Math 135B, Spring 2010.
May 19, 2010.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________________

ID#: ______________________________

Instructions: Each of the 4 problems is worth 25 points. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Calculators, books or notes are not allowed. Unless you are directed to do so, or it is required for further work, do *not* evaluate complicated expressions to give the result as a decimal number.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. A machine has two parts, A and B. Each part is checked at the beginning of each day. If at least one of the two parts is working, the machine is in working order and neither part is replaced. If neither part is working, the machine is in repair for the day, and both parts are replaced with working parts. If both A and B are working, each will fail independently with probability 0.1 the next day. If A is working but B is not, then A fails with probability 0.2 the next day; and if B is working but A is not, then B fails with probability 0.3 the next day.

(a) Determine the transition probability matrix for the day-to-day status of the machine, on states 1, 2, 3, 4, which code, in order, the parts that are working: A and B, A but not B, B but not A, neither B nor A.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.81 & 0.09 & 0.09 & 0.01 \\ 0 & 0.8 & 0 & 0.2 \\ 0 & 0 & 0.7 & 0.3 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

9

(b) Assume that today is Wednesday and the machine is in repair. Write an expression for the probability of the following event: the machine will be in repair again on this Sunday and working the following Sunday. Do not evaluate.

$$P^4_{44} \left(1 - P^7_{44} \right)$$

8

$$\left(P^n_{44} = 44\text{'th entry of } P^n \right)$$

(c) Explain how you would compute the proportion of days on which part A is working. Do not carry out the computation.

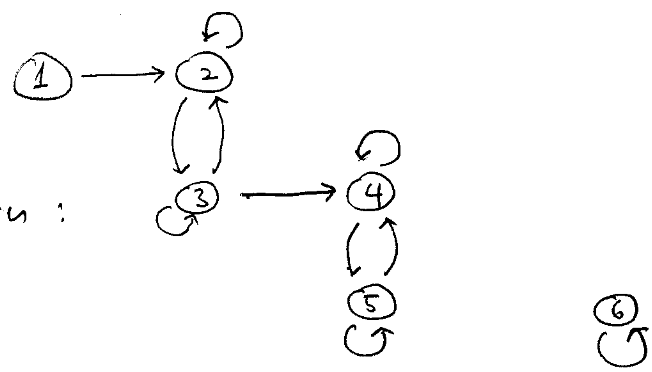
P is irreducible, so it has a unique invariant distribution $\pi = [\pi_1, \pi_2, \pi_3, \pi_4]$, given by $\pi P = \pi$. The answer is $\underline{\pi_1 + \pi_2}$.

8

2. Determine the transient and recurrent classes of the Markov chain with the following transition matrix:

$$\begin{array}{c l}
 & \begin{array}{cccccc} \;1 & \;\;2 & \;\;3 & \;\;4 & \;\;5 & \;6 \end{array} \\
\begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} &
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0.3 & 0.7 & 0 & 0 & 0 \\
0 & 0.1 & 0.8 & 0.1 & 0 & 0 \\
0 & 0 & 0 & 0.1 & 0.9 & 0 \\
0 & 0 & 0 & 0.1 & 0.9 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}.$$

Arrows indicate nonzero probabilities:



{1} transient

{2,3} transient (because not closed)

{4,5} recurrent (because closed)

{6} recurrent (absorbing)

3. In a branching process, an individual has $0, 1, 2, \overset{\text{or}}{\vee} 3$ descendants, each with equal probability $\frac{1}{4}$. Start the process at generation 0 with a single ancestor.
(a) Compute the expected population size at generation 3.

$$\mu = \frac{0+1+2+3}{4} = \frac{3}{2} \qquad \underline{\text{Answer}}: \left(\frac{3}{2}\right)^3$$

6

(b) Write an expression for the probability that the branching process dies out at or before generation 3. Do not evaluate.

$$\varphi(s) = \frac{1}{4}\left(1 + s + s^2 + s^3\right)$$

$$\varphi(0) = \frac{1}{4}$$

$$\varphi(\varphi(0)) = \underbrace{\frac{1}{4}\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3\right)}$$

6

$\underline{\text{Answer}}: \varphi(\varphi(\varphi(0))) = \varphi(\smile)$.

(c) Compute the probability that the branching process ever dies out. (Help: $s^3 + s^2 - 3s + 1 = (s-1)(s^2+2s-1)$). As $\mu > 1$, the prob. is the solution $s \in (0,1)$ to

$$\varphi(s) = s.$$

$$\frac{1}{4}\left(1 + s + s^2 + s^3\right) = s$$

$$s^3 + s^2 - 3s + 1 = 0$$

$$(s-1)(s^2+2s-1) = 0$$

$$s = \frac{-2+\sqrt{8}}{2} = \underline{\underline{\sqrt{2} - 1 = \pi_0}}$$

7

(d) Compute the probability that the process dies out conditioned on the event that population at generation 3 consists of 4 individuals.

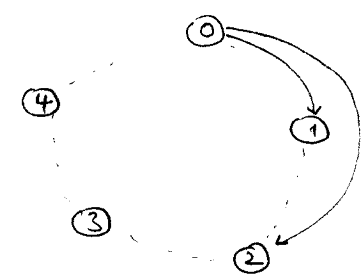
$$\underline{\underline{\left(\sqrt{2} - 1\right)^4}}$$

6

4. A walker walks on the points 0, 1, 2, 3, and 4, arranged on a circle in the clockwise order. Each time, she jumps by 1 in the clockwise direction with probability p and by 2 in the clockwise direction with probability $1-p$. Here, $p \in (0,1)$. (For example, from 3 she jumps to 4 with probability p and to 0 with probability $1-p$.)

(a) Write down the transition matrix P.

$$P = \begin{bmatrix} 0 & p & 1-p & 0 & 0 \\ 0 & 0 & p & 1-p & 0 \\ 0 & 0 & 0 & p & 1-p \\ 1-p & 0 & 0 & 0 & p \\ p & 1-p & 0 & 0 & 0 \end{bmatrix}$$



6

(b) For all states i and j, determine the limit, as $n \to \infty$, of P^n_{ij}. Carefully verify the conditions of any theorems you use. (Help: this requires no long calculations!)

The Markov chain is irreducible, and aperiodic (can get from 0 to 0 in 5 steps and in 4 steps). ~~The~~ P is doubly stochastic, so the invariant distribution is $[\frac{1}{5}\ \frac{1}{5}\ \frac{1}{5}\ \frac{1}{5}\ \frac{1}{5}]$ 6

Answer: $\underline{\underline{\frac{1}{5}}}$, indep. of i and j.

(c) Assume the walker starts at 1. Compute the expected number of jumps she makes before she returns to 1.

$$\underline{\underline{\frac{1}{\pi_1} = 5}}$$

(d) Let $S_\sigma = \{1,3\}$ be the set of odd states. Compute the average number of time steps the walker remains in S_σ after she enters S_σ.

Let σ = ave no. of steps the walker remains in S_σ
e = —//— in $S \setminus S_\sigma$ 7

Then $\frac{\sigma}{\sigma + e} = \pi_1 + \pi_3 = \frac{2}{5}$

$$\frac{1}{\sigma+e} = \pi_1 \cdot P_{12} + \pi_3 (P_{34} + P_{30}) = \frac{1}{5}(2p+1-p) = \frac{1+p}{5}$$

Thus $\sigma = \frac{2}{5} \cdot \frac{5}{1+p} = \underline{\underline{\frac{2}{1+p}}}$.