

## Math 21B-B - Homework Set 4

### Section 5.6:

1. (a)  $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x \, dx$

Let  $u = \tan x$ ,  $du = \sec^2 x \, dx$ .

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \tan x \sec^2 x \, dx &= \int_{\tan 0}^{\tan \frac{\pi}{4}} u \, du \\ &= \int_0^1 u \, du \\ &= \left. \frac{1}{2} u^2 \right|_0^1 \\ &= \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 \\ &= \frac{1}{2} \end{aligned}$$

(b)  $\int_{-\frac{\pi}{4}}^0 \tan x \sec^2 x \, dx$

$$\begin{aligned} \int_{-\frac{\pi}{4}}^0 \tan x \sec^2 x \, dx &= \int_{\tan -\frac{\pi}{4}}^{\tan 0} u \, du \\ &= \int_{-1}^0 u \, du \\ &= \left. \frac{1}{2} u^2 \right|_{-1}^0 \\ &= \frac{1}{2}(0)^2 - \frac{1}{2}(-1)^2 \\ &= -\frac{1}{2} \end{aligned}$$

2. (a)  $\int_0^{\sqrt{7}} t (t^2 + 1)^{1/3} \, dt$

Let  $u = t^2 + 1$ ,  $du = 2t \, dt$ .

$$\begin{aligned}
\int_0^{\sqrt{7}} t(t^2+1)^{1/3} dt &= \int_1^8 \frac{1}{2} u^{1/3} dt \\
&= \frac{3}{8} u^{4/3} \Big|_1^8 \\
&= \frac{3}{8} \cdot 8^{4/3} - \frac{3}{8} \cdot 1^{4/3} \\
&= 6 - \frac{3}{8} \\
&= \frac{45}{8}
\end{aligned}$$

(b)  $\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3} dt$

$$\begin{aligned}
\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3} dt &= \int_8^0 \frac{1}{2} u^{1/3} du \\
&= -\int_0^8 \frac{1}{2} u^{1/3} du \\
&= -\frac{45}{8} \quad \text{by (a.)}
\end{aligned}$$

3. (a)  $\int_0^1 \frac{x^3}{\sqrt{x^4+9}} dx$

Let  $u = x^4 + 9$ ,  $du = 4x^3 dx$

$$\begin{aligned}
\int_0^1 \frac{x^3}{\sqrt{x^4+9}} dx &= \int_9^{10} \frac{1}{4\sqrt{u}} du \\
&= \int_9^{10} \frac{1}{4} u^{-1/2} du \\
&= \frac{1}{4} \cdot 2 u^{1/2} \Big|_9^{10} \\
&= \frac{1}{2} \sqrt{10} - \frac{1}{2} \sqrt{9} \\
&= \frac{1}{2} \sqrt{10} - \frac{3}{2}
\end{aligned}$$

(b)  $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}} dx$

$$\begin{aligned}
\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}} dx &= \int_{10}^9 \frac{1}{4\sqrt{u}} du \\
&= - \int_9^{10} \frac{1}{4\sqrt{u}} du \\
&= \frac{3}{2} - \frac{1}{2} \sqrt{10} \quad \text{by (b.)}
\end{aligned}$$

4. (a)  $\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta} \cos^2(\theta^{3/2}) d\theta$

Let  $u = \theta^{3/2}$ ,  $du = \frac{3}{2} \theta^{1/2} = \frac{3}{2} \sqrt{\theta} d\theta$

$$\begin{aligned}
\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta} \cos^2(\theta^{3/2}) d\theta &= \int_0^\pi \frac{2}{3} \cos^2 u du \\
&= \int_0^\pi \frac{2}{3} \left[ \frac{1}{2} + \frac{1}{2} \cos(2u) \right] du \\
&= \int_0^\pi \left[ \frac{1}{3} + \frac{1}{3} \cos(2u) \right] du \\
&= \left[ \frac{u}{3} + \frac{1}{3} \cdot \frac{1}{2} \sin(2u) \right] \Big|_0^\pi \\
&= \left[ \frac{\pi}{3} + 0 \right] - [0 + 0] \\
&= \frac{\pi}{3}
\end{aligned}$$

(b)  $\int_{-1}^{-1/2} t^{-2} \sin^2 \left( 1 + \frac{1}{t} \right) dt$

Let  $u = 1 + \frac{1}{t}$ ,  $du = -\frac{1}{t^2} dt$

$$\begin{aligned} \int_{-1}^{-1/2} t^{-2} \sin^2 \left( 1 + \frac{1}{t} \right) dt &= \int_0^{-1} -\sin^2 u \, du \\ &= \int_{-1}^0 \sin^2 u \, du \\ &= \int_{-1}^0 \left[ \frac{1}{2} - \frac{1}{2} \cos(2u) \right] du \\ &= \left[ \frac{u}{2} - \frac{1}{2} \cdot \frac{1}{2} \sin(2u) \right] \Big|_{-1}^0 \\ &= (0 - 0) - \left( -\frac{1}{2} - \frac{1}{4} \sin(-2) \right) \\ &= \frac{1}{2} - \frac{1}{4} \sin(2) \end{aligned}$$

