PREREQUISITE REVIEW 4.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–6, describe how the graph of g is related to the graph of f.

1. $g(x) = f(x + 2)$

2. $g(x) = -f(x)$

3. $g(x) = -1 + f(x)$

4. $g(x) = f(-x)$

5. $g(x) = f(x - 1)$

6. $g(x) = f(x) + 2$

In Exercises 7–10, discuss the continuity of the function.

7. $f(x) = \dfrac{x^2 + 2x - 1}{x + 4}$

8. $f(x) = \dfrac{x^2 - 3x + 1}{x^2 + 2}$

9. $f(x) = \dfrac{x^2 - 3x - 4}{x^2 - 1}$

10. $f(x) = \dfrac{x^2 - 5x + 4}{x^2 + 1}$

In Exercises 11–16, solve for x.

11. $2x - 6 = 4$

12. $3x + 1 = 5$

13. $(x + 4)^2 = 25$

14. $(x - 2)^2 = 8$

15. $x^2 + 4x - 5 = 0$

16. $2x^2 - 3x + 1 = 0$

EXERCISES 4.1

In Exercises 1 and 2, evaluate each expression.

1. (a) $5(5^3)$ (b) $27^{2/3}$
(c) $64^{3/4}$ (d) $81^{1/2}$
(e) $25^{3/2}$ (f) $32^{2/5}$

2. (a) $\left(\frac{1}{5}\right)^3$ (b) $\left(\frac{1}{8}\right)^{1/3}$
(c) $64^{2/3}$ (d) $\left(\frac{5}{8}\right)^2$
(e) $100^{3/2}$ (f) $4^{5/2}$

In Exercises 3–6, use the properties of exponents to simplify the expression.

3. (a) $(5^2)(5^3)$ (b) $(5^2)(5^{-3})$
(c) $(5^2)^2$ (d) 5^{-3}

4. (a) $\dfrac{5^3}{5^6}$ (b) $\left(\dfrac{1}{5}\right)^{-2}$
(c) $(8^{1/2})(2^{1/2})$ (d) $(32^{3/2})\left(\frac{1}{2}\right)^{3/2}$

5. (a) $\dfrac{5^3}{25^2}$ (b) $(9^{2/3})(3)(3^{2/3})$
(c) $[(25^{1/2})(5^2)]^{1/3}$ (d) $(8^2)(4^3)$

6. (a) $(4^3)(4^2)$ (b) $\left(\frac{1}{4}\right)^2(4^2)$
(c) $(4^6)^{1/2}$ (d) $[(8^{-1})(8^{2/3})]^3$

In Exercises 7–10, evaluate the function. If necessary, use a graphing utility, rounding your answers to three decimal places.

7. $f(x) = 2^{x-1}$
(a) $f(3)$ (b) $f\left(\frac{1}{2}\right)$
(c) $f(-2)$ (d) $f\left(-\frac{3}{2}\right)$

8. $f(x) = 3^{x+2}$
(a) $f(-4)$ (b) $f\left(-\frac{1}{2}\right)$
(c) $f(2)$ (d) $f\left(-\frac{5}{2}\right)$

9. $g(x) = 1.05^x$
(a) $g(-2)$ (b) $g(120)$
(c) $g(12)$ (d) $g(5.5)$

10. $g(x) = 1.075^x$
(a) $g(1.2)$ (b) $g(180)$
(c) $g(60)$ (d) $g(12.5)$

In Exercises 11–18, solve the equation for x.

11. $3^x = 81$

12. $5^{x+1} = 125$

13. $\left(\frac{1}{3}\right)^{x-1} = 27$

14. $\left(\frac{1}{5}\right)^{2x} = 625$

15. $4^3 = (x + 2)^3$

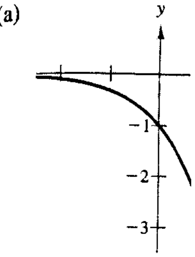
16. $4^2 = (x + 2)^2$

17. $x^{3/4} = 8$

18. $(x + 3)^{4/3} = 16$

In Exercises 19–24, m
are labeled (a)–(f).]

(a)



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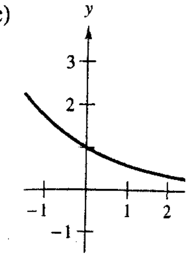

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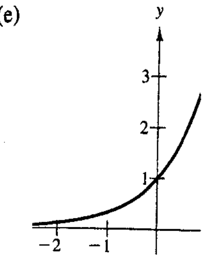


19. $f(x) = 3^x$

21. $f(x) = -3^x$

23. $f(x) = 3^{-x} - 1$

In Exercises 25–34, sket

25. $f(x) = 6^x$

27. $f(x) = \left(\frac{1}{5}\right)^x = 5^{-x}$

29. $y = 3^{-x^2}$

31. $y = 3^{-|x|}$

33. $s(t) = \frac{1}{4}(3^{-t})$

35. ***Population Grow***
the United States fi
by the exponential

$P(t) = 251.27(1$

where t is the time
1992. Use the mode
(a) 2006 and (b) 20

er sections. You will

1. If necessary, use a graph
ree decimal places.

$f(\frac{1}{2})$
$f(-\frac{3}{2})$

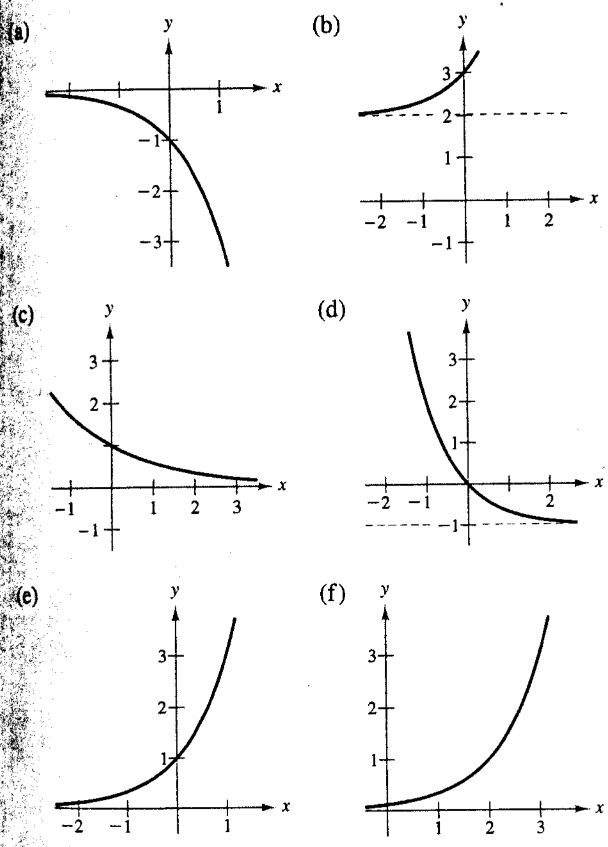
$f(-\frac{1}{2})$
$f(-\frac{5}{2})$

$g(120)$
$g(5.5)$

$g(180)$
$g(12.5)$

for x.
$5^{x+1} = 125$
$(\frac{1}{5})^{2x} = 625$
$4^2 = (x + 2)^2$
$(x + 3)^{4/3} = 16$

In Exercises 19–24, match the function with its graph. [The graphs are labeled (a)–(f).]



19. $f(x) = 3^x$
20. $f(x) = 3^{-x/2}$
21. $f(x) = -3^x$
22. $f(x) = 3^{x-2}$
23. $f(x) = 3^{-x} - 1$
24. $f(x) = 3^x + 2$

In Exercises 25–34, sketch the graph of the function.

25. $f(x) = 6^x$
26. $f(x) = 4^x$
27. $f(x) = \left(\frac{1}{5}\right)^x = 5^{-x}$
28. $f(x) = \left(\frac{1}{4}\right)^x = 4^{-x}$
29. $y = 3^{-x^2}$
30. $y = 2^{-x^2}$
31. $y = 3^{-|x|}$
32. $y = 3^{|x|}$
33. $s(t) = \frac{1}{4}(3^{-t})$
34. $s(t) = 2^{-t} + 3$

35. ***Population Growth*** The population P (in millions) of the United States from 1992 through 2002 can be modeled by the exponential function

$$P(t) = 251.27(1.0118)^t$$

where t is the time in years, with $t = 2$ corresponding to 1992. Use the model to estimate the population in the years (a) 2006 and (b) 2012. *(Source: U.S. Census Bureau)*

36. ***Sales*** The sales S (in millions of dollars) for Starbucks from 1994 through 2003 can be modeled by the exponential function

$$S(t) = 116.59(1.3295)^t$$

where t is the time in years, with $t = 4$ corresponding to 1994. Use the model to estimate the sales in the years (a) 2006 and (b) 2012. *(Source: Starbucks Corp.)*

37. ***Property Value*** Suppose that the value of a piece of property doubles every 15 years. If you buy the property for \$64,000, its value t years after the date of purchase should be

$$V(t) = 64{,}000(2)^{t/15}.$$

Use the model to approximate the value of the property (a) 5 years and (b) 20 years after it is purchased.

