# Math 21B - Homework Set 6

**Section 6.5:**

1. A force of $90N$ stretches a spring 1 m beyond its natural length. We can find the spring constant $k$ by using Hooke's Law:

$$90 = k$$

Thus we know that the force it takes to move the spring $x$ meters beyond its natural length is given by $F(x) = 90x$. To find the work it takes to stretch the spring 5 m beyond its natural length we take the following integral:

$$\begin{aligned} W &= \int_0^5 F(x)\,dx \\ &= \int_0^5 90x\,dx \\ &= 45x^2\Big|_0^5 \\ &= 1125J \end{aligned}$$

2. Note that since the force is acting *toward* the origin, $F(x) = -\frac{k}{x^2}$. To find the work done as the particle moves from point b to point a, we can use the following integral:

$$\begin{aligned} W &= \int_b^a -\frac{k}{x^2}\,dx \\ &= \frac{k}{x}\Big|_b^a \\ &= \frac{k}{a} - \frac{k}{b} \\ &= \frac{k(b-a)}{ab} \end{aligned}$$

3. a. Work to empty the tank by pumping the water back to ground level.

Consider a horizontal "slab" of water at level $y$ with width $\Delta y$. The force $F_{slab}$ to lift the slab is given by:

$$\begin{aligned} F_{slab} &= 62.4 \cdot V_{slab} \\ &= 62.4 \cdot \Delta y \cdot 10 \cdot 12 \\ &= 62.4 \cdot 120\,\Delta y\,lb \end{aligned}$$

To compute the work needed to pump this slab out of the tank, we recall that $F_{slab}$ must act over a distance of $y$ft. Thus we have:

$$W_{slab} = F_{slab} \cdot d = 62.4 \cdot 120y\Delta y \, ft \cdot lb$$

To approximate the total work $W$ necessary to empty the tank, we could use a Riemann sum $f(y) = 7488y$ over the interval $0 \le y \le 20$.

$$W = \sum_{0}^{20} 64.2 \cdot 120y\Delta y \, ft \cdot lb$$

To find the exact value, we take the limit of the this sum over progressively finer partitions.

$$\begin{aligned} W &= \int_0^{20} 64.2 \cdot 120y \, dy \\ &= \int_0^{20} 7488y \, dy \\ &= 3744y^2 \Big|_0^{20} \\ &= 3744 \cdot 400 \\ &= 1497600 \, ft \cdot lb \end{aligned}$$

b. The pump moves $250 \, ft - lb/sec$, the time it will take to empty the tank is:

$$\text{time} = \frac{1497600 \, ft \cdot lb}{250 \, ft - lb/sec} = 5990.4 \, sec \approx \text{1hr 40min}$$

c. The amount of work it takes to empty out the first half of the tank is given by:

$$\begin{aligned} \text{Work} &= \int_0^{10} 7488y \, dy \\ &= 3744y^2 \Big|_0^{10} \\ &= 3744 \cdot 100 \\ &= 374400 \, \text{ft-lb} \end{aligned}$$

To see how much time this amount of work will take we use:

$$t = \frac{374400}{250} = 1497.6 \, \text{sec}$$

The last thing to note is that $1497.6 \, \text{sec} \approx 25 \, \text{min}$.

d. i. If water weighs 62.26 lb/ft$^3$

$$\begin{aligned} W &= \int_0^{20} 62.26 \cdot 120y \, dy \\ &= \int_0^{20} 7471.2y \, dy \\ &= 3735.2y^2 \Big|_0^{20} \\ &= 3735.6 \cdot 400 \\ &= 1494240 \, \text{ft-lb} \end{aligned}$$

$$\begin{aligned} t &= \frac{1494240}{250} \\ &= 5976.96 \, \text{sec} \\ &\approx 1\text{hr } 40\text{min} \end{aligned}$$

ii. If water weighs 62.59 lb/ft$^3$

$$\begin{aligned} W &= \int_0^{20} 62.59 \cdot 120y \, dy \\ &= \int_0^{20} 7510.8y \, dy \\ &= 3755.4y^2 \Big|_0^{20} \\ &= 3755.4 \cdot 400 \\ &= 1502160 \, \text{ft-lb} \end{aligned}$$

$$\begin{aligned} t &= \frac{1502160}{250} \\ &= 6008.64 \, \text{sec} \\ &\approx 1\text{hr } 40\text{min} \end{aligned}$$

