IN SEARCH OF A HIDDEN CURVE

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Abstract. Three enumeration problems are discussed in these lectures. One is a detailed account on simple Hurwitz numbers, explaining how the problem was solved by discovering a key curve. This key ignites the generating functions of Hurwitz numbers and drives them into polynomials. The unexpected polynomiality then brings us simple proofs of the Witten-Kontsevich theorem and the λ_g-theorem of Faber-Pandharipande. An analogous problem associated with Catalan numbers is also presented, which has a simpler feature in terms of analysis. The third enumeration problem is a quest of identifying a hidden curve behind the Apéry sequence, which remains to be discovered.

It is examined how a counting problem leads to a *spectral curve* through a differential equation, of Picard-Fuchs type but can be nonlinear. The counting problem is a geometric question associated with the *genus* 0*, one marked point* case. Going from the $(0, 1)$-case to arbitrary (g, n)-case is explained as a process of *quantization* of the spectral curve.

This perspective of quantization is discussed in a geometric setting, when the differential equations have holomorphic coefficients, in terms of Higgs bundles, opers, and Gaiotto's conformal limit construction. In this context, however, there are no counting problems behind the scene.

Contents

0. Introduction 2
0.1. The story begins with Catalan numbers and their Laplace transform 2
0.2. A miraculous integer sequence 4
0.3. The unique quantization for the holomorphic cases 5
0.4. A quick guide of the contents 7
1. Prelude 8
1.1. Theme 1: Mirror symmetry of Catalan numbers 8
1.2. Theme 2: When is a space a moduli space? 12
1.3. Theme 3: Finding a curve from a period matrix 14
1.4. Theme 4: The τ-functions 15
2. Fugue of the theme of spectral curves 17
2.1. The Laplace transform 17
2.2. Simple Hurwitz numbers [51] 19
2.3. Fugue in two themes: The Laplace transform of the simple Hurwitz numbers 23
2.4. Fugue in three themes: Two quantum curves for the Lambert curve 28
2.5. Fugue in four themes: New proofs of Witten-Kontsevich and the λ_g formulas 31
2.6. Coda: From a tree counting to the Lambert curve 33
3. Variations of spectral curves in Higgs bundles and opers 35
3.1. Variation 1: Moduli spaces of Higgs bundles and character varieties 35
3.2. Variation 2: Higher order linear differential operators on a curve C 40
3.3. Variation 3: Spectral curves and opers 42

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4. Finale Incompiuto: The Geometry that knows the irrationality of $\zeta(3)$ 44
4.1. The Mystery Formula 44
4.2. A quick review of Apéry's ideas 45
4.3. The differential equations behind the scene 47
Appendix A. The Lagrange Inversion Formula 49
References 50

0. Introduction

0.1. **The story begins with Catalan numbers and their Laplace transform.** Let us start with two differential equations

$$\left((x^2-4)\,\frac{d^2}{dx^2}+x\,\frac{d}{dx}-1\right)z(x)=0, \tag{0.1}$$

$$\left(\hbar^2\frac{d^2}{dx^2}+\hbar x\frac{d}{dx}+1\right)\Psi(x,\hbar)=0. \tag{0.2}$$

Question 0.1. *What is the common algebraic curve hidden behind these two equations?*

The first equation is easy to solve. Obviously x itself is a solution, and $\sqrt{x^2-4}$ also solves it. We can thus choose

$$z(x)=\frac{x\pm\sqrt{x^2-4}}{2}$$

as a basis for all solutions. It then reminds us of the quadratic formula for $z^2-xz+1=0$, and a possible relation to the second equation through the *Weyl quantization* $\begin{cases} z\longmapsto -\hbar\frac{d}{dx},\\ x\longmapsto x.\end{cases}$ Equivalently, we can find the same polynomial from the *semi-classical limit* of (0.2):

$$\lim_{\hbar\to 0} e^{-\frac{S_0(x)}{\hbar}}\left(\hbar^2\frac{d^2}{dx^2}+\hbar x\frac{d}{dx}+1\right)e^{\frac{S_0(x)}{\hbar}}=\big(S_0(x)'\big)^2+xS_0(x)'+1,\quad z=-S_0(x)'.$$

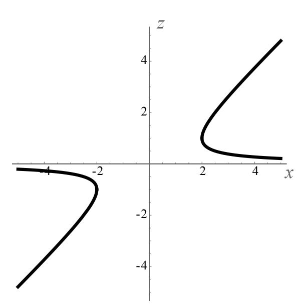

Figure 0.1. The mirror dual spectral curve Σ of the Catalan numbers.

The curve in common appears to be simply $x=z+\frac{1}{z}$. Is there anything significant here? Indeed, this curve will tell us a rich story of geometry that is *not* obvious at all from the shape of these differential equations. As we see in the main text (1.2), $z(x)$ is an unconventional generating function of Catalan numbers C_m : $1,2,5,14,42,132,429,1430,4862,16796\ldots$ One interpretation of these numbers is the count of *cell-decompositions* of a two-dimensional

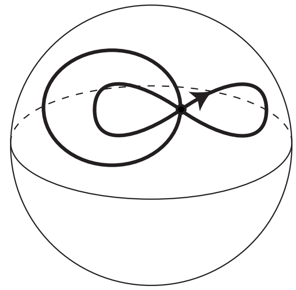

FIGURE 0.2. A cell-decomposition of a curve of genus 0 with one 0-cell. The arrow on a half-edge indicates no rotation symmetry is allowed.

sphere S^2 with one 0-cell and m 1-cells, while rotation symmetry is not allowed to avoid complication coming from automorphism count (see Remark 2.29).

From this point of view, the counting can be generalized to cell-decompositions on an arbitrary compact oriented surface of genus g with n 0-cells. This is the story of Section 1.1, based on the author's papers [17, 31, 33]. The unexpected result is that the generating functions $F_{g,n}(t_1, \dots, t_n)$ of the numbers of these cell-decompositions are ***Laurent polynomials*** of degree $6g - 6 + 3n$ when $2g - 2 + n > 0$. This *polynomiality* is surprising, and is a key to relate the story with many different subjects of geometry. We will show that these Laurent polynomials know, as their special values, the Euler characteristic $\chi(\mathcal{M}_{g,n})$ of the moduli spaces of smooth pointed curves first calculated by Harer-Zagier [48], and even lead to the Witten-Kontsevich intersection theory of tautological cotangent classes on $\overline{\mathcal{M}}_{g,n}$ [23, 54, 95] through their asymptotic behavior.

What do these geometric quantities of moduli spaces have anything to do with the differential equation (0.2) above? We will see that its solution $\psi(x, \hbar)$ is the *generating function* of these generating functions $F_{g,n}$ for all $g \geq 0$ and $n > 0$. In this context, since the curve $x = z + 1/z$ is the **semi-classical limit** of (0.2), the differential equation is called the **quantum curve** [1, 22, 50] associated with the classical curve $x = z + 1/z$. The Catalan numbers are the count of cell-decompositions of the $(g, n) = (0, 1)$ geometry. Hence going from $z(x)$ to $F_{g,n}(t_1, \dots, t_n)$ for arbitrary (g, n) is the process of *quantization*.