5.  $\int_2^4 \frac{dx}{x \ln x}$

Let  $u = \ln x$ ,  $du = \frac{1}{x} dx$

$$\begin{aligned} \int_2^4 \frac{dx}{x \ln x} &= \int_{\ln 2}^{\ln 4} \frac{du}{u} \\ &= \ln u \Big|_{\ln 2}^{\ln 4} \\ &= \ln(\ln 4) - \ln(\ln 2) \\ &= \ln \left( \frac{\ln 4}{\ln 2} \right) \\ &= \ln \left( \frac{2 \ln 2}{\ln 2} \right) \\ &= \ln 2 \end{aligned}$$

6.  $\int_0^{\ln \sqrt{3}} \frac{e^x dx}{1 + e^{2x}}$

Let  $u = e^x$ ,  $du = e^x dx$

$$\begin{aligned} \int_0^{\ln \sqrt{3}} \frac{e^x dx}{1 + e^{2x}} &= \int_1^{\sqrt{3}} \frac{du}{1 + u^2} \\ &= \tan^{-1} u \Big|_1^{\sqrt{3}} \\ &= \frac{\pi}{3} - \frac{\pi}{4} \\ &= \frac{\pi}{12} \end{aligned}$$

7.  $y = x\sqrt{4-x^2}$

We note that the function is negative on  $[-2, 0]$  and positive on  $[0, 2]$ .  
Therefore, we get:

$$\text{Area} = -\int_{-2}^0 x\sqrt{4-x^2} dx + \int_0^2 x\sqrt{4-x^2} dx$$

Let  $u = 4 - x^2$ ,  $du = -2x dx$ .

$$\begin{aligned} \text{Area} &= -\int_{-2}^0 x\sqrt{4-x^2} dx + \int_0^2 x\sqrt{4-x^2} dx \\ &= -\int_0^4 -\frac{1}{2}u^{1/2} du + \int_4^0 -\frac{1}{2}u^{1/2} du \\ &= \int_0^4 u^{1/2} du \\ &= \frac{2}{3}u^{3/2}\Big|_0^4 \\ &= \frac{2}{3} \cdot 8 - \frac{2}{3} \cdot 0 \\ &= \frac{16}{3} \end{aligned}$$

8.  $y = \frac{1}{2}\sec^2 t$  and  $y = -4\sin^2 t$

We note that the first function is positive on  $[-\frac{\pi}{3}, \frac{\pi}{3}]$  and the second function is negative on this interval. Therefore, we get:

$$\begin{aligned} \text{Area} &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\sec^2 t + 4\sin^2 t dt \\ &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\sec^2 t + 4\left(\frac{1}{2} - \frac{1}{2}\cos(2t)\right) dt \\ &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\sec^2 t + 2 - 2\cos(2t) dt \\ &= \frac{1}{2}\tan t + 2t - \sin(2t)\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\ &= \left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) - \left(-\frac{\sqrt{3}}{2} - \frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right) \\ &= \frac{4\pi}{3} \end{aligned}$$

9.  $x = 2y^2 - 2y$  and  $x = 12y^2 - 12y^3$

We will take the integral with respect to  $y$ . We note that the first function is negative (negative  $x$ -values) on  $[0, 1]$ , while the second function is positive. Therefore, we get:

$$\begin{aligned} \text{Area} &= \int_0^1 [-(2y^2 - 2y) + (12y^2 - 12y^3)] dy \\ &= \int_0^1 -12y^3 + 10y^2 + 2y dy \\ &= -3y^4 + \frac{10}{3}y^3 + y^2 \Big|_0^1 \\ &= \left(-3 + \frac{10}{3} + 1\right) - (0 + 0 + 0) \\ &= \frac{4}{3} \end{aligned}$$

10.  $y = x^2$  and  $x + y = 2$

We can integrate this with respect to either variable ( $x$  or  $y$ ). For practice, let's set it up both ways:

wrt  $x$ : We want to find the area of the region under  $y = x^2$  on  $[0, 1]$  and under  $y = 2 - x$  on  $[1, 2]$ . Therefore, we get the sum:

$$\begin{aligned} \text{Area} &= \int_0^1 x^2 dx + \int_1^2 2 - x dx \\ &= \frac{1}{3}x^3 \Big|_0^1 + 2x - \frac{1}{2}x^2 \Big|_1^2 \\ &= \left[\frac{1}{3} - 0\right] + \left[(4 - 2) - \left(2 - \frac{1}{2}\right)\right] \\ &= \frac{5}{6} \end{aligned}$$

wrt  $y$ : We want to find the area of (region bounded by  $x = 2 - y$  and the  $y$ -axis on  $[0, 1]$ ) - (region bounded by  $x = \sqrt{y}$  and the  $y$ -axis on  $[0, 1]$ ).