38. ***Inflation Rate*** Suppose that the annual rate of inflation averages 5% over the next 10 years. With this rate of inflation, the approximate cost C of goods or services during any year in that decade will be given by

$$C(t) = P(1.05)^t, \quad 0 \le t \le 10$$

where t is time in years and P is the present cost. If the price of a movie theater ticket is presently \$6.95, estimate the price 10 years from now.

39. ***Depreciation*** After t years, the value of a car that originally cost \$16,000 depreciates so that each year it is worth $\frac{3}{4}$ of its value for the previous year. Find a model for $V(t)$, the value of the car after t years. Sketch a graph of the model and determine the value of the car 4 years after it was purchased.

40. ***Radioactive Decay*** After t years, the initial mass of 16 grams of a radioactive element whose half-life is 30 years is given by

$$y = 16\left(\tfrac{1}{2}\right)^{t/30}, \quad t \ge 0.$$

(a) Use a graphing utility to graph the function.
(b) How much of the initial mass remains after 50 years?
(c) Use the *zoom* and *trace* features of a graphing utility to find the time required for the mass to decay to an amount of 1 gram.

41. ***Radioactive Decay*** After t years, the initial mass of 23 grams of a radioactive element whose half-life is 45 years is given by

$$y = 23\left(\tfrac{1}{2}\right)^{t/45}, \quad t \ge 0.$$

(a) Use a graphing utility to graph the function.
(b) How much of the initial mass remains after 75 years?
(c) Use the *zoom* and *trace* features of a graphing utility to find the time required for the mass to decay to an amount of 1 gram.

PREREQUISITE REVIEW 4.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–4, discuss the continuity of the function.

1. $f(x) = \dfrac{3x^2 + 2x + 1}{x^2 + 1}$

2. $f(x) = \dfrac{x + 1}{x^2 - 4}$

3. $f(x) = \dfrac{x^2 - 6x + 5}{x^2 - 3}$

4. $g(x) = \dfrac{x^2 - 9x + 20}{x - 4}$

In Exercises 5–12, find the limit.

5. $\lim\limits_{x\to\infty} \dfrac{25}{1 + 4x}$

6. $\lim\limits_{x\to\infty} \dfrac{16x}{3 + x^2}$

7. $\lim\limits_{x\to\infty} \dfrac{8x^3 + 2}{2x^3 + x}$

8. $\lim\limits_{x\to\infty} \dfrac{x}{2x}$

9. $\lim\limits_{x\to\infty} \dfrac{3}{2 + (1/x)}$

10. $\lim\limits_{x\to\infty} \dfrac{6}{1 + x^{-2}}$

11. $\lim\limits_{x\to\infty} 2^{-x}$

12. $\lim\limits_{x\to\infty} \dfrac{7}{1 + 5x}$

EXERCISES 4.2

In Exercises 1–4, use the properties of exponents to simplify the expression.

1. (a) $(e^3)(e^4)$ (b) $(e^3)^4$
(c) $(e^3)^{-2}$ (d) e^0

2. (a) $\left(\dfrac{1}{e}\right)^{-2}$ (b) $\left(\dfrac{e^5}{e^2}\right)^{-1}$
(c) $\dfrac{e^5}{e^3}$ (d) $\dfrac{1}{e^{-3}}$

3. (a) $(e^2)^{5/2}$ (b) $(e^2)(e^{1/2})$
(c) $(e^{-2})^{-3}$ (d) $\dfrac{e^5}{e^{-2}}$

4. (a) $(e^{-3})^{2/3}$ (b) $\dfrac{e^4}{e^{-1/2}}$
(c) $(e^{-2})^{-4}$ (d) $(e^{-4})(e^{-3/2})$

In Exercises 5–12, solve the equation for *x*.

5. $e^{-3x} = e$

6. $e^x = 1$

7. $e^{\sqrt{x}} = e^3$

8. $e^{-1/x} = \sqrt{e}$

9. $x^{2/3} = \sqrt[3]{e^2}$

10. $\dfrac{x^2}{2} = e^2$

11. $3x^3 = 9e^3$

12. $x^{-2} = \dfrac{2}{e^2}$

In Exercises 13–18, match the function with its graph. [The graphs are labeled (a)–(f).]

13. $f(x) = e^{2x+1}$

14. $f(x) = e^{-x/2}$

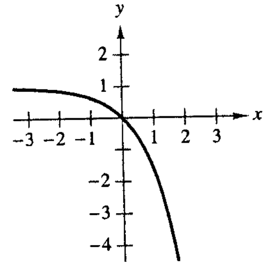
15. $f(x) = e^{x^2}$

16. $f(x) = e^{-1/x}$

17. $f(x) = e^{\sqrt{x}}$

18. $f(x) = -e^x + 1$

(a)



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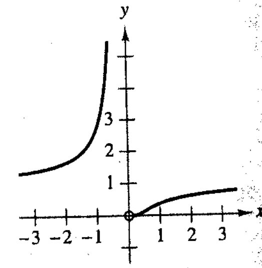

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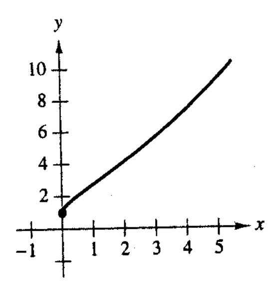

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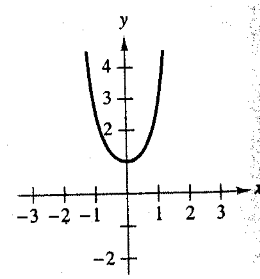
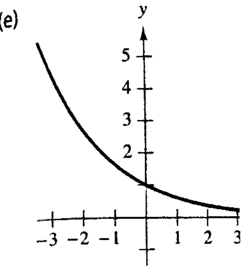


(e)



In Exercises 19–22, sketch

19. $h(x) = e^{x-2}$

21. $g(x) = e^{1-x}$

In Exercises 23–26, use a gı
sure to choose an approp

23. $N(t) = 500e^{-0.2t}$

24. $A(t) = 500e^{0.15t}$

25. $g(x) = \dfrac{2}{1 + e^{x^2}}$

26. $g(x) = \dfrac{10}{1 + e^{-x}}$

In Exercises 27–30, use a
Determine whether the fı
and discuss the continuity

27. $f(x) = \dfrac{e^x + e^{-x}}{2}$

28. $f(x) = \dfrac{e^x - e^{-x}}{2}$

29. $f(x) = \dfrac{2}{1 + e^{1/x}}$

30. $f(x) = \dfrac{2}{1 + 2e^{-0.2x}}$

Compound Interest I
to determine the balance
years, compounded *n* tim

n	1	2	4	12
A				

31. $P = \$1000$, $r = 3\%$,

32. $P = \$2500$, $r = 5\%$,

33. $P = \$1000$, $r = 3\%$,

34. $P = \$2500$, $r = 5\%$,

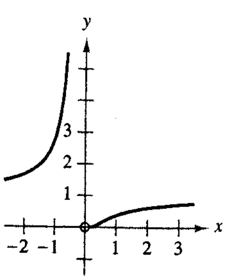
er sections. You will

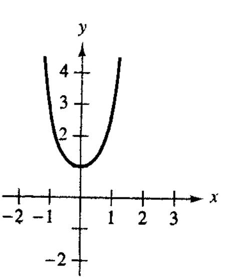
vith its graph. [The graphs

$f(x) = e^{-x/2}$

$f(x) = e^{-1/x}$

$f(x) = -e^x + 1$





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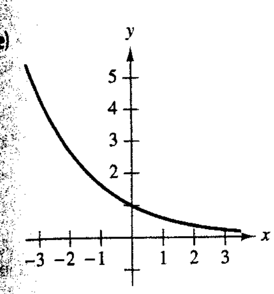
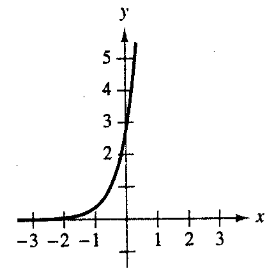


(f)



In Exercises 19–22, sketch the graph of the function.

19. $h(x) = e^{x-2}$
20. $f(x) = e^{2x}$
21. $g(x) = e^{1-x}$
22. $j(x) = e^{-x+2}$

In Exercises 23–26, use a graphing utility to graph the function. Be sure to choose an appropriate viewing window.

23. $N(t) = 500e^{-0.2t}$
24. $A(t) = 500e^{0.15t}$
25. $g(x) = \dfrac{2}{1 + e^{x^2}}$
26. $g(x) = \dfrac{10}{1 + e^{-x}}$

In Exercises 27–30, use a graphing utility to graph the function. Determine whether the function has any horizontal asymptotes and discuss the continuity of the function.