At the same time, the fact that generating functions $F_{g,n}(t_1, \dots, t_n)$ know the ψ-class intersection numbers of $\overline{\mathcal{M}}_{g,n}$ indicates that they give the Gromov-Witten invariants $GW_{g,n}(\bullet)$ of a point. Therefore, from the point of view of Mariño [65] and Bouchard-Klemm-Mariño-Pasquetti [13], we understand that the spectral curve $x = z + 1/z$ is the *mirror* B-model, corresponding to the Gromov-Witten invariants on the A-model. From the Catalan numbers to this point of view of mirror symmetry is the theme of Section 1.1.

The polynomiality of some quantities appearing in enumerative problems associated with the moduli spaces $\overline{\mathcal{M}}_{g,n}$ was discovered in the author's earlier paper [76], which was the key to the solution [36] of the Hurwitz number conjecture of Bouchard and Mariño [14]. The polynomiality of the generating functions of simple Hurwitz numbers established in [76] presented another surprise: it gives simple few-line proofs of Witten's conjecture [95] and the λ_g-conjecture of Faber-Pandharipande [38, 39].

The stories coming out of Hurwitz numbers are weaved in Section 2. The hidden curve in this context is the **Lambert curve** (2.15), whose role in discovering the polynomiality is explained in detail. Bouchard and Mariño [14] identified the Lambert curve through the limiting process of the *mirror curves* of toric Calabi-Yau 3-folds [13], and using Lagrange's Inversion Formula [94]. Appendix gives an elementary analysis that leads to this identification of the Lambert curve. In Section 2.6, we will show that the Lambert curve is actually a simple consequence of *tree counting*, and the formula for the curve, i.e., the *Lambert function*, directly follows from a combinatorial identity of trees.

Our point is that all these known results are obtained by finding the hidden curve behind the scene, the **spectral curve** of the counting problem. And the spectral curve is always the generating function of the type $(0,1)$-invariants of the problem. The **quantum curve** is a family of $\hbar$-deformations of a differential operator whose *semi-classical limit* is the spectral curve. We will explain this relation for the case of Catalan numbers and Hurwitz numbers in detail in the main text.

Remark 0.2. We will use the terminology **Laplace Transform** in a slightly more general context.

- The process from the sequence $\{C_m\}_{m=0}^{\infty}$ of Catalan numbers to their *generating function* and its inverse function

$$z(x) = \sum_{m=0}^{\infty} \frac{C_m}{x^{2m+1}}, \qquad x = z + \frac{1}{z}$$

  is considered as the **Laplace transform** in these lectures. This is an idea developed over the years (see for example, [17, 33, 72, 73, 75, 76]). The rationale behind it is that when a sequence satisfies a combinatorial relation, we can take the Laplace transform of the relation. Often the result becomes a system of differential equations. Therefore, the **Laplace transform changes combinatorics to geometry**.
- This effect is explained in detail in Section 2 using Hurwitz numbers. The starting sequence is the number of *rooted trees* on k nodes and their generating function,

$$y(x) = \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} x^k.$$

  We will show that a simple combinatorial identity of tree counting allows us to identify its inverse function, the Lambert curve, $x = ye^{-y}$, in Section 2.6.
- Both $y(x)$ and $z(x)$ above land on $\mathbb{P}^1$, so we use an automorphism of $\mathbb{P}^1$ to bring these variables to the "right" coordinate, which we write as $t \in \mathbb{P}^1$. The key of Section 1.1 is that the Laplace transform of the genus g, n marked point version of the Catalan numbers $C_{g,n}(\mu_1, \dots, \mu_n)$ for $2g - 2 + n > 0$ is a Laurent polynomial in the t-variables. The special values of these Laurent polynomials are identified as the Euler characteristic $\chi(\mathcal{M}_{g,n})$ of the moduli space of smooth pointed curves [72], and the asymptotic behavior of these Laurent polynomials recovers the ψ-class intersection numbers on the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ [17].
- In Section 2.3 it is explained that the Laplace transform of the (g,n)-Hurwitz number in the stable region $2g - 2 + n > 0$ is a polynomial in the t-variables. We will prove that the Laplace transform of the combinatorial formula, the *cut-and-join* equation of [46, 91], is a differential equation, and that it automatically proves the Witten-Kontsevich theorem on the ψ-class intersection numbers and the Faber-Pandharipande theorem on the λ_g-conjecture [76].
- So what is the Laplace transform, after all? Our thesis in Section 2.1 is that the **Laplace transform is the mirror symmetry**.

0.2. **A miraculous integer sequence.** In the final Section 4, we explore stories from uncharted territories. Let $H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$ be the m-th harmonic number. We use the convention $H_0 = 0$. For every $n \geq 0$, define

$$\widetilde{GW}_{0,1}(n) := (-1)^n (n!)^2 \sum_{\substack{\ell+m=n \\ \ell,m\geq 0}} \frac{(2\ell+m)!(\ell+2m)!}{(\ell!)^5(m!)^5}\Big(1+(m-\ell)\big(H_{2\ell+m}+2H_{\ell+2m}-5H_m\big)\Big)$$

and

$$A_n := \sum_{\ell=0}^{n} \binom{n}{\ell}^2 \binom{n+\ell}{\ell}^2.$$

Then we have (see [45, 96]):

$$A_n = \widetilde{GW}_{0,1}(n) \qquad \text{for all } n. \tag{0.3}$$

The first few terms are: $1, 5, 73, 1445, 33001, 819005, 21460825, 584307365, 16367912425, \ldots$. What is unbelievable about this equality is that it is not obvious at all that $\widetilde{GW}_{0,1}(n)$ as defined by the formula gives a **positive integer** sequence, let alone it is equal to the second line. What we know about (0.3) so far includes the following:

- The formula comes from the **mirror symmetry**, a geometric relation between the Gromov-Witten invariants of a 3-dimensional variety and the Gauss-Manin connection associated with its mirror partner. In this particular context, it is proved in Golyshev-Zagier [45]. We refer to Zagier [96] for an inspiring account of this interplay.
- The quantity $\widetilde{GW}_{0,1}(n)$, the **quantum period**, was obtained by Coates, Corti, Galkin, and Kasprzyk [18], which is essentially the same as the genus 0, 1-marked point case of the degree n Gromov-Witten invariant of a Fano 3-fold known as V_{12}. The generating function of these numbers, after appropriate modifications including the Borel-Laplace transform, satisfies a **Quantum Differential Equation**. A brief background will be explained in Section 4.
- The sequence $\{A_n\}_{n=0}^{\infty}$ was discovered by Apéry in 1978 [3] in his **proof of the irrationality** of a special value $\zeta(3)$ of the Riemann zeta function. A quick review of his proof, including the significance of the integer sequence $\{A_n\}_{n=0}^{\infty}$, is presented in Section 4.2.
- The differential equation that determines the generating function of $\{A_n\}_{n=0}^{\infty}$ was discovered to be the **Picard-Fuchs equation** of a particular 1-parameter family of K3 surfaces by Beukers and Peters [9].

What we can hope to see happens, yet still not established, is the following:

- There should be a **spectral curve** hidden behind the scene, determined by the type $(0,1)$ Gromov-Witten invariants of V_{12}. This should essentially be the same object as the Picard-Fuchs equation mentioned above, being on the B-model side.
- There should be an extension of the **quantization procedure**, explained in Section 3.3 for differential equations with holomorphic coefficients, to differential equations with *irregular singular points*. This is the story of **quantum curves**.

