Math 21B-B - Homework Set 7

Section 7.1:

1. (a)

$$\int \frac{2y}{y^2-25}\,dy = \int \frac{1}{u}\,du \qquad (u = y^2-25, \quad du = 2y\,dy)$$
$$= \ln|u| + C$$
$$= \ln|y^2-25| + C$$

(b)

$$\int \frac{\sec y \tan y}{2+\sec y}\,dy = \int \frac{1}{u}\,du \qquad (u = 2+\sec y, \quad du = \sec y \tan y\,dy)$$
$$= \ln|u| + C$$
$$= \ln|2+\sec y| + C$$

(c)

$$\int \frac{\sec x}{\sqrt{\ln(\sec x + \tan x)}}\,dx = \int \frac{1}{\sqrt{u}}\,du$$
$$(u = \ln(\sec x + \tan x),\ du = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}\,dx = \sec x\,dx)$$
$$= 2\sqrt{u} + C$$
$$= 2\sqrt{\ln(\sec x + \tan x)} + C$$

(d)

$$\int \frac{e^{\sqrt{r}}}{\sqrt{r}}\,dr = \int 2e^u\,du \qquad (u = \sqrt{r}, \quad du = \frac{1}{2\sqrt{r}}\,dr)$$
$$= 2e^u + C$$
$$= 2e^{\sqrt{r}} + C$$

(e)

$$\int \frac{e^{-1/x^2}}{x^3}\,dx = \int \frac{1}{2}\,e^u\,du \qquad (u = -\frac{1}{x^2}, \quad du = \frac{2}{x^3}\,dx)$$
$$= \frac{1}{2}\,e^u + C$$
$$= \frac{1}{2}\,e^{-1/x^2} + C$$

2. (a) $\dfrac{dy}{dx} = 1 + \dfrac{1}{x}, \quad y(1) = 3$

$$\begin{aligned} y &= \int 1 + \frac{1}{x} \qquad\qquad \left(y = \int \frac{dy}{dx}\, dx\right) \\ &= x + \ln|x| + C \end{aligned}$$

We will use the initial condition $y(1) = 3$ to find the value of C.

$$\begin{aligned} y(1) = 3 \quad &\Rightarrow \quad 1 + \ln 1 + C = 3 \\ &\Rightarrow \quad C = 2 \end{aligned}$$

Thus we get $y = x + \ln|x| + 2$.

(b) $\dfrac{d^2y}{dx^2} = \sec^2 x, \quad y(0) = 0 \quad y'(0) = 1$

$$\begin{aligned} \frac{dy}{dx} &= \int \sec^2 x\, dx \qquad\qquad \left(\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\, dx\right) \\ &= \tan x + C_1 \end{aligned}$$

We will use the initial condition $y'(0) = 1$ to find C_1.

$$\begin{aligned} y'(0) = 1 \quad &\Rightarrow \quad \tan(0) + C_1 = 1 \\ &\Rightarrow \quad C_1 = 1 \end{aligned}$$

Thus we get $\dfrac{dy}{dx} = \tan x + 1$. We now integrate $\dfrac{dy}{dx}$ to find y.

$$\begin{aligned} y &= \int \tan x + 1\, dx \qquad\qquad \left(y = \int \frac{dy}{dx}\, dx\right) \\ &= \ln|\sec x| + x + C_2 \end{aligned}$$

We will use the initial condition $y(0) = 0$ to find C_2.

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad \ln|\sec(0)| + 0 + C_2 = 0 \\ &\Rightarrow \quad \ln 1 + 0 + C_2 = 0 \\ &\Rightarrow \quad C_2 = 0 \end{aligned}$$

Thus we get $y = \ln|\sec x| + x$.