27. $f(x) = \dfrac{e^x + e^{-x}}{2}$
28. $f(x) = \dfrac{e^x - e^{-x}}{2}$
29. $f(x) = \dfrac{2}{1 + e^{1/x}}$
30. $f(x) = \dfrac{2}{1 + 2e^{-0.2x}}$

Compound Interest In Exercises 31–34, complete the table to determine the balance A for P dollars invested at rate r for t years, compounded n times per year.

n	1	2	4	12	365	Continuous compounding
A						

31. $P = \$1000$, $r = 3\%$, $t = 10$ years
32. $P = \$2500$, $r = 5\%$, $t = 20$ years
33. $P = \$1000$, $r = 3\%$, $t = 40$ years
34. $P = \$2500$, $r = 5\%$, $t = 40$ years

Compound Interest In Exercises 35–38, complete the table to determine the amount of money P that should be invested at rate r to produce a final balance of \$100,000 in t years.

t	1	10	20	30	40	50
P						

35. $r = 4\%$, compounded continuously
36. $r = 3\%$, compounded continuously
37. $r = 5\%$, compounded monthly
38. $r = 6\%$, compounded daily

39. ***Effective Rate*** Find the effective rate of interest corresponding to a nominal rate of 9% per year compounded (a) annually, (b) semiannually, (c) quarterly, and (d) monthly.

40. ***Effective Rate*** Find the effective rate of interest corresponding to a nominal rate of 7.5% per year compounded (a) annually, (b) semiannually, (c) quarterly, and (d) monthly.

41. ***Present Value*** How much should be deposited in an account paying 7.2% interest compounded monthly in order to have a balance of \$15,503.77 three years from now?

42. ***Present Value*** How much should be deposited in an account paying 7.8% interest compounded monthly in order to have a balance of \$21,154.03 four years from now?

43. ***Future Value*** Find the future value of an \$8000 investment if the interest rate is 4.5% compounded monthly for 2 years.

44. ***Future Value*** Find the future value of a \$6000 investment if the interest rate is 6.25% compounded monthly for 3 years.

45. ***Demand*** The demand function for a product is modeled by

$$p = 5000\left(1 - \frac{4}{4 + e^{-0.002x}}\right).$$

Find the price of the product if the quantity demanded is (a) $x = 100$ units and (b) $x = 500$ units. What is the limit of the price as x increases without bound?

46. ***Demand*** The demand function for a product is modeled by

$$p = 10{,}000\left(1 - \frac{3}{3 + te^{-0.001t}}\right).$$

Find the price of the product if the quantity demanded is (a) $x = 1000$ units and (b) $x = 1500$ units. What is the limit of the price as x increases without bound?

47. ***Probability*** The average time between incoming calls at a switchboard is 3 minutes. If a call has just come in, the probability that the next call will come within the next t minutes is

$$P(t) = 1 - e^{-t/3}.$$

Find the probability of each situation.

(a) A call comes in within $\frac{1}{2}$ minute.

(b) A call comes in within 2 minutes.

(c) A call comes in within 5 minutes.

48. ***Consumer Awareness*** An automobile gets 28 miles per gallon at speeds of up to and including 50 miles per hour. At speeds greater than 50 miles per hour, the number of miles per gallon drops at the rate of 12% for each 10 miles per hour. If s is the speed (in miles per hour) and y is the number of miles per gallon, then

$$y = 28e^{0.6-0.012s}, \quad s > 50.$$

Use this information to complete the table. What can you conclude?

Speed (s)	50	55	60	65	70
Miles per gallon (y)					

49. ***Sales*** The sales S (in millions of dollars) for Avon Products from 1994 through 2003 can be modeled by

$$S = 3557.12e^{0.0475t}$$

where t is time in years, with $t = 4$ corresponding to 1994. *(Source: Avon Products Inc.)*

(a) Find the sales in 1995, 2000, and 2003.

(b) Using the data points from part (a), would a linear model fit the data? Explain your reasoning.

(c) Use the exponential growth model to estimate when the sales will exceed 10 billion dollars.

50. ***Population*** The population P (in thousands) of Las Vegas, Nevada from 1970 to 2000 can be modeled by

$$P = 115.49e^{0.0445t}$$

where t is the time in years, with $t = 0$ corresponding to 1970. *(Source: U.S. Census Bureau)*

(a) Find the populations in 1970, 1980, 1990, and 2000.

(b) Explain why the data do not fit a linear model.

(c) Use the model to estimate when the population will exceed 750,000.

51. ***Biology*** The population y of a bacterial culture is modeled by the logistic growth function

$$y = \frac{925}{1 + e^{-0.3t}}$$

where t is the time in days.

(a) Use a graphing utility to graph the model.

(b) Does the population have a limit as t increases without bound? Explain your answer.

(c) How would the limit change if the model were

$$y = \frac{1000}{1 + e^{-0.3t}}?$$

Explain your answer. Draw some conclusions about this type of model.

52. ***Biology: Cell Division*** Suppose that you have a single imaginary bacterium able to divide to form two new cells every 30 seconds. Make a table of values for the number of individuals in the population over 30-second intervals up to 5 minutes. Graph the points and use a graphing utility to fit an exponential model to the data. *(Source: Adapted from Levine/Miller, Biology: Discovering Life, Second Edition)*

53. ***Learning Theory*** In a learning theory project, the proportion P of correct responses after n trials can be modeled by

$$P = \frac{0.83}{1 + e^{-0.2n}}.$$

(a) Use a graphing utility to estimate the proportion of correct responses after 10 trials. Verify your result analytically.

(b) Use a graphing utility to estimate the number of trials required to have a proportion of correct responses of 0.75.

(c) Does the proportion of correct responses have a limit as n increases without bound? Explain your answer.

54. ***Learning Theory*** In a typing class, the average number N of words per minute typed after t weeks of lessons can be modeled by

$$N = \frac{95}{1 + 8.5e^{-0.12t}}.$$

(a) Use a graphing utility to estimate the average number of words per minute typed after 10 weeks. Verify your result analytically.

(b) Use a graphing utility to estimate the number of weeks required to achieve an average of 70 words per minute.

(c) Does the number of words per minute have a limit as t increases without bound? Explain your answer.

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he **normal probability**

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distribution.

Function

ction

Original function

function.

First derivative

Second derivative

Simplify.

nine that $x = \pm 1$. By
at these x-values yield

PREREQUISITE REVIEW 4.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–4, factor the expression.

1. $x^2e^x - \frac{1}{2}e^x$

2. $(xe^{-x})^{-1} + e^x$

3. $xe^x - e^{2x}$

4. $e^x - xe^{-x}$

In Exercises 5–8, find the derivative of the function.

5. $f(x) = \dfrac{3}{7x^2}$

6. $g(x) = 3x^2 - \dfrac{x}{6}$

7. $f(x) = (4x - 3)(x^2 + 9)$

8. $f(t) = \dfrac{t-2}{\sqrt{t}}$

In Exercises 9 and 10, find the relative extrema of the function.

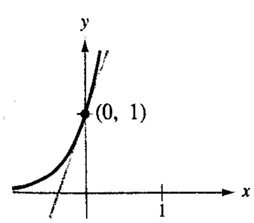
9. $f(x) = \frac{1}{8}x^3 - 2x$

10. $f(x) = x^4 - 2x^2 + 5$

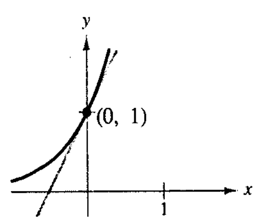
EXERCISES 4.3

In Exercises 1–4, find the slope of the tangent line to the exponential function at the point (0, 1).

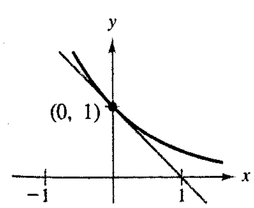
1. $y = e^{3x}$



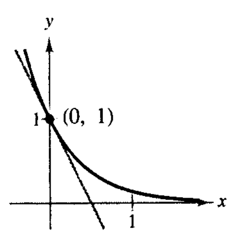
2. $y = e^{2x}$



3. $y = e^{-x}$



4. $y = e^{-2x}$



In Exercises 5–16, find the derivative of the function.

5. $y = e^{4x}$

6. $y = e^{1-x}$

7. $y = e^{-x^2}$

8. $f(x) = e^{1/x}$

9. $f(x) = e^{-1/x^2}$

10. $g(x) = e^{\sqrt{x}}$

11. $f(x) = (x^2 + 1)e^{4x}$

12. $y = 4x^3e^{-x}$

13. $f(x) = \dfrac{2}{(e^x + e^{-x})^3}$

14. $f(x) = \dfrac{(e^x + e^{-x})^4}{2}$

15. $y = xe^x - 4e^{-x}$

16. $y = x^2e^x - 2xe^x + 2e^x$

In Exercises 17–22, determine an equation of the tangent line to the function at the given point.

Function	*Point*
17. $y = e^{-2x+x^2}$	$(0, 1)$
18. $g(x) = e^{x^3}$	$\left(-1, \dfrac{1}{e}\right)$
19. $y = x^2e^{-x}$	$\left(2, \dfrac{4}{e^2}\right)$
20. $y = \dfrac{x}{e^{2x}}$	$\left(1, \dfrac{1}{e^2}\right)$
21. $y = (e^{2x} + 1)^3$	$(0, 8)$
22. $y = (e^{4x} - 2)^2$	$(0, 1)$

In Exercises 23–26, find dy/dx implicitly.