Therefore we get the integral:

$$\begin{aligned}\text{Area} &= \int_0^1 2 - y - \sqrt{y} dy \\ &= 2y - \frac{1}{2}y^2 - \frac{2}{3}y^{\frac{3}{2}} \Big|_0^1 \\ &= \left(2 - \frac{1}{2} - \frac{2}{3}\right) - (0 - 0 - 0) \\ &= \frac{5}{6}\end{aligned}$$

11.  $y = 2x - x^2$  and  $y = -3$

We first find the intersection points of the two functions:

$$\begin{aligned}2x - x^2 = -3 &\Leftrightarrow 0 = x^2 - 2x - 3 \\ &\Leftrightarrow 0 = (x - 3)(x + 1) \\ &\Leftrightarrow x = 3 \text{ or } x = -1\end{aligned}$$

We know  $2x - x^2 \geq -3$  on  $[-1, 3]$ . Therefore we get the following integral:

$$\begin{aligned}\text{Area} &= \int_{-1}^3 (2x - x^2) - (-3) dx \\ &= \int_{-1}^3 -x^2 + 2x + 3 dx \\ &= -\frac{1}{3}x^3 + x^2 + 3x \Big|_{-1}^3 \\ &= (-9 + 9 + 9) - \left(\frac{1}{3} + 1 - 3\right) \\ &= 9 + \frac{5}{3} \\ &= \frac{32}{3}\end{aligned}$$

12.  $y = \sqrt{|x|}$  and  $5y = x + 6$  (How many intersection points are there?)

We first find the intersection points of the two functions:

$$\begin{aligned}5\sqrt{|x|} = x + 6 &\Leftrightarrow 25|x| = x^2 + 12x + 36 \\ &\Leftrightarrow 0 = x^2 + 12x - 25|x| + 36\end{aligned}$$

Therefore we have two cases to deal with:

$$x \geq 0$$

$$\begin{aligned} 0 = x^2 + 12x - 25|x| + 36 &\Leftrightarrow 0 = x^2 - 13x + 36 \\ &\Leftrightarrow 0 = (x - 4)(x - 9) \\ &\Leftrightarrow x = 4 \text{ or } x = 9 \end{aligned}$$

$$x \leq 0$$

$$\begin{aligned} 0 = x^2 + 12x - 25|x| + 36 &\Leftrightarrow 0 = x^2 + 37x + 36 \\ &\Leftrightarrow 0 = (x + 1)(x + 36) \\ &\Leftrightarrow x = -1 \text{ or } x = -36 \end{aligned}$$

**Notice:**  $x = -36$  is not an intersection point (you can check this). You have to be careful when you are dealing with square roots. In solving we changed from  $y = \sqrt{|x|}$  to  $y^2 = |x|$ . In the first expression  $-\infty \leq x \leq \infty$  and  $0 \leq y \leq \infty$  while in the second expression  $-\infty \leq x, y \leq \infty$ . So while the expressions look the same, they have subtle differences and we must check that our solutions are, in fact, intersection points.

Thus our intersection points are at  $x = -1$ ,  $x = 4$  and  $x = 9$ . We can check that  $\sqrt{|x|} \leq \frac{x}{5} + \frac{6}{5}$  on  $[-1, 4]$  and that  $\frac{x}{5} + \frac{6}{5} \leq \sqrt{|x|}$  on  $[4, 9]$ . Therefore we get the integrals:

$$\begin{aligned} \text{Area} &= \int_{-1}^4 \left( \frac{x}{5} + \frac{6}{5} \right) - (\sqrt{|x|}) \, dx + \int_4^9 (\sqrt{|x|}) - \left( \frac{x}{5} + \frac{6}{5} \right) \, dx \\ &= \int_{-1}^0 \left[ \frac{x}{5} + \frac{6}{5} - \sqrt{-x} \right] \, dx + \int_0^4 \left[ \frac{x}{5} + \frac{6}{5} - \sqrt{x} \right] \, dx + \int_4^9 \left[ \sqrt{x} - \frac{x}{5} - \frac{6}{5} \right] \, dx \\ &= \left[ \frac{x^2}{10} + \frac{6x}{5} + \frac{2(-x)^{3/2}}{3} \right] \Big|_{-1}^0 + \left[ \frac{x^2}{10} + \frac{6x}{5} - \frac{2x^{3/2}}{3} \right] \Big|_0^4 + \left[ \frac{2x^{3/2}}{3} - \frac{x^2}{10} - \frac{6x}{5} \right] \Big|_4^9 \\ &= \left[ 0 - \left( \frac{1}{10} - \frac{6}{5} + \frac{2}{3} \right) \right] + \left[ \left( \frac{16}{10} + \frac{24}{5} - \frac{16}{3} \right) - 0 \right] + \left[ \left( 18 - \frac{81}{10} - \frac{54}{5} \right) - \left( \frac{16}{3} - \frac{16}{10} - \frac{24}{5} \right) \right] \\ &= \frac{13}{30} + \frac{16}{15} + \frac{1}{6} \\ &= \frac{5}{3} \end{aligned}$$