0.3. **The unique quantization for the holomorphic cases.** The correspondence between classical systems and quantum systems is never one-to-one. If we imagine the classical system to be the limit of a quantum system as the Planck constant $\hbar \to 0$, then we can add anything multiplied with $\hbar$ to the quantum equation. This addition vanishes in this limit.

It is therefore rather counter intuitive that there is a bijective, and even biholomorphic, correspondence between spectral curves as classical systems and quantum curves as quantum systems, discovered in collaboration by the author [27, 32], prompted by the idea of Gaiotto

[41]. Suppose we have a holomorphic n-th order ($n \geq 1$) linear ordinary differential operator P *globally defined* on a smooth complete complex algebraic curve C of genus $g(C) > 1$, whose leading coefficient does not vanish anywhere on C. One interpretation of the above result proves that it determines a unique algebraic curve $\pi : \Sigma \longrightarrow C$, known as a *spectral curve* of a *Higgs* bundle, that recovers P through the *conformal limit construction* of Gaiotto [41]. The passage from P to Σ includes:

- Identification of the canonical (i.e., unique) $\hbar$-deformation family $P^\hbar$ of P such that P appears in this family at $\hbar = 1$; and
- Calculating its *semi-classical limit* as $\hbar \to 0$ that selects Σ as the classical geometric object corresponding to this family of deformations.

This is counter intuitive because a family $P^\hbar$ can determine P at $\hbar = 1$, but not in the other way around. Also, from classical Σ to a quantum $P^\hbar$ is usually never unique.

In terms of a local coordinate z of a local neighborhood $U \subset C$, P is a noncommutative polynomial in d/dz of degree n with coefficients in holomorphic functions in $\mathcal{O}_C(U)$. The $\hbar$-deformation changes it to a 1-dimensional Schrödinger operator on U, which patches together to a globally defined operator $P^\hbar$. The process of semi-classical limit is equivalent to taking the "total symbol" of this differential operator. After a suitable choice of its coordinate and a conjugation action of the operator with a locally non-vanishing function, we can write

$$P = \left(\frac{d}{dz}\right)^n + \sum_{k=2}^{n} a_k(z)\left(\frac{d}{dz}\right)^{n-k}, \qquad a_k \in \mathcal{O}_C(U) \tag{0.4}$$

on U. Then the equation $P\psi = 0$ is equivalent to $\nabla_z \Psi = 0$, where

$$\nabla_z := \frac{d}{dz} + \begin{bmatrix} 0 & a_2 & \cdots & a_{n-1} & a_n \\ -1 & & & & \\ & -1 & & & \vdots \\ & & \ddots & & \\ & & & -1 & 0 \end{bmatrix} \quad \text{and} \quad \Psi = \begin{bmatrix} \psi^{(n-1)} \\ \psi^{(n-2)} \\ \vdots \\ \psi' \\ \psi \end{bmatrix}. \tag{0.5}$$

Therefore, locally an n-th order differential equation $P\psi = 0$ is always equivalent to the flatness equation $\nabla_z \Psi = 0$ with respect to a holomorphic connection ∇_z. In Section 3, we will translate the condition that P is globally defined on C into a set of properties of this connection. The requirements include P acts on the line bundle $K_C^{-\frac{n-1}{2}}$, and the connection ∇ acts on a vector bundle that has a full-flag filtration with **Griffiths transversality**. Such a connection is known as an **oper** [6, 7].

The statement we mentioned above is thus equivalent to the assertion that *every oper on C corresponds in a one-to-one manner to a spectral curve Σ covering C.* The $\hbar$-deformation of ∇ is the $\hbar$-connection $\nabla^\hbar$ of Deligne, and the parameter $\hbar$ is identified as an element $\hbar \in H^1(C, K_C)$ (see [32]), which determines the *extension classes* of line bundles associated with the full-flag filtration. In [27], the collaborators and the author constructed a globally defined linear differential operator on a curve C, or an oper, from an arbitrary spectral curve of a G-Hitchin system on a base curve C, where G is a simple Lie group of adjoint type. This is a biholomorphic construction from the moduli space of spectral curves to the moduli space of opers. Therefore, it means that the spectral curve is uniquely chosen from a given oper. And because of the passage explained above, this also implies that a higher order differential operator uniquely identifies a *quantization* of the spectral curve, i.e., the $\hbar$-family of deformations of the starting operator, or equivalently, a Deligne's $\hbar$-connection.

There is an important difference between holomorphic differential operators P on a curve C of genus $g(C) > 1$, and the examples coming from enumeration problems mentioned above. When P, and its corresponding oper ∇, is holomorphic, any *solution* to the equation $P\psi = 0$ is everywhere holomorphic. So we do not expect it to contain any new geometric information of something that goes beyond the given context, such as topology of $\overline{\mathcal{M}}_{g,n}$ as in the examples. We expect that when we consider P with *irregular singularities*, a whole new story begins. This is an active area of research in geometry. We refer to [2, 19, 20, 42].

Question 0.3. *Is there an analogous correspondence between differential operators with irregular singular points and singular spectral curves, both defined over C?*

Even for $C = \mathbb{P}^1$, if such a correspondence is established, then it should tell us a lot of stories behind some deep mysteries, such as the geometry behind the irrationality of $\zeta(3)$.

0.4. **A quick guide of the contents.**

- The story given in Section 1.1 illustrates the model of the theory: The expansion of a solution of a differential equation (= quantum curve) around its irregular singular point contains a profound amount of geometric information not obvious from the given setting.
- A story of Hurwitz numbers is presented in Section 2 with some details. In this case the quantum curve corresponding to the spectral curve is a difference-differential operator, or a differential operator of an infinite-order. This is due to the fact that the spectral curve, the Lambert curve in this case, is an analytic curve, not an algebraic curve. So far we do not have any counterpart generalization of the theorem of [27] for difference operators.
- What we mean by a globally defined high order linear differential operator on a curve C of genus $g(C) > 1$ is explained in Section 3.2. There and in the following Section 3.3, we will encounter the geometric meaning of the parameter $\hbar$, how it determines the *unique* quantization from the starting classical spectral curve, and how the **projective coordinate system** appears in the global construction.
- An analogy of two differential equations described in (0.1), a Picard-Fuchs equation, and an Hermite-Weber equation (0.2), appears in a context of Gromov-Witten invariants of a Fano 3-fold. This is also deeply and mysteriously related to the integer sequence playing a key role in the *irrationality proof* of $\zeta(3)$. This open-ended story is presented at the end of these lectures in Section 4.

These are the notes based on the author's series of talks delivered in Köln, Hamburg, Osaka, Oxford, Zürich, Riverside, Kyoto, Hiroshima, Kobe, Les Diablerets, Madrid, and Seattle in the last two years or so. They are designed to tell a story of the exploration: "*In Search of a Hidden Curve.*"

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1. Prelude

1.1. **Theme 1: Mirror symmetry of Catalan numbers.** Mathematics thrives on mysteries. *Mirror symmetry* has been a great mystery for a long time, and has served as a driving force in many areas of mathematics. Even after three and a half decades since its conception in physics, it still produces new challenges to mathematicians. One of the starting points of this forever expanding universe of research frontier is the 1991 discovery of Candelas, De La Ossa, Green and Parkes [15]. From this paper's scope, we learn that the following four subjects of mathematics,

- combinatorial counting problems,
- algebraic geometry over $\mathbb{C}$, and more lately over $\mathbb{F}_q$ and p-adic fields,
- Picard-Fuchs differential equations, and
- nonlinear integrable systems

are deeply intertwined in a manner beyond the realm of classical mathematics, and that an interplay of these different subjects leads us to new insight and further understanding of mathematics. Some of the developments of mathematics stemming out of mirror symmetry are compactly characterized as *quantum* mathematics.