23. $xe^x + 2ye^x = 0$

24. $x^2y - xe^x + 2 = 0$

25. $x^2e^{-x} + 2y^2 - xy = 0$

26. $e^{xy} + x^2 - y^2 = 10$

In Exercises 27–30, find the second derivative.

27. $f(x) = 2e^{3x} + 3e^{-2x}$

28. $f(x) = (1 + 2x)e^{4x}$

29. $f(x) = 5e^{-x} - 2e^{-5x}$

30. $f(x) = (3 + 2x)e^{-3x}$

In Exercises 31–34, graph and analyze the function. Include extrema, points of inflection, and asymptotes in your analysis.

31. $f(x) = \dfrac{1}{2 - e^{-x}}$

32. $f(x) = \dfrac{e^x - e^{-x}}{2}$

33. $f(x) = x^2e^{-x}$

34. $f(x) = xe^{-x}$

In Exercises 35 and 36, use a graphing utility to graph the function. Determine any asymptotes of the graph.

35. $f(x) = \dfrac{8}{1 + e^{-0.5x}}$ **36.** $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

Depreciation In Exercises 37 and 38, the value V (in dollars) of an item is a function of the time t (in years).

(a) Sketch the function over the interval $[0, 10]$. Use a graphing utility to verify your graph.

(b) Find the rate of change of V when $t = 1$.

(c) Find the rate of change of V when $t = 5$.

(d) Use the values $(0, V(0))$ and $(10, V(10))$ to find the linear depreciation model for the item.

(e) Compare the exponential function and the model from part (d). What are the advantages of each?

37. $V = 15{,}000e^{-0.6286t}$ **38.** $V = 500{,}000e^{-0.2231t}$

39. ***Forestry*** To estimate the defoliation p (in percent of foliage) caused by gypsy moths during a year, a forester counts the number x (in thousands) of egg masses on $\frac{1}{40}$ of an acre the preceding fall. The defoliation is modeled by

$$p = \frac{300}{3 + 17e^{-1.57x}}.$$

(Source: National Forest Service)

(a) Use a graphing utility to graph the model.

(b) Estimate the percent of defoliation if 2000 egg masses are counted.

(c) Estimate the number of egg masses for which the amount of defoliation is increasing most rapidly.

40. ***Learning Theory*** The average typing speed N (in words per minute) after t weeks of lessons is modeled by

$$N = \frac{95}{1 + 8.5e^{-0.12t}}.$$

Find the rates at which the typing speed is changing when (a) $t = 5$ weeks, (b) $t = 10$ weeks, and (c) $t = 30$ weeks.

41. ***Compound Interest*** The balance A (in dollars) in a savings account is given by $A = 5000e^{0.08t}$, where t is measured in years. Find the rates at which the balance is changing when (a) $t = 1$ year, (b) $t = 10$ years, and (c) $t = 50$ years.

42. ***Ebbinghaus Model*** The *Ebbinghaus Model* for human memory is $p = (100 - a)e^{-bt} + a$, where p is the percent retained after t weeks. (The constants a and b vary from one person to another.) If $a = 20$ and $b = 0.5$, at what rate is information being retained after 1 week? After 3 weeks?

43. ***Agriculture*** The yield V (in pounds per acre) for an orchard at age t (in years) is modeled by

$$V = 7955.6e^{-0.0458/t}.$$

At what rate is the yield changing when $t = 5$ years? When $t = 10$ years? When $t = 25$ years?

44. ***Employment*** From 1995 to 2002, the numbers y (in millions) of employed people in the United States can be modeled by

$$y = 115.46 + 1.592t + 0.0552t^2 - 0.00004e^t$$

where $t = 5$ corresponds to 1995. *(Source: U.S. Bureau of Labor Statistics)*

(a) Use a graphing utility to graph the model.

(b) Use the graph to estimate the rates of change in the number of employed people in 1995, 1998, and 2002.

(c) Confirm the results of part (b) analytically.

45. ***Probability*** A survey of high school seniors from a certain school district who took the SAT has determined that the mean score on the mathematics portion was 650 with a standard deviation of 12.5.

(a) Assuming the data can be modeled by a normal probability density function, find a model for these data.

(b) Use a graphing utility to graph the model. Be sure to choose an appropriate viewing window.

(c) Find the derivative of the model.

(d) Show that $f' > 0$ for $x < \mu$ and $f' < 0$ for $x > \mu$.

46. ***Probability*** A survey of a college freshman class has determined that the mean height of females in the class is 64 inches with a standard deviation of 3.2 inches.

(a) Assuming the data can be modeled by a normal probability density function, find a model for these data.

(b) Use a graphing utility to graph the model. Be sure to choose an appropriate viewing window.

(c) Find the derivative of the model.

(d) Show that $f' > 0$ for $x < \mu$ and $f' < 0$ for $x > \mu$.

47. Use a graphing utility to graph the normal probability density function with $\mu = 0$ and $\sigma = 2, 3$, and 4 in the same viewing window. What effect does the standard deviation σ have on the function? Explain your reasoning.

48. Use a graphing utility to graph the normal probability density function with $\sigma = 1$ and $\mu = -2, 1$, and 3 in the same viewing window. What effect does the mean μ have on the function? Explain your reasoning.

49. ***Athletics*** A parachutist jumps from a plane and opens the parachute at a height of 2000 feet. The height of the parachutist is $h = 1950 + 50e^{-1.6t} - 20t$, where h is the height (in feet) and t is the time (in seconds) since the parachute was opened.

(a) Find dh/dt and use a graphing utility to graph dh/dt.

(b) Evaluate dh/dt for $t = 0, 1, 5, 10$, and 20.

(c) Interpret your results for parts (a) and (b).

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$\ln e = 1$

$\ln \frac{1}{e} = -1$

$\ln 2 \approx 0.693$

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PREREQUISITE REVIEW 4.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–8, use the properties of exponents to simplify the expression.

1. $(4^2)(4^{-3})$ **2.** $(2^3)^2$ **3.** $\dfrac{3^4}{3^{-2}}$ **4.** $\left(\dfrac{3}{2}\right)^{-3}$

5. e^0 **6.** $(3e)^4$ **7.** $\left(\dfrac{2}{e^3}\right)^{-1}$ **8.** $\left(\dfrac{4e^2}{25}\right)^{-3/2}$

In Exercises 9–12, solve for x.

9. $0 < x + 4$ **10.** $0 < x^2 + 1$

11. $0 < \sqrt{x^2 - 1}$ **12.** $0 < x - 5$

In Exercises 13 and 14, find the balance in the account after 10 years.

13. $P = \$1900$, $r = 6\%$, compounded continuously

14. $P = \$2500$, $r = 3\%$, compounded continuously

EXERCISES 4.4

In Exercises 1–8, write the logarithmic equation as an exponential equation, or vice versa.

1. $\ln 2 = 0.6931 \ldots$

2. $\ln 8.4 = 2.1282 \ldots$

3. $\ln 0.2 = -1.6094 \ldots$

4. $\ln 0.056 = -2.8824 \ldots$

5. $e^0 = 1$

6. $e^2 = 7.3891 \ldots$

7. $e^{-3} = 0.0498 \ldots$

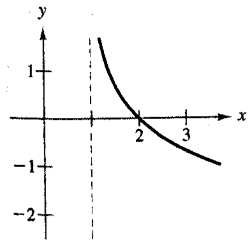
8. $e^{0.25} = 1.2840 \ldots$

In Exercises 9–12, match the function with its graph. [The graphs are labeled (a)–(d).]

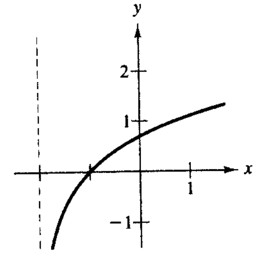
9. $f(x) = 2 + \ln x$ **10.** $f(x) = -\ln x$

11. $f(x) = \ln(x + 2)$ **12.** $f(x) = -\ln(x - 1)$

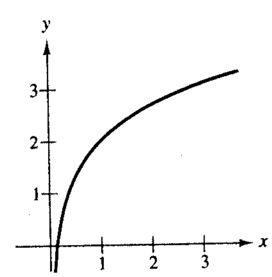
(a)



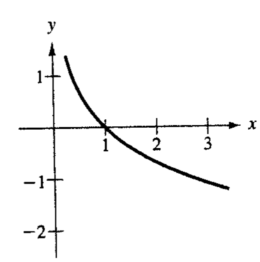
(b)



(c)



(d)



In Exercises 13–18, sketch the graph of the function.

13. $y = \ln(x - 1)$ **14.** $y = \ln|x|$

15. $y = \ln 2x$ **16.** $y = 5 + \ln x$

17. $y = 3 \ln x$ **18.** $y = \frac{1}{4} \ln x$

In Exercises 19–22, analytically show that the functions are inverse functions. Then use a graphing utility to show this graphically.

19. $f(x) = e^{2x}$

$g(x) = \ln \sqrt{x}$

20. $f(x) = e^x - 1$

$g(x) = \ln(x + 1)$

21. $f(x) = e^{2x-1}$

$g(x) = \frac{1}{2} + \ln \sqrt{x}$

22. $f(x) = e^{x/3}$

$g(x) = \ln x^3$

In Exercises 23–28, apply the inverse properties of logarithmic and exponential functions to simplify the expression.

