Math 135B, Spring 2010.
June 5, 2010.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________________

ID#: ______________________

Instructions: Each of the 6 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Calculators, books or notes are not allowed. Unless you are directed to do so, or it is required for further work, do *not* evaluate complicated expressions to give the result as a decimal number.

Make sure that you have a total of 7 pages (including this one) with 6 problems.

1	
2	
3	
4	
5	
6	
~~7~~	
TOTAL	

1. Assume that $X_0 = x$, where x is a (deterministic) number in $[0,1]$. Moreover, let $X_1, X_2, \ldots$ be independent uniform random variables on $[0,1]$. Let N be the first *increase time*, that is, the smallest $n \geq 1$ for which $X_n > X_{n-1}$. Thus EN depends only on x; denote $g(x) = EN$.

(a) Compute $g(x)$. You may, but are not required to, follow the following steps.

(a.1) Derive an integral equation for $g(x)$ by conditioning on X_1.

(a.2) Differentiate both sides of the equation derived in (a.1).

(a.3) Solve the differential equation obtained in (a.2).

$$g(x) = \underbrace{1}_{\text{make one step}} + \underbrace{\int_0^x g(z)\,dz}_{\text{make next steps only when}}$$

$X_1 \leq x$ (in which case you start over)

$$g'(x) = g(x), \quad g(x) = Ce^x, \quad g(0) = 1,$$

so $\boxed{g(x) = e^x}$.

15

(b) Assume now that X_0 is also uniform on $[0,1]$, and compute EN.

Now $EN = \int_0^1 g(x)\,dx$, by conditioning on X_0.

10

Thus $EN = \int_0^1 e^x\,dx = \underline{\underline{e-1}}$.

2. You are a coach of a good basketball team, but the owner is very unforgiving. If you lose three games in a row in the coming season, you are automatically fired. So, whenever you lose two games in a row, you bribe the referees in the next game, *ensuring* that your team wins it. Otherwise, your team wins any game independently with probability 0.8. The season is about to begin.

(a) Determine the transition probability matrix of the Markov chain whose state is the number of consecutive games you have lost before the coming game.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.8 & 0 & 0.2 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

8

Initial distr. is $[1 \quad 0 \quad 0]$.

(b) Write down an expression for the probability that your team wins game 5 and loses game 10.

$$P_{00}^{5} \cdot \left(P_{01}^{5} + P_{02}^{5} \right)$$

8

(c) ~~Determine~~ Compute the proportion of games (in a long season) your team wins, and the proportion of games your team wins honestly. Give the answers as simple fractions.

The proportion of wins is π_0, while the proportion of honest wins is $\pi_0 \cdot P_{00} + \pi_1 P_{10} = (\pi_0 + \pi_1) \cdot 0.8$

Here $[\pi_0, \pi_1, \pi_2] P = [\pi_0, \pi_1, \pi_2]$, so

9

$\pi_0 \cdot 0.2 = \pi_1$

$\pi_1 \cdot 0.2 = \pi_2$

$\pi_1 = 5\pi_2$, $\pi_0 = 5\pi_1 = 25\pi_2$

$\pi_0 + \pi_1 + \pi_2 = 1$, so $31\pi_2 = 1$, $\underline{\pi_2 = \frac{1}{31}, \pi_1 = \frac{5}{31}, \pi_0 = \frac{25}{31}}$

<u>Answers</u>: $\frac{25}{31}$ and $\frac{30}{31} \cdot 0.8 = \frac{24}{31}$.

3. A random walker on the nonnegative integers starts at 0, and then at each step adds either 0, 1, 2, or 3 to her position, each with probability 1/4.
(a) Let p_n be the probability that the walker ever hits n. Compute $\lim_{n\to\infty} p_n$.

$$\lim_{n\to\infty} p_n = \frac{1}{E[\text{step} \mid \text{step} > 0]}$$

$$= \frac{P(\text{step} > 0)}{1\cdot\frac{1}{4} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{4}} = \frac{\frac{3}{4}}{\frac{3}{2}} = \underline{\underline{\frac{1}{2}}}.$$

13

No cnd 10

(b) Let q_n be the probability that the walker ever hits both n and $n+2$. Compute $\lim_{n\to\infty} q_n$. .

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} p_n \cdot p_2 = \frac{1}{2} p_2$$

To compute p_2: $p_0 = 1$;

$p_1 = \frac{1}{4} p_1 + \frac{1}{4}$, so $p_1 = \frac{1}{3}$;

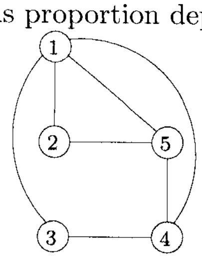
$p_2 = \frac{1}{4} p_2 + \frac{1}{4} p_1 + \frac{1}{4} = \frac{1}{4} p_2 + \frac{1}{12} + \frac{1}{4} = \frac{1}{4} p_2 + \frac{1}{3}$,

so $p_2 = \frac{4}{9}$.