To illustrate how these items listed above appear and interact together, let us begin with a naïve question:

Question 1.1. *What is the mirror symmetric dual of the Catalan numbers?*

Here our usage of the terminology *mirror symmetry* is not conventional. At least at this moment, our question is not directly interpreted from the point of veiw of the *homological mirror symmetry* of [55]. Catalan numbers $C_m = \frac{1}{(m+1)}\binom{2m}{m}$, a sequence of positive integers, exhibit solutions to many different combinatorial problems.

The first aim of these lectures is to present a few problems that have been solved when a hidden *curve* behind the scenes was discovered. These curves are commonly called *spectral curves*. As we see below, as soon as the spectral curve is identified, it acts as a catalyst to let the subject in question weave the whole story in front of us. And the important common feature of these curves is that they are *complex Lagrangians* in a complex symplectic surface.

In [33], we proposed that a *spectral curve*

$$\Sigma = \left\{ (x,z) \;\middle|\; x = z + \frac{1}{z} \right\} \tag{1.1}$$

is the mirror dual to the Catalan numbers. It is not hard to see why this curve has something to do with Catalan numbers: The inverse function $z = z(x)$ of the equation $x = z + 1/z$ that satisfies $\lim_{x\to\infty} z(x) = 0$ is an unconventional generating function of Catalan numbers

$$z(x) = \sum_{m=0}^{\infty} C_m \, \frac{1}{x^{2m+1}}. \tag{1.2}$$

This is an absolutely convergent series for $|x| > 2$. First, we observe that this curve immediately defines a differential equation. Since x and $z(x)$ satisfy a polynomial relation $z^2 - xz + 1 = 0$, the x-derivatives $z'(x)$, $z''(x)$, $z'''(x), \ldots$ are all in the extension field $\mathbb{C}(x, z)$ of degree 2 over the field of rational functions $\mathbb{C}(x)$. Therefore, z, z', and z'' are linearly dependent over the polynomial ring $\mathbb{C}[x]$. The simplest linear relation is

$$\left((x^2-4)\,\frac{d^2}{dx^2} + x\,\frac{d}{dx} - 1\right) z(x) = 0. \tag{1.3}$$

This differential equation is equivalent to the recursion formula

$$C_m = \frac{2(2m-1)}{m+1} C_{m-1}, \qquad C_0 = 1,$$

with respect to the generating function $z(x)$. The differential equation (1.3) has three regular singular points at $x = -2, 2, \infty$. (The definition of this terminology is given below in Definition 1.2.) The analytic continuations of $z(x)$ to $\mathbb{C}$ from the neighborhood of $\infty \in \mathbb{P}^1$ are

$$z(x)_{\pm} = \frac{x \pm \sqrt{x^2-4}}{2} = \left(\frac{x + \sqrt{x^2-4}}{2}\right)^{\pm 1},$$

which produce obvious solutions $x = z(x)_+ + z(x)_-$ and $\sqrt{x^2-4}$ to (1.3). Thus the monodromy property of the differential equation (1.3) is very simple at each singular point, which is the consequence of the regular singularity at these points.

Even though this example is trivial in many sense, the point here is that (1.3) is a *Picard-Fuchs* equation associated with the projection $\pi : \Sigma \longrightarrow \mathbb{P}^1$ defined by the algebraic equation $x = z + 1/z$. This is a trivial *Landau-Ginzburg* model. A coordinate transformation $x = 4t - 2$ brings this differential equation to one of the Euler-Gauß hypergeometric differential equations,

$$\left(t(1-t)\frac{d^2}{dt^2} + (-\frac{1}{2} + t)\frac{d}{dt} - 1\right) y(t) = 0 \tag{1.4}$$

with regular singular points at $0, 1, \infty$. The cohomology $H^0(\pi^{-1}(x), \mathbb{Z})$ defines a Gauss-Manin connection in a local system over $\mathbb{P}^1$.

Another differential equation that is determined by the spectral curve (1.1) is a Schrödinger equation (cf. the *quantum curve* of [28, 29])

$$\left(\hbar^2 \frac{d^2}{dx^2} + \hbar x \frac{d}{dx} + 1\right) \Psi(x, \hbar) = 0. \tag{1.5}$$

This equation has only one irregular singular point at ∞, and no regular singular points. Hence a solution is an entire function on $\mathbb{C}$ with an essential singularity at ∞.

Let us briefly review the mechanism of semi-classical limit here. The WKB method allows us to find an asymptotic solution of (1.5) in terms of the exponential of a *Laurent series* expansion in $\hbar$. We impose that

$$\psi(x,\hbar) = \exp\left(\frac{s_0(x)}{\hbar}\right)\exp\left(\sum_{m=1}^{\infty}\hbar^{m-1}s_m(x)\right)$$

is a solution to (1.5) and derive differential equations for each $s_m(x)$. The idea here is that as $\hbar \to 0$, the function $s_0(x)$ has a dominant importance, which should recover the classical behavior of the quantized equation (1.5). Since the above expression has no meaning as a series in $\hbar$ because each term $\hbar^n$ is an infinite sum. So we rewrite the equation as

$$\left[\exp\left(-\frac{s_0(x)}{\hbar}\right)\left(\hbar^2\frac{d^2}{dx^2}+\hbar x\frac{d}{dx}+1\right)\exp\left(\frac{s_0(x)}{\hbar}\right)\right]\exp\left(\sum_{m=1}^{\infty}\hbar^{m-1}s_m(x)\right) = 0.$$

Then the operator acting on $\exp\left(\sum_{m=1}^{\infty}\hbar^{m-1}s_m(x)\right)$ is

$$\left(\hbar^2\frac{d^2}{dx^2}+\hbar x\frac{d}{dx}+1\right)+2\hbar s_0'(x)\frac{d}{dx}+\hbar s_0''(x)+\big(s_0'(x)\big)^2+xs_0(x)',$$

hence the equation produces only non-negative powers of $\hbar$. The semi-classical limit is the limit of $\hbar \to 0$ at this stage. Clearly, the operator converges to $\big(s_0'(x)\big)^2+xs_0(x)'+1$, which is multiplied to $e^{s_1(x)}$. Therefore, the equation becomes an algebraic equation $\big(s_0'(x)\big)^2+xs_0(x)'+1=0$, which is the same as $z^2-xz+1=0$ by substituting $s_0(x)'=-z$.

What we find is that the semi-classical limit in this context is equivalent to replacing $\begin{cases}\hbar\frac{d}{dx}\longrightarrow -z,\\ x\longrightarrow x\end{cases}$ in (1.5). Hence the spectral curve (1.1) is recovered from the quantum curve (1.5). In other words, the quantum curve (1.5) is the result of *Weyl quantization* of the spectral curve $z^2-xz+1=0$.

Since we are already using the terminology of regular singular and irregular singular points of differential equations, let us explain what they are.