23. $\ln e^{x^2}$ **24.** $\ln e^{2x-1}$

25. $e^{\ln(5x+2)}$ **26.** $e^{\ln\sqrt{x}}$

27. $-1 + \ln e^{2x}$ **28.** $-8 + e^{\ln x^3}$

In Exercises 29 and 30, use the properties of logarithms and the fact that $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$ to approximate the logarithm. Then use a calculator to confirm your approximation.

29. (a) $\ln 6$ (b) $\ln \frac{3}{2}$

(c) $\ln 81$ (d) $\ln \sqrt{3}$

30. (a) $\ln 0.25$ (b) $\ln 24$

(c) $\ln \sqrt[3]{12}$ (d) $\ln \frac{1}{72}$

In Exercises 31–40, use the properties of logarithms to write the expression as a sum, difference, or multiple of logarithms.

31. $\ln \frac{2}{3}$ **32.** $\ln \frac{1}{5}$

33. $\ln xyz$ **34.** $\ln \dfrac{xy}{z}$

35. $\ln \sqrt{x^2+1}$ **36.** $\ln \sqrt{\dfrac{x^3}{x+1}}$

37. $\ln [z(z-1)^2]$ **38.** $\ln \left(x\sqrt[3]{x^2+1}\right)$

39. $\ln \dfrac{3x(x+1)}{(2x+1)^2}$ **40.** $\ln \dfrac{2x}{\sqrt{x^2-1}}$

In Exercises 41–50, write the expression as the logarithm of a single quantity.

41. $\ln(x-2) - \ln(x+2)$

42. $\ln(2x+1) + \ln(2x-1)$

43. $3\ln x + 2\ln y - 4\ln z$

44. $2\ln 3 - \frac{1}{2}\ln(x^2+1)$

45. $3[\ln x + \ln(x+3) - \ln(x+4)]$

46. $\frac{1}{3}[2\ln(x+3) + \ln x - \ln(x^2-1)]$

47. $\frac{3}{2}[\ln x(x^2+1) - \ln(x+1)]$

48. $2\left[\ln x + \frac{1}{4}\ln(x+1)\right]$

49. $\frac{1}{3}\ln(x+1) - \frac{2}{3}\ln(x-1)$

50. $\frac{1}{2}\ln(x-2) + \frac{3}{2}\ln(x+2)$

In Exercises 51–68, solve for x or t.

51. $e^{\ln x} = 4$ **52.** $e^{\ln x^2} - 9 = 0$

53. $\ln x = 0$ **54.** $2\ln x = 4$

55. $e^{x+1} = 4$ **56.** $e^{-0.5x} = 0.075$

57. $300e^{-0.2t} = 700$ **58.** $400e^{-0.0174t} = 1000$

59. $4e^{2x-1} - 1 = 5$ **60.** $2e^{-x+1} - 5 = 9$

61. $\dfrac{10}{1+4e^{-0.01x}} = 2.5$ **62.** $\dfrac{50}{1+12e^{-0.02x}} = 10.5$

63. $5^{2x} = 15$ **64.** $2^{1-x} = 6$

65. $500(1.07)^t = 1000$ **66.** $400(1.06)^t = 1300$

67. $1000\left(1 + \dfrac{0.07}{12}\right)^{12t} = 3000$

68. $2000\left(1 + \dfrac{0.06}{12}\right)^{12t} = 10{,}000$

69. ***Compound Interest*** A deposit of \$1000 is made in an account that earns interest at an annual rate of 5%. How long will it take for the balance to double if the interest is compounded (a) annually, (b) monthly, (c) daily, and (d) continuously?

70. ***Compound Interest*** Complete the table, which shows the time t necessary for P dollars to triple if the interest is compounded continuously at the rate of r.

r	2%	4%	6%	8%	10%	12%	14%
t							

71. ***Chemistry: Carbon Dating*** The remnants of an ancient fire in a cave in Africa showed a $^{14}_{6}C$ (carbon-14) decay rate of 3.1 counts per minute per gram of carbon. Assuming that the decay rate of $^{14}_{6}C$ in freshly cut wood (corrected for changes in the $^{14}_{6}C$ content of the atmosphere) is 13.6 counts per minute per gram of carbon, calculate the age of the remnants. The half-life of $^{14}_{6}C$ is 5715 years. Use the integrated first-order rate law,

$$\ln(N/N_0) = -kt$$

where N is the number of nuclides present at time t, N_0 is the number of nuclides present at time 0, $k = 0.693/5715$, and t is the time of the fire. *(Source: Adapted from Zumdahl,* Chemistry, *Sixth Edition)*

72. ***Demand*** The demand function for a product is given by

$$p = 250 - 0.8e^{0.005x}$$

where p is the price per unit and x is the number of units sold. Find the numbers of units sold for prices of (a) $p = \$200$ and (b) $p = \$125$.

73. ***Population Growth*** The population P (in thousands) of Orlando, Florida from 1980 through 2000 can be modeled by

$$P = 821.95e^{0.0358t}$$

where $t = 0$ corresponds to 1980. *(Source: U.S. Census Bureau)*

(a) According to this model, what was the population of Orlando in 2000?

(b) According to this model, in what year will Orlando have a population of 2,500,000?

74. ***Population (***
Houston, Tex
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Carbon Dating
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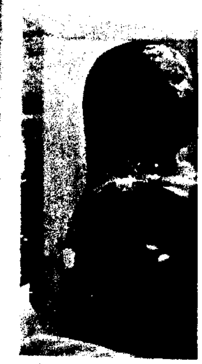
75. $R = 0.32 \times 1($

77. $R = 0.22 \times 1($



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79. ***Learning Th***
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(a) What was t

(b) What was t

(c) After how

74. *Population Growth* The population P (in thousands) of Houston, Texas from 1980 through 2000 can be modeled by

$$P = 2734.07e^{0.0210t}$$

where $t = 0$ corresponds to 1980. *(Source: U.S. Census Bureau)*

(a) According to this model, what was the population of Houston in 2000?

(b) According to this model, in what year will Houston have a population of 6,000,000?

Carbon Dating In Exercises 75–78, you are given the ratio of carbon atoms in a fossil. Use the information to estimate the age of the fossil. In living organic material, the ratio of radioactive carbon isotopes to the total number of carbon atoms is about 1 to 10^{12}. (See Example 2 in Section 4.1.) When organic material dies, its radioactive carbon isotopes begin to decay, with a half-life of about 5715 years. So, the ratio R of carbon isotopes to carbon-14 atoms is modeled by

$$R = 10^{-12}\left(\tfrac{1}{2}\right)^{t/5715}$$

where t is the time (in years) and $t = 0$ represents the time when the organic material died.

75. $R = 0.32 \times 10^{-12}$

76. $R = 0.27 \times 10^{-12}$

77. $R = 0.22 \times 10^{-12}$

78. $R = 0.13 \times 10^{-12}$

AP/Wide World Photos

In 1995, archeologist Johan Reinhard discovered the frozen remains of a young Incan woman atop Mt. Ampato in Peru. Carbon dating was used to estimate the age of the "Ice Maiden" at 500 years.

79. *Learning Theory* Students in a mathematics class were given an exam and then retested monthly with equivalent exams. The average score S (on a 100-point scale) for the class can be modeled by

$$S = 80 - 14\ln(t + 1), \qquad 0 \le t \le 12$$

where t is the time in months.

(a) What was the average score on the original exam?

(b) What was the average score after 4 months?

(c) After how many months was the average score 46?

80. *Research Project* Use a graphing utility to graph

$$y = 10\ln\left(\frac{10 + \sqrt{100 - x^2}}{10}\right) - \sqrt{100 - x^2}$$

over the interval $(0, 10]$. This graph is called a *tractrix* or *pursuit curve*. Use your school's library, the Internet, or some other reference source to find information about a tractrix. Explain how such a curve can arise in a real-life setting.

81. Demonstrate that

$$\frac{\ln x}{\ln y} \ne \ln\frac{x}{y} = \ln x - \ln y$$

by completing the table.

x	y	$\dfrac{\ln x}{\ln y}$	$\ln\dfrac{x}{y}$	$\ln x - \ln y$
1	2			
3	4			
10	5			
4	0.5			

82. Complete the table using $f(x) = \dfrac{\ln x}{x}$.

x	1	5	10	10^2	10^4	10^6
$f(x)$						

(a) Use the table to estimate the limit: $\lim_{x\to\infty} f(x)$.

(b) Use a graphing utility to estimate the relative extrema of f.

In Exercises 83 and 84, use a graphing utility to verify that the functions are equivalent for $x > 0$.

83. $f(x) = \ln\dfrac{x^2}{4}$

$g(x) = 2\ln x - \ln 4$

84. $f(x) = \ln\sqrt{x(x^2 + 1)}$

$g(x) = \frac{1}{2}[\ln x + \ln(x^2 + 1)]$

True or False? In Exercises 85–90, determine whether the statement is true or false given that $f(x) = \ln x$. If it is false, explain why or give an example that shows it is false.

