Math 21B - Homework Set 9

Section 8.4:

1. $\displaystyle\int \frac{x+4}{x^2+5x-6}\,dx = \int \frac{x+4}{(x+6)(x-1)}\,dx$

$$\frac{x+4}{(x+6)(x-1)} = \frac{A}{x+6} + \frac{B}{x-1} \quad \Leftrightarrow \quad x+4 = (A+B)x + (-A+6B)$$

$$\Leftrightarrow \quad \begin{cases} 1 = A+B \\ 4 = -A+6B \end{cases}$$

$$\Leftrightarrow \quad A = \frac{2}{7}, \quad B = \frac{5}{7}$$

Therefore, going back to our integral we get:

$$\int \frac{x+4}{x^2+5x-6}\,dx = \int \left(\frac{2}{7}\cdot\frac{1}{x+6} + \frac{5}{7}\cdot\frac{1}{x-1}\right)dx$$
$$= \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C$$

2. $\displaystyle\int \frac{x+3}{2x^3-8x}\,dx = \int \frac{x+3}{2x(x-2)(x+2)}\,dx$

$$\frac{x+3}{2x(x-2)(x+2)} = \frac{A}{2x} + \frac{B}{x-2} + \frac{C}{x+2} \quad \Leftrightarrow \quad x+3 = (A+2B+2C)x^2 + (4C-4B)x - 4A$$

$$\Leftrightarrow \quad \begin{cases} 0 = A+2B+2C \\ 1 = 4C-4B \\ 3 = -4A \end{cases}$$

$$\Leftrightarrow \quad A = -\frac{3}{4}, \quad B = \frac{1}{16}, \quad C = \frac{5}{16}$$

Therefore, going back to our integral we get:

$$\int \frac{x+3}{2x^3-8x}\,dx = \int \left(-\frac{3}{4}\cdot\frac{1}{2x} + \frac{1}{16}\cdot\frac{1}{x-2} + \frac{5}{16}\cdot\frac{1}{x+2}\right)dx$$
$$= -\frac{3}{8}\ln|x| + \frac{1}{16}\ln|x-2| + \frac{5}{16}\ln|x+2| + C$$

3. $\displaystyle\int_0^1 \frac{x^3}{x^2+2x+1}\,dx = \int_0^1 \left(x-2+\frac{3x+2}{x^2+2x+1}\right)dx \qquad \textit{(by long division)}$

$$\frac{3x+2}{x^2+2x+1} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \qquad \Leftrightarrow \qquad 3x+2 = Ax + (A+B)$$

$$\Leftrightarrow \qquad \begin{cases} 3 = A \\ 2 = A+B \end{cases}$$

$$\Leftrightarrow A = 3, \quad B = -1$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int_0^1 \frac{x^3}{x^2+2x+1}\,dx &= \int_0^1 \left(x-2+\frac{3}{x+1}-\frac{1}{(x+1)^2}\right)dx \\
&= \left(\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right)\Bigg|_0^1 \\
&= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) \\
&= 3\ln 2 - 2
\end{aligned}$$

4. $\displaystyle\int_0^1 \frac{dx}{(x+1)(x^2+1)}$

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \qquad \Leftrightarrow \qquad 1 = (A+B)x^2 + (B+C)x + (C+A)$$

$$\Leftrightarrow \qquad \begin{cases} 0 = A+B \\ 0 = B+C \\ 1 = C+A \end{cases}$$

$$\Leftrightarrow \qquad A = \frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = \frac{1}{2}$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int_0^1 \frac{dx}{(x+1)(x^2+1)} &= \int_0^1 \left(\frac{1}{2}\cdot\frac{1}{x+1} + \frac{1}{2}\cdot\frac{1-x}{x^2+1}\right)dx \\
&= \frac{1}{2}\int_0^1 \left(\frac{1}{x+1} + \frac{1}{x^2+1} - \frac{x}{x^2+1}\right)dx \\
&= \frac{1}{2}\left(\ln|x+1| + \tan^{-1} x - \frac{1}{2}\ln|x^2+1|\right)\Bigg|_0^1 \\
&= \frac{1}{2}\left(\ln 2 + \frac{\pi}{4} - \frac{\ln 2}{2}\right) \\
&= \frac{1}{4}\ln 2 + \frac{\pi}{8}
\end{aligned}$$

5. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$

$$\frac{3t^2+t+4}{t(t^2+1)} = \frac{A}{t} + \frac{Bx+C}{t^2+1} \quad \Leftrightarrow \quad 3t^2+t+4 = (A+B)t^2 + Ct + A$$

$$\Leftrightarrow \quad \begin{cases} 3 = A+B \\ 1 = C \\ 4 = A \end{cases}$$

$$\Leftrightarrow \quad A = 4, \quad B = -1, \quad C = 1$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dx &= \int_1^{\sqrt{3}} \left(\frac{4}{t} + \frac{1-t}{t^2+1}\right) dt \\
&= \int_1^{\sqrt{3}} \left(\frac{4}{t} + \frac{1}{t^2+1} - \frac{t}{t^2+1}\right) dt \\
&= \left(4\ln|t| + \tan^{-1} t - \frac{1}{2}\ln|t^2+1|\right)\Bigg|_1^{\sqrt{3}} \\
&= \left(4\ln(\sqrt{3}) + \frac{\pi}{3} - \frac{1}{2}\ln 4\right) - \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) \\
&= 2\ln 3 + \frac{\pi}{3} - \ln 2 - \frac{\pi}{4} + \frac{1}{2}\ln 2 \\
&= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12}
\end{aligned}$$