3. The linearization $L(x)$ satisfies

$$\begin{aligned} L(x) &= f(0) + f'(0) \cdot x \\ &= \ln(1+0) + \frac{1}{0+1} \cdot x \\ &= x \end{aligned}$$

4. The linearization $L(x)$ satisfies

$$\begin{aligned} L(x) &= f(0) + f'(0) \cdot x \\ &= e^0 + e^0 \cdot x \\ &= 1 + x \end{aligned}$$

Section 7.2:

1. (a) $\dfrac{dp}{dh} = kp \quad (k\,\text{constant}) \qquad\qquad p = p_0$ when $h = 0$

   Using the Law of Exponential Change (p.428) we know that $p = p_0\, e^{kh}$. In the problem we are given that $p(0) = 1013$ and $p(20) = 90$. We will use these initial conditions to find the values of p_0 and k.

$$\begin{aligned} p(0) = 1013 \quad &\Rightarrow \quad p_0\, e^0 = 1013 \\ &\Rightarrow \quad p_0 = 1013 \end{aligned}$$

   Thus, $p(h) = 1013e^{kh}$

$$\begin{aligned} p(20) = 90 \quad &\Rightarrow \quad 1013\, e^{20k} = 90 \\ &\Rightarrow \quad e^{20k} = \frac{90}{1013} \\ &\Rightarrow \quad 20k = \ln\left(\frac{90}{1013}\right) \\ &\Rightarrow \quad k = \frac{1}{20} \ln\left(\frac{90}{1013}\right) \approx -0.121 \end{aligned}$$

   Thus, $p(h) \approx 1013e^{-0.121h}$

   (b) $p(50) = 1013\, e^{(-0.121)(50)} \approx 2.389$ milibars

   (c) $900 = 1013\, e^{-0.121h}$

   $\Leftrightarrow \frac{900}{1013} = e^{-0.121h}$

   $\Leftrightarrow \ln\left(\frac{900}{1013}\right) = -0.121h$

   $\Leftrightarrow h \approx 0.977$ km

2. $\frac{dV}{dt} = -\frac{1}{40} V \qquad \Rightarrow \qquad V = V_0\, e^{-t/40}$

We want to find t such that $V(t) = 0.1V_0$.

$$\begin{aligned} 0.1V_0 = V_0\, e^{-t/40} \qquad &\Rightarrow \qquad 0.1 = e^{-t/40} \\ &\Rightarrow \qquad \ln(0.1) = -\frac{t}{40} \\ &\Rightarrow \qquad t = -40\ln(0.1) \approx 92.1 \text{ sec} \end{aligned}$$

3. We will let the population of the bacteria colony be given by $p(t) = e^{kt}$ where t in measured in hours (we are told $p_0 = 1$).

   We know that the p doubles every half hour. Thus $p(0.5) = 2$. We can use this information to find the value of k.

$$\begin{aligned} 2 = e^{0.5k} \qquad &\Rightarrow \qquad \ln(2) = 0.5k \\ &\Rightarrow \qquad k = 2\ln(2) \\ &\Rightarrow \qquad k = \ln(4) \end{aligned}$$

   Thus we have that $p(t) = e^{\ln(4)t}$. In 24 hours there are $p(24) = e^{\ln(4)\cdot 24} = 4^{24} \approx 2.81475 \times 10^{14}$ bacteria.

4. (a) Let $A(t)$ be the account balance at time t, in thousands of dollars. Then A satisfies the differential equation

$$\frac{dA}{dt} = rA + 1 \qquad (1)$$

   (The second term on the right-hand side is 1 because you are investing at a rate of \$1,000 per year and the units of A are thousands of dollars.) To solve (2), divide both sides of the equation by $rA + 1$ and then integrate:

$$\begin{aligned} \frac{\frac{dA}{dt}}{rA+1} &= 1 \\ \int \frac{\frac{dA}{dt}}{rA+1}\, dt &= \int dt \\ \int \frac{dA}{rA+1} &= \int dt \quad (\text{since } dA = \tfrac{dA}{dt} dt) \\ \frac{1}{r}\ln(rA+1) &= t + C \\ \ln(rA+1) &= rt + rC \\ rA + 1 &= e^{rt+rC} = Be^{rt}, \end{aligned}$$

   where we write B for the constant e^{rC}. Solving for A gives

$$A = \frac{1}{r}(Be^{rt} - 1).$$

Next, we solve for B using the initial condition $A(0) = 1$:

$$\begin{aligned} 1 &= \frac{1}{r}(Be^{r\cdot 0} - 1) \\ 1 &= 20(B-1) \\ \Rightarrow B &= \frac{21}{20}\,. \end{aligned}$$

