Math 21B - Homework Set 1

Section 5.1:

1. $f(x) = x^3$ between $x = 0$ and $x = 1$.

   a. Estimate using lower sum with two rectangles of equal width:

   If we want two rectangles of equal width, we will let $\Delta x = \frac{1}{2}$. The function $f(x)$ is increasing on $[0, 1]$, so the height of each rectangle is given by the value of f at its left endpoint.

$$f(0) = 0$$
$$f\left(\frac{1}{2}\right) = \frac{1}{8}$$

   Thus we get:

$$\begin{aligned} A &\approx 0 \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{2} \\ &= \frac{1}{16} \end{aligned}$$

   b. Estimate using lower sum with four rectangles of equal width:

   We will let $\Delta x = \frac{1}{4}$ and the heighths of the rectangles are given by the value of f at their respective left endpoints.

$$f(0) = 0$$
$$f\left(\frac{1}{4}\right) = \frac{1}{64}$$
$$f\left(\frac{1}{2}\right) = \frac{1}{8}$$
$$f\left(\frac{3}{4}\right) = \frac{27}{64}$$

   Thus we get:

$$\begin{aligned} A &\approx 0 \cdot \frac{1}{4} + \frac{1}{64} \cdot \frac{1}{4} + \frac{1}{8} \cdot \frac{1}{4} + \frac{27}{64} \cdot \frac{1}{4} \\ &= \frac{36}{256} \\ &= \frac{9}{64} \end{aligned}$$

c. Estimate using upper sum with two rectangles of equal width:

We will let $\Delta x = \frac{1}{2}$ and the heighths of the rectangles are given by the value of f at their respective right endpoints.

$$f\left(\frac{1}{2}\right) = \frac{1}{8}$$
$$f(1) = 1$$

Thus we get:

$$\begin{aligned} A &\approx \frac{1}{8} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} \\ &= \frac{9}{16} \end{aligned}$$

d. Estimate using upper sum with four rectangles of equal width:

We will let $\Delta x = \frac{1}{4}$ and the heighths of the rectangles are given by the value of f at their respective right endpoints.

$$f\left(\frac{1}{4}\right) = \frac{1}{64}$$
$$f\left(\frac{1}{2}\right) = \frac{1}{8}$$
$$f\left(\frac{3}{4}\right) = \frac{27}{64}$$
$$f(1) = 1$$

Thus we get:

$$\begin{aligned} A &\approx \frac{1}{64} \cdot \frac{1}{4} + \frac{1}{8} \cdot \frac{1}{4} + \frac{27}{64} \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} \\ &= \frac{100}{256} \\ &= \frac{25}{64} \end{aligned}$$

2. $f(x) = \frac{1}{x}$ between $x = 1$ and $x = 5$.

a. Estimate using lower sum with two rectangles of equal width:

If we want two rectangles of equal width, we will let $\Delta x = 2$. The function $f(x)$ is decreasing on $[1, 5]$, so the height of each rectangle is given by the value of f at its right endpoint.

$$f(3) = \frac{1}{3}$$
$$f(5) = \frac{1}{5}$$

Thus we get:

$$\begin{aligned} A &\approx \frac{1}{3} \cdot 2 + \frac{1}{5} \cdot 2 \\ &= \frac{16}{15} \end{aligned}$$

b. Estimate using lower sum with four rectangles of equal width:

We will let $\Delta x = 1$ and the heighths of the rectangles are given by the value of f at their respective right endpoints.

$$f(2) = \frac{1}{2}$$
$$f(3) = \frac{1}{3}$$
$$f(4) = \frac{1}{4}$$
$$f(5) = \frac{1}{5}$$

Thus we get:

$$\begin{aligned} A &\approx \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{1}{5} \cdot 1 \\ &= \frac{77}{60} \end{aligned}$$

c. Estimate using upper sum with two rectangles of equal width:

We will let $\Delta x = 2$ and the heighths of the rectangles are given by the value of f at their respective left endpoints.

$$f(1) = 1$$
$$f(3) = \frac{1}{3}$$

Thus we get:

$$
\begin{aligned}
A &\approx 1 \cdot 2 + \frac{1}{3} \cdot 2 \\
&= \frac{8}{3}
\end{aligned}
$$

d. Estimate using upper sum with four rectangles of equal width:

We will let $\Delta x = 1$ and the heighths of the rectangles are given by the value of f at their respective left endpoints.

$$
\begin{aligned}
f(1) &= 1 \\
f(2) &= \frac{1}{2} \\
f(3) &= \frac{1}{3} \\
f(4) &= \frac{1}{4}
\end{aligned}
$$

Thus we get:

$$
\begin{aligned}
A &\approx 1 \cdot 1 + \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 + \frac{1}{4} \cdot 1 \\
&= \frac{25}{12}
\end{aligned}
$$

3. • 2 rectangles

We will let $\Delta x = \frac{1}{2}$. To get the height of the rectangles we will use:

$$
\begin{aligned}
f\left(\frac{1}{4}\right) &= \frac{1}{16} \\
f\left(\frac{3}{4}\right) &= \frac{9}{16}
\end{aligned}
$$

Thus we get:

$$
\begin{aligned}
A &\approx \frac{1}{16} \cdot \frac{1}{2} + \frac{9}{16} \cdot \frac{1}{2} \\
&= \frac{10}{32} \\
&= \frac{5}{16}
\end{aligned}
$$

- 4 rectangles
  We will let $\Delta x = \frac{1}{4}$. To get the height of the rectangles we will use:

$$f\left(\frac{1}{8}\right) = \frac{1}{64}$$
$$f\left(\frac{3}{8}\right) = \frac{9}{64}$$
$$f\left(\frac{5}{8}\right) = \frac{25}{64}$$
$$f\left(\frac{7}{8}\right) = \frac{49}{64}$$