4. a. Let $\rho$ be the $x$-coordinate of the second electron. Then $r^2 = (\rho - 1)^2$.

$$\begin{aligned} W &= \int_{-1}^{0} F(r)\,dr \\ &= \int_{-1}^{0} \frac{23 \times 10^{-29}}{(\rho - 1)^2}\,d\rho \\ &= \left(23 \times 10^{-29}\right) \cdot -\frac{1}{\rho - 1}\Big|_{-1}^{0} \\ &= \frac{1}{2}\left(23 \times 10^{-29}\right) \\ &= 11.5 \times 10^{-29} \end{aligned}$$

b. We will use the fact that $W = W_1 + W_2$ where $W_1$ is the work against the fixed electron $(-1, 0)$ and $W_2$ is the work against the second fixed electron $(1, 0)$. We will let $\rho$ be the $x$-coordinate of the third electron. Then ${r_1}^2 = (\rho + 1)^2$ and ${r_2}^2 = (\rho - 1)^2$.

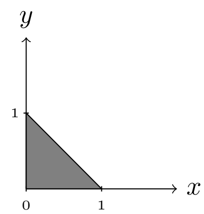
$$\begin{aligned} W_1 &= \int_{3}^{5} \frac{23 \times 10^{-29}}{(\rho + 1)^2}\,d\rho \\ &= -\frac{23 \times 10^{-29}}{\rho + 1}\Big|_{3}^{5} \\ &= -(23 \times 10^{-29})\left(\frac{1}{6} - \frac{1}{4}\right) \\ &= \frac{23}{12} \times 10^{-29} \end{aligned}$$

$$\begin{aligned} W_2 &= \int_{3}^{5} \frac{23 \times 10^{-29}}{(\rho - 1)^2}\,d\rho \\ &= -\frac{23 \times 10^{-29}}{\rho - 1}\Big|_{3}^{5} \\ &= -(23 \times 10^{-29})\left(\frac{1}{4} - \frac{1}{2}\right) \\ &= \frac{23}{4} \times 10^{-29} \end{aligned}$$

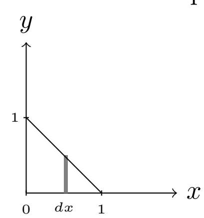
$$\begin{aligned} W &= W_1 + W_2 \\ &= \frac{23}{12} \times 10^{-29} + \frac{23}{4} \times 10^{-29} \\ &= \frac{23}{3} \times 10^{-29} \end{aligned}$$

**Mass Problems:**

1. Find the mass of the triangular region below. All lengths are in meters, and the density of the region is given by $\delta(x) = x$ grams/m$^2$. (Note that since the object is 2-dimensional, its density is its mass per unit area.)




**Solution.** Since the density is a function of $x$, we divide the region into thin vertical strips of thickness $dx$ as shown in the following figure:



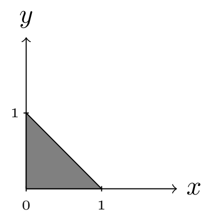

(Question: What kind of strips would we use if the density were a function of $y$?)
Since the height of the strip is $1 - x$, its area is $(1 - x)dx$, and hence its mass is

$$\begin{aligned} dM &= (\text{density of strip}) \times (\text{area of strip}) \\ &= x(1-x)dx. \end{aligned}$$

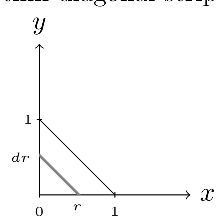
Hence the total mass of the triangle is

$$\begin{aligned} M &= \int dm \\ &= \int_0^1 x(1-x)dx \\ &= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 \\ &= \frac{1}{6} \text{ grams.} \end{aligned}$$

2. Find the mass of the triangular region below. All lengths are in meters, and the density of the region is given by $\delta(x, y) = e^{(x+y)^2}$ grams/m$^2$.