Definition 1.2. Let

$$\left(\frac{d^2}{dx^2}+a_1(x)\frac{d}{dx}+a_2(x)\right)\Psi(x)=0 \tag{1.6}$$

be a second order differential equation defined around a neighborhood of $x=0$ on a small disc $|x|<\epsilon$ with meromorphic coefficients $a_1(x)$ and $a_2(x)$ with poles at $x=0$. Denote by k (reps. ℓ) the order of the pole of $a_1(x)$ (resp. $a_2(x)$) at $x=0$. If $k\le 1$ and $\ell\le 2$, then (1.6) has a **regular singular point** at $x=0$. Otherwise, consider the *Newton polygon* of the order of poles of the coefficients of (1.6). It is the upper part of the convex hull of three points $(0,0),(1,k),(2,\ell)$. As a convention, if $a_j(x)$ is identically 0, then we assign $-\infty$ as its pole order. Let $(1,r)$ be the intersection point of the Newton polygon and the line $x=1$, $r=\begin{cases}k & 2k\ge\ell,\\ \frac{\ell}{2} & 2k\le\ell.\end{cases}$ If $r>1$, (1.6) has an **irregular singular point of class** $r-1$.

Remark 1.3. Jacob [53] recently discovered a 2-parameter family of (1.1) associated with different combinatorial objects. How the story of this section changes with this new family is a subject of future investigation.

The significance of (1.5) is in the expression of its solution, asymptotically expanded at its essential singularity:

$$\Psi(x,\hbar) = \exp\left(\sum_{g\geq 0, n>0} \frac{1}{n!}\hbar^{2g-2+n} F_{g,n}(x,\dots,x)\right), \tag{1.7}$$

where $F_{g,n}(x,\dots,x)$ is the *principal specialization* of the symmetric functions

$$F_{0,1}(x) = -\frac{1}{2}z(x)^2 + \log z(x), \tag{1.8}$$

$$F_{0,2}(x_1,x_2) = -\log\big(1 - z(x_1)z(x_2)\big), \tag{1.9}$$

$$F_{g,n}(x_1,\dots,x_n) = \sum_{\mu_1,\dots,\mu_n>0} \frac{C_{g,n}(\mu_1,\dots,\mu_n)}{\mu_1\cdots\mu_n} \prod_{i=1}^n x_i^{-\mu_i}, \quad 2g-2+n>0. \tag{1.10}$$

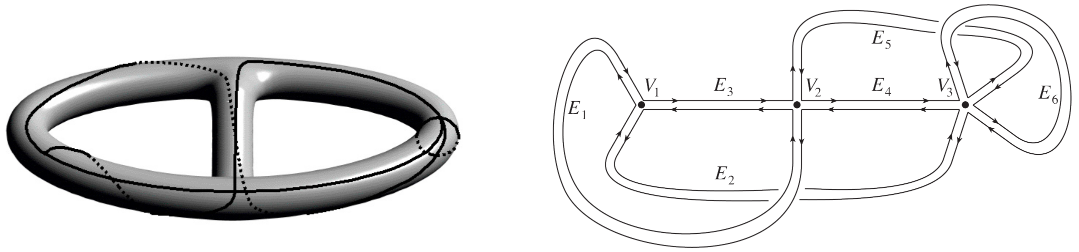

FIGURE 1.1. 1-Skeleton of a Cell Decomposition and a Cell Graph

Here comes the relation to enumeration. The coefficients of the expansion $C_{g,n}(\mu_1,\dots,\mu_n)$ of (1.10) are the generalized Catalan numbers of type (g,n) that count the numbers of **cell graphs** of genus g and n labeled vertices of degrees $(\mu_1,\dots,\mu_n)$ [30, 31, 33, 71, 72]. A cell graph is the 1-skeleton of a cell decomposition of a connected oriented surface of genus g with n labeled 0-cells. Its *dual graph* is commonly called a *ribbon graph* on which faces are labeled. To avoid the difficulties of counting automorphisms of graphs, we impose that no cyclic rotations of incident edges at any vertex are allowed, as in Figure 0.1. This explains the denominator $\mu_1\cdots\mu_n$ of (1.10) (see Remark 2.29). For $(g,n)=(0,1)$, the unique vertex has to have an even degree, and $C_{0,1}(2m) = C_m$ is the m-th Catalan number. The surprising properties of the functions $F_{g,n}$ are the following (see for example, [31, Theorem 2.7]).

Theorem 1.4 ([17, 31, 32, 33, 72])**.** *For the case of $2g = 2+n > 0$, substitute each x_i in (1.10) with* $\begin{cases} x_i = z_i + \frac{1}{z_i}, \\ z_i = \frac{t_i+1}{t_i-1}, \end{cases}$ *and write the result as $F_{g,n}(t_1,\dots,t_n)$ incorporating these substitutions. Then for every (g,n) in the range of $2g-2+n>0$, the following holds:*

- *$F_{g,n}(t_1,\dots,t_n)$ is a Laurent polynomial in the t-variables of degree $6g-6+3n$.*
- *$F_{g,n}(t_1,\dots,t_n) = F_{g,n}(1/t_1,\dots,1/t_n)$.*
- *$F_{g,n}(1,\dots,1) = (-1)^n\chi(\mathcal{M}_{g,n})$.*
- *The restriction to the highest degree terms of $F_{g,n}(t_1,\dots,t_n)$ is a homogeneous polynomial*

$$F^{top}_{g,n}(t_1,\dots,t_n) = \frac{(-1)^n}{2^{2g-2+n}} \sum_{\substack{d_1+\cdots+d_n \\ =3g-3+n}} \langle\tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n} \prod_{i=1}^n (2d_i-1)!!\left(\frac{t_i}{2}\right)^{2d_i+1}. \tag{1.11}$$

Here, $\mathcal{M}_{g,n}$ is the moduli space of smooth n-pointed algebraic curves of genus g, $\overline{\mathcal{M}}_{g,n}$ its compactification, i.e., the Deligne-Mumford stack of stable curves of finite type (g,n), and

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} c_1(\mathbb{L}_1)^{d_1} \cdots c_1(\mathbb{L}_n)^{d_n}$$

is the intersection number of the first Chern classes of the tautological line bundles $\mathbb{L}_i \longrightarrow \overline{\mathcal{M}}_{g,n}$ that is determined by the i-th marked point on stable curves (see (2.8) below).

