MSRI SUMMER SCHOOL: 4-MANIFOLD CONSTRUCTIONS PROBLEM SESSION 3

(1) Using the definition of B_p as $(\#_{p-1}\mathbb{CP}^2) \setminus \overline{C}_p$, and the fact that ∂B_p is a lens space, show that $H_i(B_p;\mathbb{Q}) \cong H_i(B^4;\mathbb{Q})$. (You can use that for a lens space L, $H_i(L;\mathbb{Q}) \cong \mathbb{Q}$ if $i = 0, 3$ and $H_i(L;\mathbb{Q}) = 0$ for $i \neq 0, 3$.) If you have extra time, you can also try calculating $H_i(B_p;\mathbb{Z})$ precisely with exact sequences, but you'll need to keep track of more information about the maps coming from the data of how $\overline{C}_p$ is embedded into $\#_{p-1}\mathbb{CP}^2$.

(2) In this problem, we will explore another example construction of an exotic 4-manifold using rational blow-down. In $E(4)$ there are nine disjoint sections of the elliptic fibration and each section is a 2-sphere of self-intersection -4. Therefore, one or more of these -4-spheres has a C_2 neighborhood which can be rationally blown down. Let X_1 denote the manifold obtained from $E(4)$ by rationally blowing down the neighborhood of one of the -4-sphere sections σ.

(a) What are the basic classes of $E(4)$? What are their Seiberg-Witten invariants?

(b) Given the embedding of C_2 into $E(4)$ which sends the core (-4)-sphere to the section σ, determine what is the restriction of each basic class of $E(4)$ to the boundary of C_2? Note that the meridian μ of σ is the generator of $H_1(\partial C_2;\mathbb{Z}) \cong \mathbb{Z}_4$, so (via Poincare duality) your answers should be written as some multiples of μ.

(c) Determine which basic classes on $E(4)$ descend to basic classes on X_1 using the criteria from lecture. Do the basic classes which descend from $E(4)$ to X_1 have equivalent or distinct images in $H_2(E(4), C_2)$ (which is identified by excision with $H_2(X_0, \partial X_0)$)? Using the fact that there is a unique extension from $(X_0, \partial X_0)$ to X_1, this determines whether those basic classes descend to equal or distinct classes in X_1. Using the Seiberg-Witten formula for rational blow-down, determine the Seiberg-Witten invariants for each of these descending classes, and write the Seiberg-Witten polynomial for X_1.

(d) Calculate the algebro-topological invariants of X_1. Show that X_1 is simply-connected. Determine $b_2^+(X_1)$ and $b_2^-(X_1)$. Is the intersection form of X_1 even or odd? Using Freedman's theorem, what "standard" manifold is homeomorphic X_1? Can you say that X_1 is exotic to (not diffeomorphic to) the standard manifold?

(e) If a 4-manifold admits a complex structure, there are certain holomorphic invariants that turn out to only depend on the Euler characteristic and signature of the 4-manifold. Motivated by the formulas for complex manifolds, we define:

$$c_1^2(X) = 3\sigma(X) + 2\chi(X)$$

$$\chi_h(X) = \frac{1}{4}\left(\sigma(X) + \chi(X)\right).$$

$c_1^2(X)$ represents the square of the first Chern class, and $\chi_h(X)$ is called the holomorphic Euler characteristic. If X is a complex surface of general type, it satisfies the Noether inequality:

$$2\chi_h(X) - 6 \leq c_1^2.$$

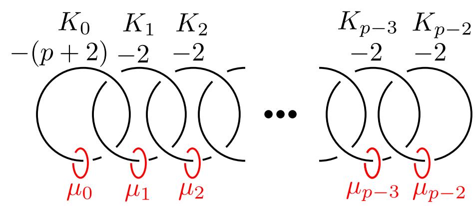
(The Enriques-Kodaira classification shows that all simply connected complex surfaces which are not general type are either elliptic surfaces which share χ and σ with some $E(n)$, or rational surfaces of the form $\mathbb{CP}^2\#_N\overline{\mathbb{CP}}^2$.)

Prove that the manifold you obtained from one rational blow-down of $E(4)$ violates the Noether inequality and thus cannot be a complex surface of general type (since its Euler characteristic and signature also differ from elliptic and rational surfaces, it cannot be any simply connected complex surface). Note: X_1 does admit a symplectic structure, so this gives an example of a symplectic 4-manifold which is not a complex surface.

(f) Challenge: Using the blow-up formula and properties of its intersection form, can you prove that X_1 is not the blow-up of another smooth 4-manifold?

(3) This problem looks at $H_1(L_p;\mathbb{Z})$ using a surgery diagram, and then calculates the restrictions of the classes E_i to L_p under the embedding of C_p into $E(n)\#(p-1)\overline{\mathbb{CP}}^2$ that we saw in lecture.

(a) The diagram below gives a handle diagram for C_p, and correspondingly, a surgery diagram for its boundary L_p.



There is a standard way to give a presentation for the first homology of a 3-manifold presented as a surgery diagram, as described in [Gompf-Stipsicz Proposition 5.3.11], as follows. The generators are the meridians μ_i of each component K_i of the surgery link. For each component K_i with surgery coefficient p_i/q_i (in our case, all $q_i = 1$), there is a relation:

$$p_i\mu_i + q_i \sum_{j\neq i} lk(K_i, K_j)\mu_j = 0.$$

Using this surgery diagram of L_p, write out the presentation as described above, and then simplify it: use the first $p-2$ relations to inductively show that

$$\mu_j = (j(p+1)+1)\mu_0 \qquad \text{for } j = 1, \dots, p-2$$

and then plug into the last relation to show that $H_1(L_p;\mathbb{Z}) \cong \mathbb{Z}_{p^2}$.

(b) Recall from the lecture that we described an embedding of C_p into $E(n)\#(p-1)\overline{\mathbb{CP}}^2$ which identifies the homology classes of the spheres generating $H_2(C_p;\mathbb{Z})$ as follows

$$u_0 = F - 2E_1 - E_2 - E_3 - \cdots - E_{p-1}$$
$$u_j = E_j - E_{j+1} \qquad j = 1, \cdots, p-2.$$

For each E_i, determine the intersection number of E_i with each of $u_0, u_1, \dots, u_{p-2}$. The restriction of E_i (or more precisely its Poincare dual in H^2) to L_p will be (Poincare dual to)

$$E_i|_{L_p} = \sum_{j=0}^{p-2}(E_i \cdot u_j)\mu_j.$$

(The intersection of L_p with a representative surface for E_i will include one (positive/negatively oriented) meridian of u_j for each transverse intersection of E_i with u_j.) Find $E_i|_{L_p}$ for $i = 1, \dots, p-1$ and show that these restrictions are the same for all i in $H_1(L_p;\mathbb{Z})$.