**Solution.** Note that we can write the density as $\delta(r) = e^{r}$ , where $r = x + y$. For $r$ in $[0, 1]$, the graph of $x + y = r$ is a diagonal line that passes through the point $(r, 0)$. This suggests dividing the triangle into thin diagonal strips as shown in the following figure:




In the picture above, the strip is bounded by coordinate axes and the lines $x + y = r$ and $x + y = r + dr$. Its area is $rdr$, which can be seen from the following calculation:
Let $A(r) = \frac{1}{2}r^2$ be the area of a triangle whose base and height are both $r$, and let $dA = A(r + dr) - A(r)$. Then

$$\begin{aligned} \text{area strip} &= dA \\ &= rdr, \end{aligned}$$

where the last line follows from differentiation. It follows that the mass of the strip is

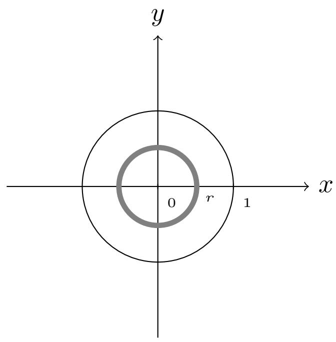
$$\begin{aligned} dM &= (\text{density of strip}) \times (\text{area of strip}) \\ &= e^{r^2 r} dr. \end{aligned}$$

Hence the total mass of the triangle is

$$\begin{aligned} M &= \int dm \\ &= \int_0^1 e^{r^2 r} dr \\ &= \frac{1}{2} e^{r^2} \Big|_0^1 \\ &= \frac{1}{2}(e - 1) \text{ grams.} \end{aligned}$$

3. A thin plate occupies the region of the plane bounded by the circle $x^2 + y^2 = 1$. Find the total mass if the density at the point $(x, y)$ is given by

$\delta(x, y) = 1/\sqrt{x^2 + y^2}$. (Hint: divide the region into thin circular rings centered at the origin.)




**Solution.** Note that we can write the density as $\delta(r) = 1/r$ , where $r = \sqrt{x^2 + y^2}$. For $r$ in $[0, 1]$, the graph of $r = \sqrt{x^2 + y^2}$ is a thin circular ring centered at the origin with radius $r$ and thickness $dr$, as shown in the figure. We want to derive an expression for its area. The area formula for a circle as a function of radius $r$ is

$$A = \pi r^2;$$

so the area for a circular ring, by differentiating the above formula, is

$$dA = 2\pi r dr.$$

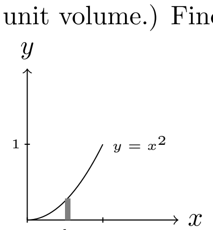
Its mass

$$\begin{aligned} dM &= \delta dA \\ &= 2\pi dr. \end{aligned}$$

Now we take the integral and find the mass for the disk

$$\begin{aligned} M &= \int_0^1 2\pi dr \\ &= 2\pi. \end{aligned}$$

4. The region bounded by the graph of $y = x^2$ and the x-axis, between 0 and 1, is revolved about the $x$-axis. The resulting solid has density given by $\delta(x) = x$. (Here the object is 3-dimensional, so its density is its mass per unit volume.) Find the total mass.



**Solution.** After the revolution, a typical vertical strip forms a circular disk with radius $y = x^2$ and thickness $dx$. Its volume

$$\begin{aligned} dV &= \pi y^2 dx \\ &= \pi x^4 dx, \end{aligned}$$

and mass

$$\begin{aligned} dM &= \delta dV \\ &= \pi x^5 dx. \end{aligned}$$

The mass for the whole solid is therefore

$$\begin{aligned} M &= \int_0^1 \pi x^5 dx \\ &= \frac{\pi}{6}. \end{aligned}$$