13.  $y = 3 - x^2$  and  $y = -1$

wrt x: We begin by finding the points of intersection:

$$\begin{aligned} 3 - x^2 = -1 &\Leftrightarrow 0 = x^2 - 4 \\ &\Leftrightarrow 0 = (x + 2)(x - 2) \\ &\Leftrightarrow x = -2 \text{ or } x = 2 \end{aligned}$$



We know  $3 - x^2 \geq -1$  on the interval  $[-2, 2]$ . Therefore we get the following integral:

$$\begin{aligned}
 \text{Area} &= \int_{-2}^2 (3 - x^2) - (-1) dx \\
 &= \int_{-2}^2 4 - x^2 dx \\
 &= 4x - \frac{1}{3}x^3 \Big|_{-2}^2 \\
 &= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) \\
 &= 2\left(8 - \frac{8}{3}\right) \\
 &= \frac{32}{3}
 \end{aligned}$$

wrt  $y$ : To find the area of the region by integrating with respect to  $y$ , we note that we want to find the area of the region bounded by  $x = \sqrt{3 - y}$  and  $x = -\sqrt{3 - y}$  on the interval  $[-1, 3]$ . (You can get this by looking at the picture of the region.) Therefore, we get the following integral:

$$\begin{aligned}
 \text{Area} &= \int_{-1}^3 (\sqrt{3 - y}) - (-\sqrt{3 - y}) dy \\
 &= 2 \int_{-1}^3 \sqrt{3 - y} dy \\
 &= 2 \left[ -\frac{2}{3}(3 - y)^{3/2} \Big|_{-1}^3 \right] \\
 &= 2 \left[ 0 - \left(-\frac{16}{3}\right) \right] \\
 &= \frac{32}{3}
 \end{aligned}$$

14. Let  $F(x)$  be an antiderivative of  $f(x) = \frac{\sin x}{x}$  for  $x > 0$ .

$$\int_1^3 \frac{\sin(2x)}{x} dx = \int_1^3 \frac{2 \sin(2x)}{2x} dx$$

Let  $u = 2x$ ,  $du = 2 dx$

$$\begin{aligned}\int_1^3 \frac{2 \sin(2x)}{2x} dx &= \int_2^6 \frac{\sin u}{u} du \\ &= \int_2^6 f(u) du \\ &= F(6) - F(2) \qquad \text{by FTC (part 2)}\end{aligned}$$

15. (a) Suppose  $f$  is an odd function

$$\Rightarrow f(x) = -f(-x)$$

Consider the following:

$$\int_{-a}^0 f(x) dx = \int_{-a}^0 -f(-x) dx$$

If we let  $u = -x$ ,  $du = -dx$ , we get the following:

$$\begin{aligned}\int_{-a}^0 -f(-x) dx &= \int_a^0 f(u) du \\ &= -\int_0^a f(u) du\end{aligned}$$

If we do a change of variables, we get:

$$\int_{-a}^0 f(x) dx = -\int_0^a f(x) dx$$

Therefore if we look at the integral  $\int_{-a}^a f(x) dx$ , we get:

$$\begin{aligned}\int_{-a}^a f(x) dx &= \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \\ &= -\int_0^a f(x) dx + \int_0^a f(x) dx \\ &= 0\end{aligned}$$

b. Consider  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, dx$

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, dx &= -\cos x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= -\cos\left(\frac{\pi}{2}\right) - \left(-\cos\left(-\frac{\pi}{2}\right)\right) \\ &= -\cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) && (\text{cosine is even}) \\ &= 0\end{aligned}$$

16.  $\int_a^b f(x) \, dx = \int_{a-c}^{b-c} f(x+c) \, dx$

Beginning with the left-hand side, make the substitution  $u = x - c$ ,  $du = dx$ . Then we get the following:

$$\begin{aligned}\int_a^b f(x) \, dx &= \int_a^b f((x-c) + c) \, dx \\ &= \int_{a-c}^{b-c} f(u+c) \, du \\ &= \int_{a-c}^{b-c} f(x+c) \, dx && (\text{change of variables})\end{aligned}$$