

MSRI SUMMER SCHOOL: LECTURE OUTLINE

1. LECTURE 1: INTRO TO 4-MANIFOLDS

Basic examples: S^4 , $S^2 \times S^2$, $\Sigma_1 \times \Sigma_2$, $\mathbb{C}\mathbb{P}^2$, $\overline{\mathbb{C}\mathbb{P}^2}$, $S^1 \times S^3$, connected sums

Algebraic topology invariants: π_1 , homology, Poincaré duality, Betti numbers, Euler characteristic

Intersection form:

- Via cup product
- Via intersections of surfaces
- Example: $S^2 \times S^2$, $\mathbb{C}\mathbb{P}^2$ (partially left as an exercise), $\overline{\mathbb{C}\mathbb{P}^2}$
- b_2^+ , b_2^- , signature
- Unimodularity
- Even versus odd forms

Freedman's Theorem: statement of topological classification in simply connected case, contrast to Donaldson diagonalization, contrast to existence of exotica

Blow-up:

- Definition from algebraic equations
- Definition as connected sum with $\overline{\mathbb{C}\mathbb{P}^2}$
- Exceptional divisor
- Total and proper transforms, self-intersections

2. LECTURE 2: FIRST CONSTRUCTIONS AND SEIBERG-WITTEN FORMULAS

Elliptic fibrations

- $E(1)$ as a blow-up of the cubic pencil
- Singular fibers in generic $E(1)$

Fiber sum:

- Define operation
- Define $E(n)$
- Surfaces in $E(n)$, homotopy invariants (exercise)

Seiberg-Witten invariants for $E(n)$

- narrow down possible basic classes with adjunction
- state Fintushel-Stern formula for invariants

Blow-up formula

Connected sum vanishing theorem

First exotic example: $E(2) \# \overline{\mathbb{C}\mathbb{P}^2}$ exotic to $\#3\mathbb{C}\mathbb{P}^2 \# 20\overline{\mathbb{C}\mathbb{P}^2}$

3. LECTURE 3: RATIONAL BLOW-DOWN

Rational blow down operation:

- Plumblings and B_p
- Rational homology balls and C_p
- $\partial B_p \cong \partial C_p$, cut and paste

Effect on topological invariants:

- Effect on π_1
- Effect on H_2

Effect on Seiberg-Witten invariants:

- Relating characteristic classes
- Statement of Seiberg-Witten rational blow-down formula

Exotic example:

- Blow-up a nodal fiber of $E(2)$ ($p - 1$) times to get an embedding of C_p and then rationally blow-down to get $E(2)_p$.
- Calculate basic classes.

4. LECTURE 4: KNOT SURGERY

Knot surgery operation:

- definition from gluing $S^1 \times$ knot complement
- as a fiber sum with $S^1 \times$ 0-surgery – comment on fibered knot case (symplectic)

Effect on homeomorphism type

Alexander polynomial of a knot (via Skein relations)

Seiberg-Witten formula for knot surgery

Ideas in the proof

5. LECTURE 5: GENERALIZATIONS

Combining constructive techniques

Finding exotica when $b_2^+ = 1$

Generalizations of rational blow-down

Geography and botany state of the art