The contrast between the Picard-Fuchs equation (1.3) and the irregular singular equation (1.5) is the *quantization* effect: the former only determines the original Catalan numbers, while the latter has the information of all (g,n)-Catalan numbers $C_{g,n}(\mu_1, \dots, \mu_n)$. This effect is often observed in the *Borel-Laplace* transform of differential equations. The passage from the spectral curve Σ of (1.1) to the quantum curve (1.5) is the inverse operation of taking the semi-classical limit. As noted in [29], the conic Σ, a nonsingular plane curve in $\mathbb{P}^2$, should be placed in the cotangent bundle $T^*\mathbb{P}^1$ to consider its quantum curve. The irregular singularity at $\infty \in \mathbb{P}^1$ of (1.5) comes from the fact that the embedding of Σ into the compactified cotangent bundle $\overline{T^*\mathbb{P}^1}$ of $\mathbb{P}^1$, or the Hirzebruch surface $\mathbb{F}^2$, has a singularity at $\infty \in \mathbb{P}^1$.

$$\begin{array}{ccccc} \Sigma & \xrightarrow{\ i\ } & \overline{T^*\mathbb{P}^1} & \xrightarrow{\ \cong\ } & \mathbb{F}^2 \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle\pi} & & \\ & & \mathbb{P}^1 & & \end{array} \tag{1.12}$$

While the Picard-Fuchs equation (1.3) is associated with the the conic $\Sigma \subset \mathbb{P}^2$ and the ramified double-sheeted covering $\pi : \Sigma \longrightarrow \mathbb{P}^1$, the irregular singular equation (1.5) represents the symplectic geometry of $T^*\mathbb{P}^1$ and the singularity of Σ in the compactified fiber of $\pi : \overline{T^*\mathbb{P}^1} = \mathbb{F}^2 \longrightarrow \mathbb{P}^1$ at $\infty \in \mathbb{P}^1$. Precisely this geometric difference is reflected in the *quantization* process, from the $(0,1)$ invariants in $z(x)$ of (1.1) to a function $\Psi(x,\hbar)$ of (1.7) that has the information of invariants for all values of (g,n).

Although we do not touch in these lectures, there is yet another coordinate change from $(t_1, t_2, t_3, \dots)$ appearing in the Catalan case to the time evolution parameters $(t_1, t_2, t_3, \dots)$ in the Lax equations (1.17) defined below. Our use of the same notations is due to the conventions of the author's previous publications. In reality, these are not the same variables. After the choice of the right coordinate transformation,

$$\Psi(\mathbf{t}, \hbar) = \exp\left(\sum_{g \ge 0, n > 0} \frac{1}{n!} \hbar^{2g-2+n} F_{g,n}(t_1, t_2, t_3, \dots) \right) \tag{1.13}$$

becomes a τ-function for the KP equations. When we restrict $F_{g,n}$ to the highest degree terms and apply the same coordinate change, this τ-function then becomes that of the KdV equations of Witten [95] and Kontsevich [54].

1.2. **Theme 2: When is a space a moduli space?** A space, or a **manifold** is a democratic space. Every point of a manifold is treated equal. No matter where you go on a manifold, no matter which point you choose, its neighborhood is *isomorphic* to that of any other point. There are no special points, or *singular* points. No point dominates other points, and no point shows its individuality. Everywhere is the same and things work smoothly on a manifold.

A **moduli space** is different. On this space, every point has its *unique* name. Each point is distinguished by its individuality. No points on a moduli space are the same. A neighborhood of each point is different from place to place. There may also be *singularities* on a moduli space. Each point has its own characteristic importance, yet it is a member of a smooth family. Since every point is different, the moduli space itself presents its miracle: it can exist, as a harmonious *space*!

We are not asking when a moduli space is a manifold. Our purpose is different. We are wondering when a given space is actually a moduli space.

A simplest example we can consider is a vector space $\mathbb{C}^n$. Can $\mathbb{C}^n$ be a moduli space? Hitchin [49] told us: *Yes*! But for some special values of the dimension n. He starts with a smooth algebraic curve C of genus $g > 1$, and defines a vector space

$$B := \bigoplus_{i=2}^{r} H^0\big(C, K_C^{\otimes i}\big) \cong \mathbb{C}^{(r^2-1)(g-1)} \tag{1.14}$$

consisting of *multi differentials* on C. Here, $K_C = \Omega_C^1$ is the canonical sheaf of C consisting of holomorphic 1-forms. This vector space is the *moduli space of spectral curves* for $SL_r(\mathbb{C})$-Higgs bundles. This example also explains the title of these lecture notes: *In search of a hidden curve*. The vector space B becomes a moduli space as soon as we find a curve C, and the dimension n factors as $n = (r^2-1)(g-1)$ for some $r > 1$, which is associated with the group $SL_r(\mathbb{C})$.

As another simple example, let us start, again, with the complex vector space $\mathbb{C}^n$ of dimension $n > 0$, but this time n is arbitrary, and we choose a set of $2n$ vectors in it linearly independent over $\mathbb{R}$. These vectors generate a $\mathbb{Z}$-submodule $\Gamma \subset \mathbb{C}^n$ of rank $2n$, a full-rank $\mathbb{Z}$-*lattice*. The quotient space $T := \mathbb{C}^n/\Gamma$ is a compact complex manifold whose underlying space is the real $2n$-dimensional torus $(S^1)^{2n}$. Now the question: When is this quotient space a *moduli space*?

We give a linear coordinate system on $\mathbb{C}^n$ and represent the $2n$ generators of Γ by a set of $2n$ column vectors $w_1, \dots, w_{2n}$ of size n. They form a complex matrix Π of size $n \times 2n$. $GL_n(\mathbb{C})$ acts on Π from the left, representing the coordinate change. Since the column vectors of the matrix Π generate the $\mathbb{Z}$-module Γ, $SL_{2n}(\mathbb{Z})$ acts on Π from the right, representing change of generators of the same lattice Γ. Since $\{w_1, \dots, w_{2n}\}$ is linearly independent over $\mathbb{R}$, there is a $\mathbb{C}$-linearly independent subset consisting of n vectors. Thus we can use a left multiplication of $GL_n(\mathbb{C})$ and a right multiplication of $SL_{2n}(\mathbb{Z})$ on Π to bring it to the special shape $[I \,|\, \Omega]$. The geometry of the quotient space $\mathbb{C}^n/\Gamma$ is encoded in this $n \times n$ matrix Ω, which we call the *period matrix* of the torus $T = \mathbb{C}^n/\Gamma$.

A classical result shows that if the period matrix satisfies the *Riemann period condition* ${}^t\Omega = \Omega$ and $Im(\Omega) > 0$, i.e., Ω is symmetric and its imaginary part is positive definite, then the torus admits a holomorphic embedding $\mathbb{C}^n/\Gamma \subset \mathbb{P}^N$ into a complex projective space of a large dimension N. When it happens, the quotient $\mathbb{C}^n/\Gamma$ can be defined by a set of homogeneous polynomial equations in $N + 1$ variables, and it becomes an *Abelian variety*. The choice of the shape $\Pi = [I \,|\, \Omega]$ brings to the quotient a *principal polarization*. For those complex tori with periods satisfying the Riemann period condition, let us use the notation $\mathbf{A} = \mathbb{C}^n/\Gamma$.

The space of all period matrices satisfying the Riemann period condition is therefore a moduli space, known as the moduli space of Abelian varieties. But our question is at a different level: we are still asking, when is an Abelian variety $\mathbf{A}$ a moduli space by itself?

