MSRI SUMMER SCHOOL: 4-MANIFOLD CONSTRUCTIONS PROBLEM SESSION 2

(1) Show that $E(n)$ is simply connected. Hint: Use the fact that $E(1)$ is simply connected and $E(n)$ is obtained by fiber summing. Inducting on n, use Seifert van Kampen on the decomposition

$$E(n) = (E(1) \setminus \nu(F)) \cup_\partial (E(n-1) \setminus \nu(F)).$$

You will need to know $\pi_1(E(n-1) \setminus \nu(F))$ and $\pi_1(E(1) \setminus \nu(F))$ are generated by meridians of F (which you can also show using Seifert van Kampen), and that these meridians are actually null-homotopic in $E(1) \setminus \nu(F)$ (find a disk that the meridian bounds using a section).

(2) Calculate Euler characteristic of $E(n)$ from the fact that it is a fiber sum of copies of $E(1)$. Use this, together with your result from the previous problem that $E(n)$ is simply connected, to calculate the Betti numbers (ranks of homology). (Hint: You'll need Poincare duality for b_3.)

(3) The goal of this problem is to calculate the intersection form on $E(2)$. (You can also generalize this to find the intersection form for $E(n)$.)
   (a) Calculate the intersection form restricted to the surfaces in $E(2)$ which we constructed in the lecture as follows:
      (i) Using the formulas for $\phi_1, \dots, \phi_8$, and the fact that $h^2 = 1$, $e_i^2 = -1$ and $h \cdot e_i = e_i \cdot e_j = 0$ for $i \neq j$, determine the intersection form restricted to $\mathbb{Z}\langle \phi_1, \dots, \phi_8 \rangle$. Using the fact that all the other generators have 0 intersection with the ϕ_i, conclude that we get 2 direct summands of this in the intersection form for $E(2)$ (and n direct summands in $E(n)$).
      (ii) Prove that the tori representing the classes f, t_1, and t_2 have self-intersection 0. Prove that $f \cdot t_i$ and $t_1 \cdot t_2 = 0$. (Hint: use the realizations by tori contained in a T^3 to construct push-offs.)
      (iii) Calculate $t_i \cdot s_i$ and $\sigma \cdot f$. Next, gain further information about intersections from the facts from lecture that $s_i^2 = -2$, $\sigma^2 = -2$ (or more generally $-n$ in $E(n)$), and that the s_i can be realized by spheres disjoint from the ϕ_i, from σ and from each other. Conclude what the intersection form is on the restriction to $\mathbb{Z}\langle \sigma, f, t_1, s_1, t_2, s_2 \rangle$.
   (b) Comparing $b_2(E(2))$ with the number of surfaces we constructed and checking unimodularity of this intersection form, conclude that these surfaces provide a basis for $H_2(E(2); \mathbb{Z})$ and you have found the intersection form for $E(2)$.

(4) Show that $E(2)\#\overline{\mathbb{CP}}^2$ is homeomorphic to $\#3\mathbb{CP}^2\#20\overline{\mathbb{CP}}^2$. [Hint: Use the intersection form for $E(2)$ from the previous problem, together with Problem 5(b) from Problem Session 1.]

(5) Here we complete the check that for a basic class K on $E(n)$, K vanishes on the summands generated by copies of $\phi_1, \dots, \phi_8$, following an argument of Stipsicz. Recall that

$$\phi_1 = e_1 - e_2, \dots, \phi_7 = e_7 - e_8, \qquad \phi_8 = e_6 + e_7 + e_8 - h.$$

Use the intersection form you calculated in problem 3 on $\mathbb{Z}\langle \phi_1, \dots, \phi_8 \rangle$ for this problem.
   (a) Show that the only elements in $\mathbb{Z}\langle \phi_1, \dots, \phi_8 \rangle$ which have square -2 are those of the form

$$e_i - e_j, \qquad \pm(h - e_i - e_j - e_k),\ i, j, k \in \{1, \cdots, 8\} \text{ distinct}$$

$$\pm\left(2h - \sum_{j=1}^{6} e_{i_j}\right) (i_j \in \{1, \dots, 8\} \text{ are 6 distinct indices})$$

$$\pm(3h - 2e_{i_1} - e_{i_2} - \cdots - e_{i_8})\ (i_j \in \{1, \dots, 8\} \text{ distinct}).$$

(b) Show that each of these elements can be represented by a sphere. (Hints: the degree genus formula tells you the genus of a generic complex curve in classes $h, 2h, 3h$ in $\mathbb{CP}^2$. Proper transforms do not change genus. For curves of the last type, instead of using a generic curve in class $3h$, take a nodal curve and choose the blow-up of e_{i_1} to occur at the node.)

(c) Show that classes of the form $h - e_i - e_j - e_k$ generate $\mathbb{Z}\langle\phi_1, \ldots, \phi_8\rangle$.
Recall from lecture that if K is a basic class, $K(x) \in \{-2, 0, 2\}$ for any class x represented by a -2 sphere. We want to show that $K(x) = 0$ for all the classes we are considering. From what you just showed, it suffices to show that $K(h - e_i - e_j - e_k) = 0$ for all $i, j, k \in \{1, \ldots, 8\}$ distinct.

(d) Suppose for contradiction that there exists a class $h - e_i - e_j - e_k$ such that $K(h - e_i - e_j - e_k) \neq 0$. Without loss of generality, we can assume $K(h - e_1 - e_2 - e_3) = 2$ (relabel the indices of the e_i and swap K for $-K$ if needed). Consider classes of the form $e_1 - e_i$, $e_2 - e_j$, and $e_3 - e_k$ for $i, j, k \in \{4, \ldots, 8\}$ distinct.
   (i) Show that if K evaluates to 2 on *any* of these classes, then there exists a class x represented by a sphere of square -2 such that $|K(x)| > 2$ (a contradiction).
   (ii) Show that if K evaluates to -2 on *all three* of these classes, then there exists a class x represented by a sphere of square -2 such that $|K(x)| > 2$ (a contradiction).
   (iii) Show that if K evaluates to 0 on *all three* of these classes, then there exists a class x represented by a sphere of square -2 such that $|K(x)| > 2$ (a contradiction).
   (iv) Give a case analysis of the possible values of K on $e_i - e_j$ for $i \in \{1, 2, 3\}$, $j \in \{4, 5, 6, 7, 8\}$, to show that no matter what values K takes, there exists a class x represented by a sphere of square -2 such that $|K(x)| > 2$ (a contradiction).