85. $f(0) = 0$

86. $f(ax) = f(a) + f(x), \quad a > 0, x > 0$

87. $f(x - 2) = f(x) - f(2), \quad x > 2$

88. $\sqrt{f(x)} = \frac{1}{2}f(x)$

89. If $f(u) = 2f(v)$, then $v = u^2$.

90. If $f(x) < 0$, then $0 < x < 1$.

PREREQUISITE REVIEW 4.5

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–6, expand the logarithmic expression.

1. $\ln(x + 1)^2$
2. $\ln x(x + 1)$
3. $\ln \dfrac{x}{x + 1}$
4. $\ln\left(\dfrac{x}{x - 3}\right)^3$
5. $\ln \dfrac{4x(x - 7)}{x^2}$
6. $\ln x^3(x + 1)$

In Exercises 7 and 8, find dy/dx implicitly.

7. $y^2 + xy = 7$
8. $x^2y - xy^2 = 3x$

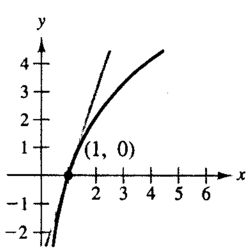
In Exercises 9 and 10, find the second derivative of f.

9. $f(x) = x^2(x + 1) - 3x^3$
10. $f(x) = -\dfrac{1}{x^2}$

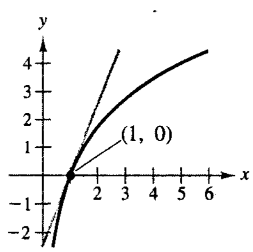
EXERCISES 4.5

In Exercises 1–4, find the slope of the tangent line to the graph of the function at the point (1, 0).

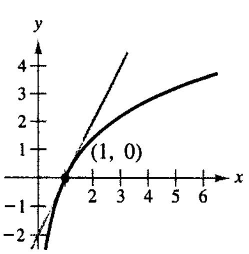
1. $y = \ln x^3$



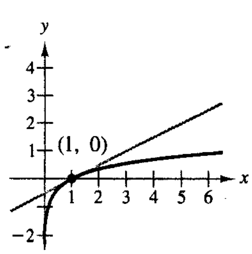
2. $y = \ln x^{5/2}$



3. $y = \ln x^2$



4. $y = \ln x^{1/2}$



In Exercises 5–26, find the derivative of the function.

5. $y = \ln x^2$
6. $f(x) = \ln 2x$
7. $y = \ln(x^2 + 3)$
8. $f(x) = \ln(1 - x^2)$
9. $y = \ln\sqrt{x^4 - 4x}$
10. $y = \ln(1 - x)^{3/2}$
11. $y = \frac{1}{2}(\ln x)^6$
12. $y = (\ln x^2)^2$
13. $f(x) = x^2 \ln x$
14. $y = \dfrac{\ln x}{x^2}$
15. $y = \ln\left(x\sqrt{x^2 - 1}\right)$
16. $y = \ln \dfrac{x}{x^2 + 1}$
17. $y = \ln \dfrac{x}{x + 1}$
18. $y = \ln \dfrac{x^2}{x^2 + 1}$
19. $y = \ln \sqrt[3]{\dfrac{x - 1}{x + 1}}$
20. $y = \ln \sqrt{\dfrac{x + 1}{x - 1}}$
21. $y = \ln \dfrac{\sqrt{4 + x^2}}{x}$
22. $y = \ln\left(x\sqrt{4 + x^2}\right)$
23. $g(x) = e^{-x} \ln x$
24. $f(x) = x \ln e^{x^2}$
25. $g(x) = \ln \dfrac{e^x + e^{-x}}{2}$
26. $f(x) = \ln \dfrac{1 + e^x}{1 - e^x}$

In Exercises 27–30, wr

27. 2^x

29. $\log_4 x$

In Exercises 31–36, u

Round to three decim

31. $\log_2 48$

33. $\log_3 \frac{1}{2}$

35. $\log_{1/5} 31$

In Exercises 37–46, fin

37. $y = 3^x$

39. $f(x) = \log_2 x$

41. $h(x) = 4^{2x-3}$

43. $y = \log_{10}(x^2 + 6$

45. $y = x2^x$

In Exercises 47–50, det

the function at the giv

Function

47. $y = x \ln x$

48. $y = \dfrac{\ln x}{x}$

49. $y = \log_3(3x + 7)$

50. $g(x) = \log_2(3x -$

In Exercises 51–54, find

51. $x^2 - 3 \ln y + y^2 =$

52. $\ln xy + 5x = 30$

53. $4x^3 + \ln y^2 + 2y$

54. $4xy + \ln(x^2y) = 7$

In Exercises 55–58, find

55. $f(x) = x \ln \sqrt{x} +$

56. $f(x) = 3 + 2 \ln x$

57. $f(x) = 5^x$

58. $f(x) = \log_{10} x$

59. ***Sound Intensity*** of decibels β and square centimeter is

$$\beta = 10 \log_{10}\Big(\frac{}{10}$$

Find the rate of cha intensity is 10^{-4} wa sections. You will

$= (\ln x^2)^2$

$= \dfrac{\ln x}{x^2}$

$= \ln \dfrac{x}{x^2 + 1}$

In Exercises 27–30, write the expression with base e.

27. 2^x **28.** 3^x

29. $\log_4 x$ **30.** $\log_3 x$

In Exercises 31–36, use a calculator to evaluate the logarithm. Round to three decimal places.

31. $\log_2 48$ **32.** $\log_5 12$

33. $\log_3 \frac{1}{2}$ **34.** $\log_7 \frac{2}{9}$

35. $\log_{1/5} 31$ **36.** $\log_{2/3} 32$

In Exercises 37–46, find the derivative of the function.

37. $y = 3^x$ **38.** $y = \left(\frac{1}{4}\right)^x$

39. $f(x) = \log_2 x$ **40.** $g(x) = \log_5 x$

41. $h(x) = 4^{2x-3}$ **42.** $y = 6^{5x}$

43. $y = \log_{10}(x^2 + 6x)$ **44.** $f(x) = 10^{x^2}$

45. $y = x2^x$ **46.** $y = x3^{x+1}$

In Exercises 47–50, determine an equation of the tangent line to the function at the given point.

Function	*Point*
47. $y = x \ln x$	$(1, 0)$
48. $y = \dfrac{\ln x}{x}$	$\left(e, \dfrac{1}{e}\right)$
49. $y = \log_3(3x + 7)$	$\left(\dfrac{2}{3}, 2\right)$
50. $g(x) = \log_2(3x - 1)$	$(11, 5)$

In Exercises 51–54, find dy/dx implicitly.

51. $x^2 - 3 \ln y + y^2 = 10$

52. $\ln xy + 5x = 30$

53. $4x^3 + \ln y^2 + 2y = 2x$

54. $4xy + \ln(x^2y) = 7$

In Exercises 55–58, find the second derivative of the function.

55. $f(x) = x \ln \sqrt{x} + 2x$

56. $f(x) = 3 + 2 \ln x$

57. $f(x) = 5^x$

58. $f(x) = \log_{10} x$

59. ***Sound Intensity*** The relationship between the number of decibels β and the intensity of a sound I in watts per square centimeter is given by

$$\beta = 10 \log_{10}\left(\frac{I}{10^{-16}}\right).$$

Find the rate of change in the number of decibels when the intensity is 10^{-4} watts per square centimeter.

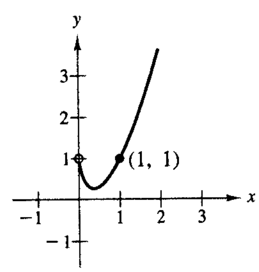
60. ***Chemistry*** The temperatures T (°F) at which water boils at selected pressures p (pounds per square inch) can be modeled by

$$T = 87.97 + 34.96 \ln p + 7.91\sqrt{p}.$$

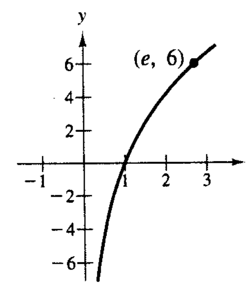
Find the rate of change of the temperature when the pressure is 60 pounds per square inch.

In Exercises 61–66, find the slope of the graph at the indicated point. Then write an equation of the tangent line at the point.

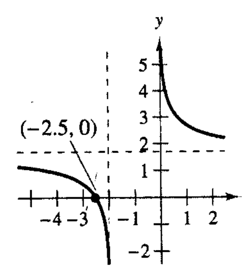
61. $f(x) = 1 + 2x \ln x$



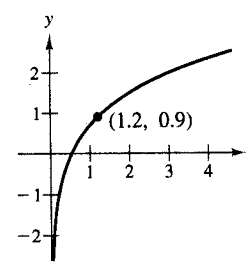
62. $f(x) = 2 \ln x^3$



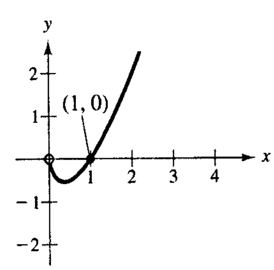
63. $f(x) = \ln \dfrac{5(x + 2)}{x}$



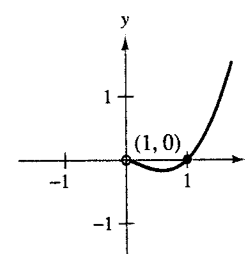
64. $f(x) = \ln\left(x\sqrt{x + 3}\right)$



65. $f(x) = x \log_2 x$



66. $f(x) = x^2 \log_3 x$



In Exercises 67–72, graph and analyze the function. Include any relative extrema and points of inflection in your analysis. Use a graphing utility to verify your results.