12

Answer: $\frac{1}{2} \cdot \frac{4}{9} = \underline{\underline{\frac{2}{9}}}$.

4. A random walker is in one of the five vertices, labeled 1, 2, 3, 4, and 5, of the graph in the picture. At each time, she moves to a randomly chosen vertex connected to her current position by an edge. (All choices are equally likely and she never stays at the same position for two successive steps.)
(a) Compute the (long-run) proportion of time the walker spends at each of the five vertices. Does this proportion depend on the walker's starting vertex?



The proportions are given by invariant probabilities:

$$\pi_1 = \frac{4}{14}$$
$$\pi_2 = \frac{2}{14}$$
$$\pi_3 = \frac{2}{14}$$
$$\pi_4 = \frac{3}{14}$$
$$\pi_5 = \frac{3}{14}$$

9

(b) Compute the proportion of time the walker is in the set $\{1, 2\}$, while she was in the set $\{3, 4, 5\}$ the previous time.

$$\pi_3 P_{31} + \pi_4 P_{41} + \pi_5 P_{51} + \pi_5 P_{52}$$

8

$$= \frac{2}{14} \cdot \frac{1}{2} + \frac{3}{14} \cdot \frac{1}{3} + \frac{3}{14} \cdot \frac{2}{3} = \frac{4}{14} = \underline{\underline{\frac{2}{7}}}$$

(c) Now assume that there are two independent walkers started at vertex 1. What is the expected time before they again meet at 1?

$$\frac{1}{\pi_1^2} = \left(\frac{14}{4}\right)^2 = \underline{\underline{\frac{49}{4}}}$$

8

5. In a branching process, an individual has three descendants (in the next generation) with probability p and no descendants with probability $1-p$. The process starts with a single individual at generation 0. (Note that both answers below will depend on p.)
(a) Compute the expected number of individuals in generation 2.

$$E\,(\text{no. of offspring}) = 3p, \qquad 10$$

so $$EX_2 = \underline{(3p)^2}.$$

(b) Compute the probability that the process ever goes extinct. Answer is $\underline{1 \text{ when } p \le \frac{1}{3}}$.

$$\varphi(s) = 1-p+ps^3$$

Solve $\varphi(s)=s$ to get the answer when $p > \frac{1}{3}$.

$$1-p+ps^3 = s$$

$$ps^3 - s + 1 - p = 0$$

$$(ps^2 + ps - (1-p))(s-1) = 0$$

$$s = \frac{-p + \sqrt{p^2 + 4p(1-p)}}{2p} \qquad 15$$

When $\underline{p > \frac{1}{3}}$, the answer is

$$\underline{\frac{-p + \sqrt{4p - 3p^2}}{2p}}$$

6. Bob has arrived at the side of the road and started to hitchhike. He is currently there alone, but other hitchhikers arrive as a Poisson process with rate 1 per hour. Moreover, cars arrive as a Poisson process with hourly rate $\mu > 1$. Any arriving car picks up a single hitchhiker, and Bob, being a nice guy, takes a ride only if nobody else waits. All hitchhikers wait until picked.

(a) Compute the probability that Bob waits *alone* during first 30 minutes (so neither a car or another hitchhiker arrives).

Combined process of cars and hitchhikers has rate $1+\mu$.

Answer: $\underline{\underline{e^{-\frac{1}{2}(1+\mu)}}}$. 3

(b) Compute the probability that Bob gets picked up by the first car (in other words, before another hitchhiker arrives). Then compute the probabilities that he gets picked by the second car, and that he gets picked by the third car.

$P(\text{picked by first car}) = \frac{\mu}{1+\mu}$ (first event in combined process is a car)

$P(\text{—〃— 2nd car}) = \frac{1}{1+\mu}\left(\frac{\mu}{1+\mu}\right)^2 \quad (= P(HCC))$ 8

$$P(\text{—〃— 3rd car}) = P(HHCCC) + P(HCHCC) = 2\cdot\left(\frac{1}{1+\mu}\right)^2\left(\frac{\mu}{1+\mu}\right)^3 = \frac{2\mu^3}{(1+\mu)^5}$$

(c) Let T be the time of Bob's wait (i.e., the time before he is finally picked by a car). Compute ET. You may assume that ET is finite. You may, but are not required to, solve this problem by conditioning on a suitable first arrival.

$$ET = \frac{1}{1+\mu} + \frac{1}{1+\mu} \cdot 2ET$$ 9

Terms: waiting time for 1st event in the combined process; only wait further if the event is an H, ...in which case the line first needs to get from 2 to 1, and then from 1 to 0 (the two have the same distribution).

Assuming $ET<\infty$, $ET\left(1-\frac{2}{1+\mu}\right) = \frac{1}{1+\mu}$, $\underline{\underline{ET = \frac{1}{\mu-1}}}$.