6. $\displaystyle\int \frac{x^4}{x^2-1}\,dx = \int \left(x^2+1+\frac{1}{x^2-1}\right) dx$

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} \quad \Leftrightarrow \quad 1 = (A+B)x + (A-B)$$

$$\Leftrightarrow \quad \begin{cases} 0 = A+B \\ 1 = A-B \end{cases}$$

$$\Leftarrow \quad A = \frac{1}{2}, \quad B = -\frac{1}{2}$$

Therefore, going back to our integral we get:

$$\begin{aligned}
\int \frac{x^4}{x^2-1}\,dx &= \int \left(x^2+1+\frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x+1}\right) dx \\
&= \frac{1}{3}x^3 + x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C \\
&= \frac{1}{3}x^3 + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C
\end{aligned}$$

Section 8.7:

1.

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{b\to 0^+}\int_b^1 \frac{1}{\sqrt{x}}\,dx\\ &= \lim_{b\to 0^+}\ 2\sqrt{x}\Big|_b^1\\ &= \lim_{b\to 0^+}\ \left(2-2\sqrt{b}\right)\\ &= 2\end{aligned}$$

2.

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx &= \lim_{b\to 1^-}\int_0^b \frac{1}{\sqrt{1-x^2}}\,dx\\ &= \lim_{b\to 1^-}\ \sin^{-1}x\Big|_0^b\\ &= \lim_{b\to 1^-}\ \left(\sin^{-1}(b)-\sin^{-1}(0)\right)\\ &= \frac{\pi}{2}\end{aligned}$$

3.

$$\begin{aligned}\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx &= \lim_{b\to 1^+}\int_b^2 \frac{1}{x\sqrt{x^2-1}}\,dx + \lim_{c\to\infty}\int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx\\ &= \lim_{b\to 1^+}\ \sec^{-1}x\Big|_b^2 + \lim_{c\to\infty}\ \sec^{-1}x\Big|_2^c\\ &= \lim_{b\to 1^+}\left[\sec^{-1}(2)-\sec^{-1}(b)\right] + \lim_{c\to\infty}\left[\sec^{-1}(c)-\sec^{-1}(2)\right]\\ &= \frac{\pi}{3}-0+\frac{\pi}{2}-\frac{\pi}{3}\\ &= \frac{\pi}{2}\end{aligned}$$

4. $\displaystyle\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta$

We will first consider the indefinite integral $\displaystyle\int 2e^{-\theta}\sin\theta\,d\theta$.

Let $u_1=\sin\theta,\quad du_1=\cos\theta\,d\theta$ and $v_1=-2e^{-\theta},\quad dv_1=2e^{-\theta}\,d\theta$.

Let $u_2 = \cos\theta, \quad du_2 = -\sin\theta\, d\theta$ and $v_2 = -2e^{-\theta}, \quad dv_2 = 2e^{-\theta}\, d\theta$.

$$\begin{aligned}
\int 2e^{-\theta}\sin\theta\, d\theta &= u_1 v_1 - \int v_1\, du_1 \\
&= -2e^{-\theta}\sin\theta + \int 2e^{-\theta}\sin\theta\, d\theta \\
&= -2e^{-\theta}\sin\theta + u_2 v_2 - \int v_2\, du_2 \\
&= -2e^{-\theta}\sin\theta - 2e^{-\theta}\cos\theta - \int 2e^{-\theta}\sin\theta\, d\theta
\end{aligned}$$

Let $I = \int 2e^{-\theta}\sin\theta\, d\theta$. Solving for I we get:

$$\begin{aligned}
I = -2e^{-\theta}\sin\theta - 2e^{-\theta}\cos\theta - I \quad &\Leftrightarrow \quad 2I = -2e^{-\theta}\sin\theta - 2e^{-\theta}\cos\theta \\
&\Leftrightarrow \quad I = -e^{-\theta}\sin\theta - e^{-\theta}\cos\theta
\end{aligned}$$

So we have that $\int 2e^{-\theta}\sin\theta\, d\theta = -e^{-\theta}\sin\theta - e^{-\theta}\cos\theta + C$. Now let's consider the improper integral $\int_0^\infty 2e^{-\theta}\sin\theta\, d\theta$.

$$\begin{aligned}
\int_0^\infty 2e^{-\theta}\sin\theta\, d\theta &= \lim_{b\to\infty}\int_0^b 2e^{-\theta}\sin\theta\, d\theta \\
&= \lim_{b\to\infty} \left(-e^{-\theta}\sin\theta - e^{-\theta}\cos\theta\right)\Big|_0^b \\
&= \lim_{b\to\infty} [(-e^{-b}\sin b - e^{-b}\cos b) - (-1)] \\
&= 1
\end{aligned}$$