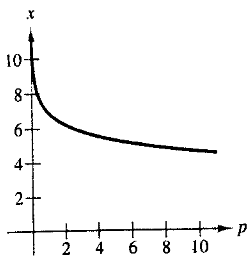
67. $y = x - \ln x$ **68.** $y = \frac{1}{2}x^2 - \ln x$

69. $y = \dfrac{\ln x}{x}$ **70.** $y = x \ln x$

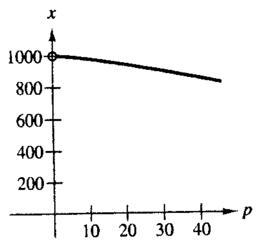
71. $y = x^2 \ln x$ **72.** $y = (\ln x)^2$

In Exercises 73–76, find dx/dp for the demand function. Interpret this rate of change when the price is \$10.

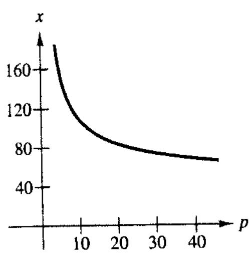
73. $x = \ln\dfrac{1000}{p}$



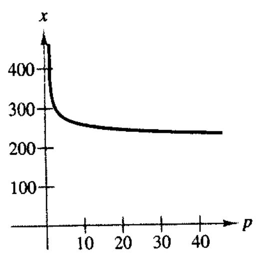
74. $x = 1000 - p \ln p$



75. $x = \dfrac{500}{\ln(p^2 + 1)}$



76. $x = 300 - 50 \ln(\ln p)$



77. Solve the demand function in Exercise 73 for p. Use the result to find dp/dx. Then find the rate of change when $p = \$10$. What is the relationship between this derivative and dx/dp?

78. Solve the demand function in Exercise 75 for p. Use the result to find dp/dx. Then find the rate of change when $p = \$10$. What is the relationship between this derivative and dx/dp?

79. ***Minimum Average Cost*** The cost of producing x units of a product is modeled by

$$C = 500 + 300x - 300 \ln x, \qquad x \geq 1.$$

(a) Find the average cost function $\overline{C}$.

(b) Analytically find the minimum average cost.

(c) Use a graphing utility to confirm your results.

80. ***Minimum Average Cost*** The cost of producing x units of a product is modeled by

$$C = 100 + 25x - 120 \ln x, \quad x \geq 1.$$

(a) Find the average cost function $\overline{C}$.

(b) Analytically find the minimum average cost.

(c) Use a graphing utility to confirm your results.

81. ***Consumer Trends*** The retail sales S (in billions of dollars per year) of e-commerce companies in the United States from 1998 to 2002 can be modeled by $S = -210.3 + 103.30 \ln t$, where $t = 8$ corresponds to 1998. *(Source: Consumer Online Report)*

(a) Use a graphing utility to graph S over the interval $[8, 12]$.

(b) Estimate the amount of sales in 2000.

(c) At what rate were the sales changing in 2000?

82. ***Home Mortgage*** The term t (in years) of a \$120,000 home mortgage at 10% interest can be approximated by

$$t = \frac{5.315}{-6.7968 + \ln x}, \quad x > 1000$$

where x is the monthly payment in dollars.

(a) Use a graphing utility to graph the model.

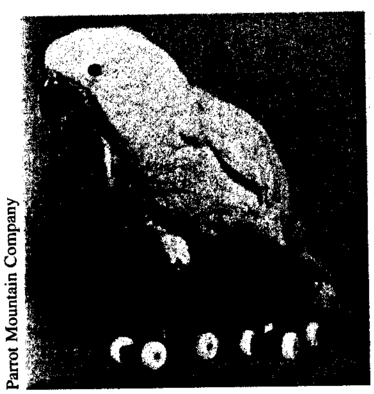
(b) Use the model to approximate the term of a home mortgage for which the monthly payment is \$1167.41. What is the total amount paid?

(c) Use the model to approximate the term of a home mortgage for which the monthly payment is \$1068.45. What is the total amount paid?

(d) Find the instantaneous rate of change of t with respect to x when $x = 1167.41$ and $x = 1068.45$.

(e) Write a short paragraph describing the benefit of the higher monthly payment.

BUSINESS CAPSULE



Parrot Mountain Company

The Parrot Mountain Company is a mail-order business started by Angel Santiago in 1990. The company offers the famous Parrot Mountain roller skates used by bird trainers and carries more than 800 bird-related items.

83. ***Research Project*** Use your school's library, the Internet, or some other reference source to research information about a mail-order company, such as that mentioned above. Collect data about the company (sales or membership over a 20-year period, for example) and find a mathematical model to represent the data.

4.6

Exponential Gr

In this section, you wi
Real-life situations t
substance or populatic
amount of the substan
sition of a radioactiv
substance at a given in
the equation below.

Rate of change of y

In this equation, k is a
tion is shown below.

Law of Exponenti

If y is a positive q
proportional to the

$y = Ce^{kt}$

where C is the **init**
Exponential grow
$k < 0$.

PROOF Because the

$\dfrac{dy}{dt} = ky.$

You can see that $y =$
obtain $dy/dt = kCe^{kt}$

$\dfrac{dy}{dt} = kCe^{kt}$

$= k(Ce^{kt})$

$= ky.$

STUDY TIP

In the model $y = C$

$y = Ce^{k(0)} = C$

PREREQUISITE REVIEW 4.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–4, solve the equation for k.

1. $12 = 24e^{4k}$

2. $10 = 3e^{5k}$

3. $25 = 16e^{-0.01k}$

4. $22 = 32e^{-0.02k}$

In Exercises 5–8, find the derivative of the function.

5. $y = 32e^{0.23t}$

6. $y = 18e^{0.072t}$

7. $y = 24e^{-1.4t}$

8. $y = 25e^{-0.001t}$

In Exercises 9–12, simplify the expression.

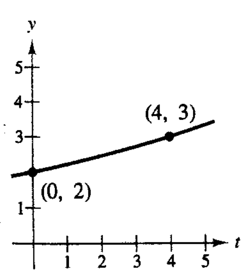
9. $e^{\ln 4}$ **10.** $4e^{\ln 3}$

11. $e^{\ln(2x+1)}$ **12.** $e^{\ln(x^2+1)}$

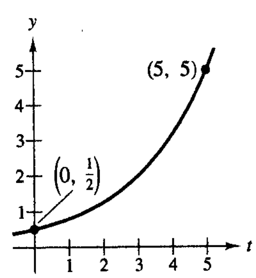
EXERCISES 4.6

In Exercises 1–6, find the exponential function $y = Ce^{kt}$ that passes through the two given points.

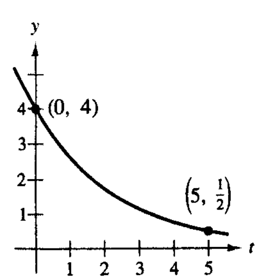
1. $y = Ce^{kt}$



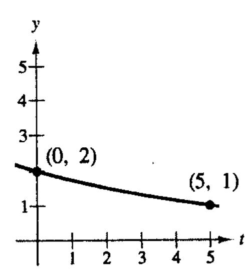
2. $y = Ce^{kt}$



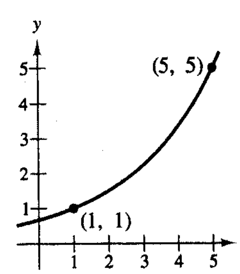
3. $y = Ce^{kt}$



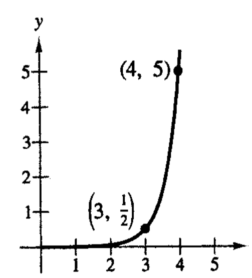
4. $y = Ce^{kt}$



5. $y = Ce^{kt}$



6. $y = Ce^{kt}$



In Exercises 7–10, use the given information to write an equation for y. Confirm your result analytically by showing that the function satisfies the equation $dy/dt = Cy$. Does the function represent exponential growth or exponential decay?

7. $\dfrac{dy}{dt} = 2y, \quad y = 10$ when $t = 0$

8. $\dfrac{dy}{dt} = -\dfrac{2}{3}y, \quad y = 20$ when $t = 0$

9. $\dfrac{dy}{dt} = -4y, \quad y = 30$ when $t = 0$

10. $\dfrac{dy}{dt} = 5.2y, \quad y = 18$ when $t = 0$

Radioactive Decay In Exercises 11–16, complete the table for each radioactive isotope.