Thus

$$A(t) = 20(\tfrac{21}{20}e^{0.05t} - 1).$$

Next we find the value of t such that $A(t) = 20$:

$$\begin{aligned} 20 &= 20(\tfrac{21}{20}e^{0.05t} - 1) \\ 1 &= \tfrac{21}{20}e^{0.05t} - 1 \\ 2 &= \tfrac{21}{20}e^{0.05t} \\ 40 &= 21e^{0.05t} \\ \ln(40) &= \ln(21) + 0.05t \\ t &= 20\Big(\ln(40) - \ln(21)\Big) \text{ years.} \end{aligned}$$

(b) Let $A(t)$ be the account balance at time t. Then A satisfies the differential equation

$$\frac{dA}{dt} = rA + 1/A \tag{2}$$

To solve (2), divide both sides of the equation by $rA + 1/y$ and then integrate:

$$\begin{aligned} \frac{\frac{dA}{dt}}{rA + 1/A} &= 1 \\ \int \frac{\frac{dA}{dt}}{rA + 1/A}\,dt &= \int dt \\ \int \frac{dA}{rA + 1/A} &= \int dt \quad (\text{since } dA = \tfrac{dA}{dt}dt) \\ \int \frac{A}{rA^2 + 1}\,dA &= t + C \\ \frac{1}{2r}\ln(rA^2 + 1) &= t + C \\ \ln(rA^2 + 1) &= 2rt + 2rC \\ rA^2 + 1 &= e^{2rt + 2rC} = Be^{2rt} \end{aligned}$$

where we write B for the constant e^{2rC}. Solving for A gives

$$A = \sqrt{\frac{1}{r}(Be^{2rt} - 1)}. \tag{3}$$

Next, we solve for B using the initial condition $A(0) = 1$:

$$\begin{aligned} 1 &= \sqrt{\frac{1}{r}(Be^{2r\cdot 0} - 1)} \\ 1 &= 10(B-1) \\ \Rightarrow B &= \frac{11}{10}\,. \end{aligned}$$

Subsituting $B = 11/10$ and $r = 1/10$ into equation (3) gives

$$\begin{aligned} A(t) &= \sqrt{10(\frac{11}{10}e^{t/5} - 1)} \\ &= \sqrt{11e^{t/5} - 10}\,. \end{aligned}$$

So

$$\begin{aligned} A(10) &= \sqrt{11e^{10/5} - 10} \\ &= \sqrt{11e^{2} - 10} \text{ dollars} \end{aligned}$$

(c) Let $r = 0.10$ and $s = 0.05$. Let $A(t)$ be the account balance at time t, in thousands of dollars. Then A satisfies the differential equation

$$\frac{dA}{dt} = rA - sA = (r-s)A \tag{4}$$

Equation (4) is the differential equation for exponential growth with rate $r - s$. With the initial condition $A(0) = 1$, the solution is

$$A(t) = e^{(r-s)t}\,.$$

The fees df collected between t and $t + dt$ is $sA(t)dt$, so the total fees collected is

$$\begin{aligned} \int df &= \int_0^{10} sA(t)\,dt \\ &= \int_0^{10} 0.05e^{0.05t}\,dt \\ &= e^{0.05t}\Big]_0^{10} \\ &= (\sqrt{e} - 1) \text{ thousand dollars.} \end{aligned}$$