An immediate answer comes, again, from the geometry of smooth algebraic curves over $\mathbb{C}$. On such a curve C of genus $g > 0$, we have g linearly independent holomorphic 1-forms. Thus $H^0(C, K_C) = \mathbb{C}^g$, which is the space of holomorphic 1-forms. The first homology group of C is $H_1(C, \mathbb{Z}) = Z^{2g}$. Let us choose a $\mathbb{C}$-basis $\{\omega_1, \dots, \omega_g\}$ for $H^0(C, K_C)$ and homology generators $\{\gamma_1, \dots, \gamma_{2g}\}$. We arrange it so that the homology basis satisfies the *symplectic* intersection property

$$\langle \gamma_i, \gamma_j \rangle = \begin{cases} 1 & j = i + g, \quad i = 1, \dots, g \\ 0 & \text{otherwise.} \end{cases} \tag{1.15}$$

Now define a complex $g \times 2g$ matrix

$$\Pi = \begin{bmatrix} \oint_{\gamma_1} \omega_1 & \cdots & \oint_{\gamma_{2g}} \omega_1 \\ \vdots & \ddots & \vdots \\ \oint_{\gamma_1} \omega_g & \cdots & \oint_{\gamma_{2g}} \omega_g \end{bmatrix}.$$

Using the actions of $GL_g(\mathbb{C})$ from the left and $Sp_{2g}(\mathbb{Z})$ from the right, we can rearrange Π in the shape $\Pi = [I \mid \Omega]$ again. Riemann discovered that Ω satisfies ${}^t\Omega = \Omega$ and $Im(\Omega) > 0$. We can thus construct an Abelian variety

$$\mathrm{Jac}(C) := H^0(C, K_C)/H_1(C, \mathbb{Z}),$$

for which Ω is the period matrix.

Therefore, a complex torus $T = \mathbb{C}^g/\Gamma$ is a moduli space if its period matrix Ω comes from the integration *periods* of holomorphic 1-forms on a smooth algebraic curve C along its homology basis, because

$$\mathrm{Jac}(C) \cong H^1(C, \mathcal{O}_C)/H^1(C, \mathbb{Z}) =: \mathrm{Pic}^0(C)$$

is the moduli space of holomorphic line bundles on C of degree 0. For each fixed $g > 1$, there are $3g - 3$ dimensional family of Jacobian varieties, while Abelian varieties form a family of $g(g + 1)/2$ dimensions. Therefore, Jacobians are very special among Abelian varieties.

So one can ask a question, which Riemann himself did: *When is a principally polarized Abelian variety a Jacobian variety*? And this question leads us again to: *In search of a hidden curve.*

1.3. **Theme 3: Finding a curve from a period matrix.** The holomorphic embedding of the Abelian variety $\mathbf{A} = \mathbb{C}^g/\Gamma$ into a projective space is constructed by the *Riemann theta function*

$$\vartheta(z, \Omega) = \sum_{n \in \mathbb{Z}^n} \exp\left(2\pi i \langle n, z \rangle + \pi i \langle n, \Omega n \rangle\right), \qquad z \in \mathbb{C}^n, \tag{1.16}$$

where $\langle \ , \ \rangle$ is the standard symmetric form on $\mathbb{C}^n$. The Riemann period condition makes the above infinite series absolutely convergent on $\mathbb{C}^n$ and satisfy quasi-periodic conditions.

Over a four decade-old work [67] shows the following:

Theorem 1.5 ([67])**.** *A principally polarized Abelian variety of period Ω is a Jacobian variety if and only if the Riemann theta function* (1.16) *satisfies the nonlinear completely integrable system of Kadomtsev-Petviashvili (KP) equations.*

The nonlinear integrable system of KP equations is compactly formulated using the formalism of Peter Lax as follows:

$$\frac{\partial L}{\partial t_n} = [B_n, L], \qquad B_n = (L^n)_+, \quad n = 1, 2, 3, \dots, \tag{1.17}$$

where the *Lax operator*

$$L = \partial + \sum_{i=1}^{\infty} u_{i+1}(x; \mathbf{t}) \partial^{-i}, \qquad \partial = \frac{\partial}{\partial x}, \quad \mathbf{t} = (t_1, t_2, t_3, \dots) \tag{1.18}$$

is a normalized (i.e., there is no constant term in L) first order pseudodifferential operator in x depending on an infinitely many parameters $(t_1, t_2, t_3, \dots)$. The notation $(L^n)_+$ indicates the differential operator part of L^n, i.e., suppressing all negative powers of ∂. If we write $u = u_2, y = t_2$ and $t = t_3$, then the first equation from the system (1.17) is of the form

$$3u_{yy} = \left(4u_t - u_{xxx} - 12uu_x\right)_x,$$

where the subscript indicates partial differentiation. This equation is discovered by Kadomtsev and Petviashlivi in plasma physics. In particular, a solution to the KdV equation

$$u_t = \frac{1}{4}u_{xxx} + 3uu_x \tag{1.19}$$

which does not depend on y gives a solution to the KP equation. A relation between the KdV equation (1.19) and algebraic curves is easily seen by considering a *traveling* wave solution with speed $-c$ constructed from the Weierstraß elliptic function $\wp(z)$:

$$u(x,t) = -\wp(x+ct) + \frac{1}{3}c. \tag{1.20}$$

The starting point of the work [67] is Issai Schur's 1905 theorem which says that if two differential operators P and Q in x satisfy the commutativity relation $[P, L] = [Q, L] = 0$ with a given pseudodifferential operator L, then P and Q automatically commute: $[P, Q] = 0$. Now, suppose a solution L of the KP system depends only on finitely many time variables so that L deforms only along a g-dimensional direction of the parameter space $\mathbf{t} \in \mathbb{C}^\infty$. Then for all other directions, the KP system (1.17) produces infinitely many differential operators P, which are linear combinations of B_n's, that commute with L. Denote by A_L the algebra generated by all these P's over $\mathbb{C}$. Schur's argument shows that A_L is a commutative ring, and by Euclidean algorithm, we can show that the transcendence degree of A_L over $\mathbb{C}$ is 1. Give a natural grading in A_L by the order of differential operators, and define

$$C = \mathrm{Proj}(gr(A_L)) = \overline{\mathrm{Spec}(A_L)} = \mathrm{Spec}(A_L) \sqcup \{\infty\}. \tag{1.21}$$

It is an algebraic curve! It is called the **spectral curve** associated with the solution $L = L(x; \mathbf{t})$. Moreover, it is proved in [67] that C is of (arithmetic) genus g, the ring of differential operators $\mathcal{D}$ in x is an A_L-module determining a line bundle $\mathcal{L}$ over C, and that the KP equations define a linear flow on the cohomology $H^1(C, \mathcal{O}_C) \cong \mathbb{C}^g$. A geometric interpretation of the KP equations in this context is that they form a compatible linear deformation family of the line bundle $\mathcal{L}$ on C along $H^1(C, \mathcal{O}_C)$ (cf. [59, 67, 70]).

A generalization of the formula (1.20) gives the Lax operator L from a Riemann theta function $\vartheta(z, \Omega)$. If is solves the KP system, then it deforms only along a finite dimensional direction. Therefore, the argument above shows that the period matrix Ω must come from an algebraic curve, based on the unique solvability of the initial value problem of the evolution equation (1.17) established in [68, 69]. The analysis of KP equations in these earlier papers were done in the formal power series level, which was sufficient for the purpose of identifying the finite-dimensional solutions (a precise argument can be found in [59]). Recently, a powerful theorem is established by Magnot and Reyes [63] that deals with differentiable (non-formal) solutions of KP equations.

This is a success story of finding a hidden curve in the jungle.

