}^2$. We will use homogeneous coordinates $\{[x:y:z]\}$ on $\mathbb{CP}^2$ where $(x,y,z)\in\mathbb{C}^3\setminus\{(0,0,0)\}$ and $[x:y:z]=[\lambda x:\lambda y:\lambda z]$ for any $\lambda\in\mathbb{C}\setminus\{0\}$. (Each point $[x:y:z]$ corresponds to a complex line in $\mathbb{C}^3$ through the origin. $\mathbb{CP}^2$ is the space of such lines.)

(a) Show that $H_2(\mathbb{CP}^2;\mathbb{Z})\cong\mathbb{Z}$ and a generator is represented by

$$\mathbb{CP}^1=\{z=0\}=\{[x:y:0]\}\subset\mathbb{CP}^2.$$

[You can use a cell decomposition of $\mathbb{CP}^2$ and cellular homology.]

(b) Consider the deformation of $\mathbb{CP}^1=\{z=0\}$ defined by

$$\Sigma_\varepsilon=\{z+\varepsilon x=0\}\subset\mathbb{CP}^2.$$

Σ_ε is a small push-off of $\mathbb{CP}^1=\Sigma_0$. Calculate the number of intersection points between Σ_ε and $\mathbb{CP}^1$ for $\epsilon\neq 0$.

(c) Use the complex structure to orient $\mathbb{CP}^1$, Σ_ε, and $\mathbb{CP}^2$. Check that each intersection between $\mathbb{CP}^1$ and Σ_ε is positive. [You may want to choose a coordinate chart near the intersection point to see the tangent spaces.] Deduce the intersection form on $\mathbb{CP}^2$.

(2) Calculate b_2^+, b_2^-, signature σ, and determine whether the intersection form is even or odd, and whether or not it is unimodular in the following examples.

(a)

$$Q_X=\begin{bmatrix}-2&1&0&0\\1&-1&1&0\\0&1&-2&1\\0&0&1&-2\end{bmatrix}$$

(b)

$$Q_X=\begin{bmatrix}0&1\\1&-n\end{bmatrix}$$

(c)

$$Q_X=H=\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

(d)

$$Q_X=E_8=\begin{bmatrix}-2&1&0&0&0&0&0&0\\1&-2&1&0&0&0&0&0\\0&1&-2&1&0&0&0&0\\0&0&1&-2&1&0&0&0\\0&0&0&1&-2&1&0&1\\0&0&0&0&1&-2&1&0\\0&0&0&0&0&1&-2&0\\0&0&0&0&1&0&0&-2\end{bmatrix}$$

(e)

$$Q_X=\begin{bmatrix}1&1&1&1\\1&-2&0&0\\1&0&-2&0\\1&0&0&-3\end{bmatrix}$$

(3) This problem is about blow-ups and proper transforms.
  (a) Consider curves in a 4-manifold which intersect transversally at a point, such that in a neighborhood B of the intersection point there exist local coordinates (x, y) (viewed as complex coordinates) so that in those coordinates the curves are given by
$$C_1 = \{y = m_1 x\}, \quad C_2 = \{y = m_2 x\}, \quad \cdots \quad , C_r = \{y = m_r x\}.$$
  Assume $m_1, m_2, \ldots, m_r$ are all distinct. Verify that the C_i intersect pairwise transversally at the origin.
  (b) Perform a blow-up at $(x, y) = (0, 0)$. (Recall this transforms the space locally to
$$\{((x, y), [u : v]) \in B \times \mathbb{CP}^1 \mid xv - yu = 0\}.)$$
  Determine the local equations for the total and proper transforms for $C_1, \ldots, C_r$.
  (c) Show that the proper transforms of $C_1, \ldots, C_r$ are disjoint.
  (d) Try repeating these steps with the curves intersecting tangentially and are modeled on $C_1 = \{y = x^2\}$, $C_2 = \{y = 2x^2\}$. How do the proper transforms intersect? Can you blow-up repeatedly to make them disjoint?
  (e) You can also see what happens when you blow-up the singular curve $C = \{y^2 = x^3\}$ at the origin. What do the total and proper transform look like? What happens if you blow-up repeatedly?

(4) If $X_1 \# X_2$ is the connected sum of 4-manifolds X_1 and X_2, show that $Q_{X_1 \# X_2} = Q_{X_1} \oplus Q_{X_2}$. Generalization: If X_1 and X_2 are manifolds with boundary, and $\partial X_1 \cong -\partial X_2$ is a homology sphere, show that the gluing of X_1 to X_2 along their common boundary has intersection form $Q_{X_1} \oplus Q_{X_2}$.

(5) This problem looks at equivalent intersection forms. We use the notation from problem 2 for standard intersection matrices.
  (a) Show that $H \oplus [1]$ is equivalent by an integer change of basis to $[1] \oplus [-1] \oplus [1]$.
  (b) Show that $E_8 \oplus [1]$ is equivalent by an integer change of basis to $[-1]^{\oplus 8} \oplus [1]$.