Problem Sheet 6

1) a) Show that if $\lim_{n\to\infty} s_n = \infty$, then $\lim_{n\to\infty} \frac{1}{s_n} = 0$.

   b) Show that if $\lim_{n\to\infty} s_n = 0$ and $s_n > 0$ for all n, then $\lim_{n\to\infty} \frac{1}{s_n} = \infty$.

2) Prove that if $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, then $\lim_{n\to\infty} s_n t_n = st$ by citing the justification for each of the following steps:

   a) $\lim_{n\to\infty} (s_n - s) = 0$ and $\lim_{n\to\infty} (t_n - t) = 0$.

   b) $\lim_{n\to\infty} (s_n - s)(t_n - t) = 0$, and so $\lim_{n\to\infty} (s_n t_n - s t_n - s_n t + st) = 0$.

   c) $\lim_{n\to\infty} s_n t_n = \lim_{n\to\infty} [(s_n t_n - s t_n - s_n t + st) + s t_n + s_n t - st] =$
   $\lim_{n\to\infty} (s_n t_n - s t_n - s_n t + st) + \lim_{n\to\infty} s t_n + \lim_{n\to\infty} s_n t - \lim_{n\to\infty} st$.

   d) $\lim_{n\to\infty} s_n t_n = 0 + st + st - st = st$.

3) If s_n is bounded, show that $\lim_{n\to\infty} \frac{s_n}{n} = 0$ using

   a) a problem from the previous sheet.

   b) the Squeeze Theorem.

4) Prove that if $s_n \leq t_n$ for all n and $\lim_{n\to\infty} s_n = \infty$, then $\lim_{n\to\infty} t_n = \infty$.

5) Use the Squeeze Theorem to find $\lim_{n\to\infty} (4^n + 5^n)^{\frac{1}{n}}$.

6) Prove that $\lim_{n\to\infty} a_n = 0$ iff $\lim_{n\to\infty} |a_n| = 0$ using the following steps:

   a) Show that if $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} |a_n| = 0$ using a problem from the previous problem sheet.

   b) Show that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$ using the Squeeze Theorem.

   (We proved this result previously using the definition of the limit.)

7) Define $\{s_n\}$ by $s_{n+1} = \sqrt{5 + s_n}$ for $n \geq 1$ and $s_1 = \sqrt{5}$.

   a) Show that $\{s_n\}$ converges.

   b) Find $\lim_{n\to\infty} s_n$.

8) Define $\{s_n\}$ by $s_{n+1} = \frac{1}{4} s_n + 15$ for $n \geq 1$ and $s_1 = 2$.

   a) Show that $\{s_n\}$ converges.

   b) Find $\lim_{n\to\infty} s_n$.