  Thus we get:

$$\begin{aligned} A &\approx \frac{1}{64}\cdot\frac{1}{4} + \frac{9}{64}\cdot\frac{1}{4} + \frac{25}{64}\cdot\frac{1}{4} + \frac{49}{64}\cdot\frac{1}{4} \\ &= \frac{84}{256} \\ &= \frac{21}{64} \end{aligned}$$

4. (a) I think this question had a misprint, and they meant to ask about the *height* after 5 sec, but I will answer the question as stated in the book. Since gravity points down, we have

$$v'(t) = -g = -32,$$

and hence $v(t) = -32t + C$. The initial condition $v(0) = 400$ implies that $C = 400$ and hence $v(t) = -32t + 400$. Hence

$$v(5) = -32 \cdot 5 + 400 = 240 \text{ ft/sec}.$$

(b) We use 5 subintervals of equal width $\Delta t = 1$. Since the velocity is decreasing, we get a lower estimate by evaluating $v(t)$ at the right endpoint of each subinterval. The lower estimate is

$$\begin{aligned} v(1)\Delta t + v(2)\Delta t + v(3)\Delta t + v(4)\Delta t + v(5)\Delta t &= \\ 368 \cdot 1 + 336 \cdot 1 + 304 \cdot 1 + 272 \cdot 1 + 240 \cdot 1 &= 1520\,\text{ft}. \end{aligned}$$

5. We will let $\Delta x = \frac{1}{2}$. To get the height of the rectangles we will use:

$$f\left(\frac{1}{4}\right) = \frac{1}{64}$$
$$f\left(\frac{3}{4}\right) = \frac{27}{64}$$
$$f\left(\frac{5}{4}\right) = \frac{125}{64}$$
$$f\left(\frac{7}{4}\right) = \frac{343}{64}$$

The area A under the graph:

$$\begin{aligned} A &\approx \frac{1}{64} \cdot \frac{1}{2} + \frac{27}{64} \cdot \frac{1}{2} + \frac{125}{64} \cdot \frac{1}{2} + \frac{343}{64} \cdot \frac{1}{2} \\ &= \frac{496}{128} \\ &= \frac{31}{8} \end{aligned}$$

So the average value of f on $[0, 2]$ is approximately $\frac{1}{2} \cdot \frac{31}{8} = \frac{31}{16}$.

Section 5.2:

1.

$$\begin{aligned} \sum_{k=1}^{2} \frac{6k}{k+1} &= \frac{6}{2} + \frac{12}{3} \\ &= 3 + 4 \\ &= 7 \end{aligned}$$

2.

$$\begin{aligned} \sum_{k=1}^{3} \frac{k-1}{k} &= \frac{0}{1} + \frac{1}{2} + \frac{2}{3} \\ &= 0 + \frac{1}{2} + \frac{2}{3} \\ &= \frac{7}{6} \end{aligned}$$

3.

$$\begin{aligned} \sum_{k=1}^{5} \sin(k\pi) &= \sin(\pi) + \sin(2\pi) + \sin(3\pi) + \sin(4\pi) + \sin(5\pi) \\ &= 0 \end{aligned}$$

4. ALL

   a. $\sum_{k=1}^{6} 2^{k-1} = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 1 + 2 + 4 + 8 + 16 + 32$

   b. $\sum_{k=0}^{5} 2^k == 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 1 + 2 + 4 + 8 + 16 + 32$

   c. $\sum_{k=-1}^{4} 2^{k+1} = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 1 + 2 + 4 + 8 + 16 + 32$

5. (a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \sum_{k=1}^{4} \frac{1}{2^k}$

   (b) $2 + 4 + 6 + 8 + 10 = \sum_{k=1}^{5} 2k$

   (c) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \sum_{k=1}^{5} \frac{(-1)^{k+1}}{k}$

6. Suppose that $\sum_{k=1}^{n} a_k = -5$ and $\sum_{k=1}^{n} b_k = 6$. Find the values of:

   a. $\sum_{k=1}^{n} 3a_k = 3 \sum_{k=1}^{n} a_k = -15$

   b. $\sum_{k=1}^{n} \frac{b_k}{6} = \frac{1}{6} \sum_{k=1}^{n} b_k = 1$

   c. $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = 1$

   d. $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k = -11$

   e. $\sum_{k=1}^{n} (b_k - 2a_k) = \sum_{k=1}^{n} b_k - 2 \sum_{k=1}^{n} a_k = 16$

7. If we want to take the upper sum using n equal subintervals, we will let $\Delta x = \frac{1}{n}$. Note that f is increasing on $[0, 1]$, so to get the upper sum, we will evaluate the function at the right endpoint of each subinterval.

$$\begin{aligned}
A &\approx f\left(\frac{1}{n}\right)\cdot\frac{1}{n} + f\left(\frac{2}{n}\right)\cdot\frac{1}{n} + \ldots + f\left(\frac{n-1}{n}\right)\cdot\frac{1}{n} + f(1)\cdot\frac{1}{n} \\
&= \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n} \\
&= \sum_{i=1}^{n}\left(\frac{3i}{n} + \frac{2i^2}{n^2}\right)\cdot\frac{1}{n} \\
&= \frac{3}{n^2}\sum_{i=1}^{n} i + \frac{2}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{3}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{2}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \frac{3+3/n}{2} + \frac{2+3/n+1/n^2}{3}
\end{aligned}$$

Taking the limits as $n \to \infty$ gives:

$$\lim_{n\to\infty} \frac{3+3/n}{2} + \frac{2+3/n+1/n^2}{3} = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}$$