5.

$$\begin{aligned}
\int_{-\infty}^\infty 2xe^{-x^2}\, dx &= \lim_{b\to-\infty}\int_b^0 2xe^{-x^2}\, dx + \lim_{c\to\infty}\int_0^c 2xe^{-x^2}\, dx \\
&= \lim_{b\to-\infty} -e^{-x^2}\Big|_b^0 + \lim_{c\to\infty} -e^{-x^2}\Big|_0^c \\
&= \lim_{b\to-\infty}\left(-1 + e^{-b^2}\right) + \lim_{c\to\infty}\left(-e^{-c^2} + 1\right) \\
&= 0
\end{aligned}$$

6.

$$\begin{aligned}
\int_0^1 x \ln x \, dx &= \lim_{b \to 0^+} \int_b^1 x \ln x \, dx \\
&= \lim_{b \to 0^+} \left[\frac{1}{2}x^2 \ln x \Big|_b^1 - \int_b^1 \frac{1}{2} x \, dx \right] && (\textit{Integration by Parts}) \\
&= \lim_{b \to 0^+} \left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 \right) \Big|_b^1 \\
&= \lim_{b \to 0^+} \left[\left(-\frac{1}{4} \right) - \left(\frac{1}{2}b^2 \ln b - \frac{1}{4}b^2 \right) \right] \\
&= -\frac{1}{4} - \frac{1}{2} \left[\lim_{b \to 0^+} \frac{\ln b}{\left(\frac{1}{b^2}\right)} \right] \\
&= -\frac{1}{4} - \frac{1}{2} \left[\lim_{b \to 0^+} \frac{\left(\frac{1}{b}\right)}{\left(\frac{-2}{b^3}\right)} \right] && (\textit{l'Hopital's Rule}) \\
&= -\frac{1}{4} + \frac{1}{4} \left[\lim_{b \to 0^+} b^2 \right] \\
&= -\frac{1}{4}
\end{aligned}$$

7.

$$\begin{aligned}
\int_{-1}^4 \frac{1}{\sqrt{|x|}} \, dx &= \lim_{b \to 0^-} \int_{-1}^b \frac{1}{\sqrt{|x|}} \, dx + \lim_{c \to 0^+} \int_c^4 \frac{1}{\sqrt{|x|}} \, dx \\
&= \lim_{b \to 0^-} \int_{-1}^b \frac{1}{\sqrt{-x}} \, dx + \lim_{c \to 0^+} \int_c^4 \frac{1}{\sqrt{x}} \, dx \\
&= \lim_{b \to 0^-} -2\sqrt{-x} \Big|_{-1}^b + \lim_{c \to 0^+} 2\sqrt{x} \Big|_c^4 \\
&= \lim_{b \to 0^-} \left(-2\sqrt{-b} + 2 \right) + \lim_{c \to 0^+} \left(4 - 2\sqrt{c} \right) \\
&= 6
\end{aligned}$$

Section 11.2:

1. $x = \cos t, y = 2 + \sin t, \quad 0 \leq t \leq 2\pi, \qquad x\text{-axis}$

$$\begin{aligned}
\text{Area} &= \int_0^{2\pi} 2\pi(2+\sin t)\sqrt{(-\sin t)^2 + (\cos t)^2}\, dt \\
&= 2\pi \int_0^{2\pi} (2+\sin t)\sqrt{\sin^2 t + \cos^2 t}\, dt \\
&= 2\pi \int_0^{2\pi} 2 + \sin t\, dt \\
&= \left. 2\pi\,(2t - \cos t)\right|_0^{2\pi} \\
&= 2\pi[(4\pi - 1) - (0 - 1)] \\
&= 8\pi^2
\end{aligned}$$

2. $x = \ln(\sec t + \tan t) - \sin t, y = \cos t, \quad 0 \leq t \leq \frac{\pi}{3}; \qquad x\text{-axis}$

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/3} 2\pi \cos t \sqrt{\left(\frac{\sec t \tan t + \sec^2 t}{\sec t + \tan t} - \cos t\right)^2 + (-\sin t)^2}\, dt \\
&= 2\pi \int_0^{\pi/3} \cos t \sqrt{\left(\frac{(\sec t)(\tan t + \sec t)}{\sec t + \tan t} - \cos t\right)^2 + \sin^2 t}\, dt \\
&= 2\pi \int_0^{\pi/3} \cos t \sqrt{(\sec t - \cos t)^2 + \sin^2 t}\, dt \\
&= 2\pi \int_0^{\pi/3} \cos t \sqrt{\sec^2 t - 2 + \cos^2 t + \sin^2 t}\, dt \\
&= 2\pi \int_0^{\pi/3} \cos t \sqrt{\sec^2 t - 1}\, dt \\
&= 2\pi \int_0^{\pi/3} \cos t \sqrt{\tan^2 t}\, dt \\
&= 2\pi \int_0^{\pi/3} \sin t\, dt \\
&= \left. -2\pi \cos t\right|_0^{\pi/3} \\
&= -2\pi\left(\frac{1}{2} - 1\right) \\
&= \pi
\end{aligned}$$