	Isotope	Half-life (in years)	Initial quantity	Amount after 1000 years	Amount after 10,000 years
11.	^{226}Ra	1599	10 grams		
12.	^{226}Ra	1599		1.5 grams	
13.	^{14}C	5715			2 grams
14.	^{14}C	5715	3 grams		
15.	^{239}Pu	24,100		2.1 grams	
16.	^{239}Pu	24,100			0.4 gram

17. ***Radioactive Decay*** What percent of a present amount of radioactive radium (^{226}Ra) will remain after 900 years?

18. ***Radioactive Decay*** Find the half-life of a radioactive material if after 1 year 99.57% of the initial amount remains.

19. ***Carbon Dating*** ^{14}C dating assumes that the carbon dioxide on the Earth today has the same radioactive content as it did centuries ago. If this is true, then the amount of ^{14}C absorbed by a tree that grew several centuries ago should be the same as the amount of ^{14}C absorbed by a similar tree today. A piece of ancient charcoal contains only 15% as much of the radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal? (The half-life of ^{14}C is 5715 years.)

20. ***Carbon Dating*** Repeat Exercise 19 for a piece of charcoal that contains 30% as much radioactive carbon as a modern piece.

In Exercises 21 and 22, find exponential models

$$y_1 = Ce^{k_1 t} \quad \text{and} \quad y_2 = C(2)^{k_2 t}$$

that pass through the points. Compare the values of k_1 and k_2. Briefly explain your results.

21. $(0, 5), (12, 20)$

22. $(0, 8), \left(20, \frac{1}{2}\right)$

23. ***Population Growth*** The number of a certain type of bacteria increases continuously at a rate proportional to the number present. There are 150 present at a given time and 450 present 5 hours later.

(a) How many will there be 10 hours after the initial time?

(b) How long will it take for the population to double?

(c) Does the answer to part (b) depend on the starting time? Explain your reasoning.

24. ***School Enrollment*** In 1960, the total enrollment in public universities and colleges in the United States was 2.3 million students. By 2000, enrollment had risen to 12.0 million students. Assume enrollment can be modeled by exponential growth. *(Source: U.S. Census Bureau)*

(a) Estimate the total enrollments in 1970, 1980, and 1990.

(b) How many years until the enrollment doubles from the 2000 figure?

(c) By what percent is the enrollment increasing each year?

Compound Interest In Exercises 25–30, complete the table for an account in which interest is compounded continuously.

	Initial investment	Annual rate	Time to double	Amount after 10 years	Amount after 25 years
25.	\$1,000	12%			
26.	\$20,000	$10\frac{1}{2}$%			
27.	\$750		$7\frac{3}{4}$ years		
28.	\$10,000		5 years		
29.	\$500			\$1292.85	
30.	\$2,000				\$6008.33

31. ***Effective Yield*** The effective yield is the annual rate i that will produce the same interest per year as the nominal rate r compounded n times per year.

(a) For a rate r that is compounded n times per year, show that the effective yield is

$$i = \left(1 + \frac{r}{n}\right)^n - 1.$$

(b) Find the effective yield for a nominal rate of 6%, compounded monthly.

32. ***Effective Yield*** The effective yield is the annual rate i that will produce the same interest per year as the nominal rate r.

(a) For a rate r that is compounded continuously, show that the effective yield is $i = e^r - 1$.

(b) Find the effective yield for a nominal rate of 6%, compounded continuously.

Effective Yield In Exercises 33 and 34, use the results of Exercises 31 and 32 to complete the table showing the effective yield for a nominal rate of r.

Number of compoundings per year	4	12	365	Continuous
Effective yield				

33. $r = 5\%$

34. $r = 7\frac{1}{2}\%$

35. ***Investment:*** ...
for an investme...
where r is the ...

36. ***Investment:*** ...
Exercise 35 to ...
ment to double...

37. ***Revenue*** Th...
\$83.3 million...
(Source: Sonic...

(a) Use an ex...
revenue in ...

(b) Use a linea...

(c) Use a grap...
(a) and (b)...

38. ***Sales*** The sal...
and wheel spor...
1990 and \$10...
Sporting Goods...

(a) Use the *reg...*
an exponent...
data.

(b) Use the exp...
in 2006.

(c) Use the line...

(d) Use a grap...
(a). Which ...

39. ***Sales*** The cu...
new product aft...
modeled by

$S = Ce^{k/t}$.

During the first ...
point for the ma...
as $t \to \infty$ is 30,0...

(a) Solve for C ...

(b) How many ...

(c) Use a graphi...

40. ***Sales*** The cun...
new product afte...
modeled by

$S = 30(1 - \ldots$

During the first ...

(a) Solve for k i...

(b) What is the ...

(c) How many u...

(d) Use a graphi...

he total enrollment in
the United States was
rollment had risen to
llment can be modeled
U.S. Census Bureau)
1970, 1980, and 1990.
lment doubles from the

it increasing each year?

-30, complete the table
unded continuously.

nount fter years	Amount after 25 years
292.85	$6008.33

ld is the annual rate i
er year as the nominal

n times per year, show

nominal rate of 6%,

ld is the annual rate i
er year as the nominal

ontinuously, show that

nominal rate of 6%,

34, use the results of
showing the effective

ntinuous

$7\frac{1}{2}\%$

35. *Investment: Rule of 70* Verify that the time necessary for an investment to double its value is approximately $70/r$, where r is the annual interest rate entered as a percent.

36. *Investment: Rule of 70* Use the Rule of 70 from Exercise 35 to approximate the times necessary for an investment to double in value if (a) $r = 10\%$ and (b) $r = 7\%$.

37. *Revenue* The revenues for Sonic Corporation were \$83.3 million in 1993 and \$446.6 million in 2003. *(Source: Sonic Corporation)*

(a) Use an exponential growth model to estimate the revenue in 2008.

(b) Use a linear model to estimate the 2008 revenue.

(c) Use a graphing utility to graph the models from parts (a) and (b). Which model is more accurate?

38. *Sales* The sales (in millions of dollars) for in-line skating and wheel sports in the United States were \$150 million in 1990 and \$1074 million in 2000. *(Source: National Sporting Goods Association)*

(a) Use the *regression* feature of a graphing utility to find an exponential growth model and a linear model for the data.

(b) Use the exponential growth model to estimate the sales in 2006.

(c) Use the linear model to estimate the sales in 2006.

(d) Use a graphing utility to graph the models from part (a). Which model is more accurate?

39. *Sales* The cumulative sales S (in thousands of units) of a new product after it has been on the market for t years are modeled by

$$S = Ce^{k/t}.$$

During the first year, 5000 units were sold. The saturation point for the market is 30,000 units. That is, the limit of S as $t \to \infty$ is 30,000.

(a) Solve for C and k in the model.

(b) How many units will be sold after 5 years?

(c) Use a graphing utility to graph the sales function.

40. *Sales* The cumulative sales S (in thousands of units) of a new product after it has been on the market for t years are modeled by

$$S = 30(1 - 3^{kt}).$$

During the first year, 5000 units were sold.

(a) Solve for k in the model.

(b) What is the saturation point for this product?

(c) How many units will be sold after 5 years?

(d) Use a graphing utility to graph the sales function.

41. *Learning Curve* The management of a factory finds that the maximum number of units a worker can produce in a day is 30. The learning curve for the number of units N produced per day after a new employee has worked t days is modeled by $N = 30(1 - e^{kt})$. After 20 days on the job, a worker is producing 19 units in a day. How many days should pass before this worker is producing 25 units per day?

42. *Learning Curve* The management in Exercise 41 requires that a new employee be producing at least 20 units per day after 30 days on the job.

(a) Find a learning curve model that describes this minimum requirement.

(b) Find the number of days before a minimal achiever is producing 25 units per day.

43. *Profit* Because of a slump in the economy, a company finds that its annual profits have dropped from \$742,000 in 1998 to \$632,000 in 2000. If the profit follows an exponential pattern of decline, what is the expected profit for 2003? (Let $t = 0$ correspond to 1998.)

44. *Revenue* A small business assumes that the demand function for one of its new products can be modeled by $p = Ce^{kx}$. When $p = \$45$, $x = 1000$ units, and when $p = \$40$, $x = 1200$ units.

(a) Solve for C and k.

(b) Find the values of x and p that will maximize the revenue for this product.

45. *Revenue* Repeat Exercise 44 given that when $p = \$5$, $x = 300$ units, and when $p = \$4$, $x = 400$ units.

46. *Forestry* The value V (in dollars) of a tract of timber can be modeled by $V = 100{,}000e^{0.75\sqrt{t}}$, where $t = 0$ corresponds to 1990. If money earns interest at a rate of 4%, compounded continuously, then the present value A of the timber at any time t is $A = Ve^{-0.04t}$. Find the year in which the timber should be harvested to maximize the present value.

47. *Forestry* Repeat Exercise 46 using the model

$$V = 100{,}000e^{0.6\sqrt{t}}.$$

48. *Earthquake Intensity* On the Richter scale, the magnitude R of an earthquake of intensity I is given by

$$R = \frac{\ln I - \ln I_0}{\ln 10}$$

where I_0 is the minimum intensity used for comparison. Assume $I_0 = 1$.

(a) Find the intensity of the 1906 San Francisco earthquake in which $R = 8.3$.

(b) Find the factor by which the intensity is increased when the value of R is doubled.

(c) Find dR/dI.