1.4. **Theme 4: The τ-functions.** In the mid 1970s, Ryogo Hirota was developing a novel mechanism to calculate exact *soliton* solutions to nonlinear PDEs such as the KdV equation, introducing the *Hirota bilinear differentiation*

$$D_x\, f(x) \bullet g(x) := \frac{\partial}{\partial s}\big(f(x+s)g(x-s)\big)\Big|_{s=0}$$

and a new dependent variable τ. For a polynomial $P(D)$ in D, the Hirota bilinear differentiation is defined to be

$$P(D_x)\, f(x) \bullet g(x) := P\left(\frac{\partial}{\partial s}\right)\big(f(x+s)g(x-s)\big)\Big|_{s=0}.$$

Then the KdV equation (1.19) becomes

$$\left(D_x^4 - 4D_xD_t\right)\ \tau \bullet \tau$$

with the substitution

$$u(x,t) = \frac{\partial^2}{\partial x^2}\ \log \tau(x,t). \tag{1.22}$$

One can easily check the effectiveness of Hirota's method by calculating examples, such as

$$\tau(x,t) = 1 + c\, e^{2(\lambda x + \lambda^3 t)}\ ,$$

which gives a 2-parameter family of 1-soliton solutions to the KdV equation via (1.22).

In the late 1970s, Mikio Sato noticed the algebraic mechanism behind Hirota's method, and discovered the *Sato Grassmannian*. He then identified the Hirota's dependent variable τ, now known as *Sato's* τ-function, as the canonical section of the determinant line bundle of the Sato Grassmannian. As a set, the Sato Grassmannian of index μ is modeled over

$$Gr(\mu) = \left\{W \subset \mathbb{C}((z))\ \big|\ \gamma : W \longrightarrow \mathbb{C}[z^{-1}] \cong \mathbb{C}((z))/\big(\mathbb{C}[[z]]\cdot z\big) \text{ is Fredholm of index } \mu\right\}.$$

The time evolution determined by the Lax formalism (1.17) becomes, in this language, the action on the unipotent element

$$\exp\left(\sum_{n=1}^{\infty} t_n \Lambda^n\right) \in GL(V)$$

on the Sato Grassmannian, where $V = \mathbb{C}((z))$ and $\Lambda \in \mathfrak{gl}_\infty$ is the maximal nilpotent element corresponding to the multiplication of z^{-1} on V (see for example, [70, Chapter 7]). The τ-function records this action through the vantage point of the determinant line bundle. This also explains why so many soliton equations are exactly solved in terms of *determinant*. Since the initial value problem of the KP equations is uniquely solvable ([68, 69]), the space of solutions to the KP equations is identified with $Gr(\mu = 0)$.

We can immediately appreciate the similarity of (1.22) and the differential relation between elliptic functions and theta functions. Indeed, the Riemann theta functions are τ-functions when the period matrix comes from a Jacobian. Of course these τ-functions are very special ones. They correspond to points $W \in Gr(0)$ determined by

$$W = H^0\big(C, \mathcal{L}(*p)\big) \subset \mathbb{C}((z)),$$

the space of meromorphic sections of $\mathcal{L}$ with arbitrary poles at p, where $p \in C$ is a non-singular point of C, z is a local parameter of C around p, and $\mathcal{L}$ is a line bundle on C of degree $g(C) - 1$. We then have (see [68, Lemma 3.7])

$$\begin{cases} H^0(C,\mathcal{L}) \cong \text{Ker } \gamma \\ H^1(C,\mathcal{L}) \cong \text{Coker } \gamma. \end{cases}$$

Generically a solution to (1.17) produces an infinite-dimensional orbit. Among infinite-dimensional orbits, there are many solutions identified in geometry. One is associated with the Catalan numbers mentioned earlier. We review another example, coming from Hurwitz theory, in the subsequent section.

2. Fugue of the theme of spectral curves

2.1. **The Laplace transform.** The concept of spectral curves as the B-model mirror to an A-model counting problem appeared in the *remodeling conjecture* for Gromov-Witten invariants of toric Calabi-Yau threefolds. This idea has been developed by Mariño [65], Bouchard-Klemm-Mariño-Pasquetti [13], and Bouchard-Mariño [14], based on the theory of topological recursion formulas of Eynard and Orantin [37]. The remodeling conjecture states that the open and closed Gromov-Witten invariants of a toric Calabi-Yau threefold can be captured by the Eynard-Orantin topological recursion as a B-model that is constructed on the mirror curve. This conjecture has been completely solved, even beyond the original scope of the conjectures, by Fang-Liu-Zong [40].

Let us now examine the idea that **mirror symmetry is the Laplace transform** in some cases, by going through the concrete example of simple Hurwitz numbers [51]. Thus our question is the following:

Question 2.1. *What is the mirror dual of simple Hurwitz numbers?*

This is the same question we asked for the Catalan numbers. Indeed, around 2010, and before the author started to work on the Catalan number case, Boris Dubrovin and he had the following conversation.

Dubrovin: *Good to see you, Motohico!*
M: *Hi Boris, good to see you, too! At last I think I am coming close to understanding what mirror symmetry is.*
Dubrovin: *Oh, yes? All right, what do you think about mirror symmetry?*
M: *It is the Laplace transform!*
Dubrovin: *Do you think so, too? But I have been saying so for the last 15 years!* (He was referring to [25, 26].)
M: *Oh, have you? But I'm not talking about the Fourier-Mukai transform or the T-duality. It's the Laplace transform in the classical complex analysis sense.*
Dubrovin: *Of course I mean the same way.*
M: *All right, then let's check if we have the same understanding. Question: What is the mirror symmetric dual of a point?*
Dubrovin: *It is the Lax operator of the KdV equations that was identified by Kontsevich.*
M: *The Lax operator is the mirror symmetric dual of a point!? Hmm. Ah! I think you mean $x = y^2$, don't you?*
Dubrovin: *What? Wait a second. Oh, yes, indeed! Now it is my turn to ask you a question. What is the mirror symmetric dual of the Weil-Petersson volume of the moduli space of bordered hyperbolic surfaces discovered by Mirzakhani?*
M: *The sine function $x = \sin y$.*
Dubrovin: *Exactly!*
M: *How about simple Hurwitz numbers?*
Dubrovin: *Oh, this one you know well: The Lambert curve!*
M: *Yes, it is. And in all these cases, the mirror symmetry is the Laplace transform.*
Dubrovin: *Of course it is.*
M: *A, ha! Then we seem to have the same understanding of the mirror symmetry.*

Dubrovin: *Apparently we do!*

This conversation took place in the lobby of MSRI, Berkeley, after the end of the day. We shook hands tightly and happily, as we started to depart. We then noticed that there was a young mathematician listening our conversation. At the end, he shouted:

Bystander: *What, what, what? With this exchange, are you saying that you understood one another? Unbelievable!*

In the spirit of the above dialogue, the answer to our question should be:

Theorem 2.2 ([14, 36, 76])**.** *The mirror dual to the simple Hurwitz numbers is the Lambert curve $x = ye^{-y}$.*

A mathematical picture has emerged in the last two decades since the discoveries of Eynard-Orantin [37], Mariño [65], Bouchard-Klemm-Mariño-Pasquetti [13] and Bouchard-Mariño [14] in physics, and many mathematical efforts including [12, 17, 33, 34, 36, 40, 61, 72, 73, 75, 76, 79, 80, 81, 82, 83, 84, 90]. As a working hypothesis, we phrase it in the form of a principle.

Principle 2.3. *For a number of interesting cases, we have the following general structure.*

- *On the A-model side of topological string theory, we have a class of mathematical problems arising from combinatorics, geometry, and topology. The common feature of these problems is that they are somehow related to a lattice point counting of a collection of polytopes.*
- *On the B-model side, we have a universal theory due to Eynard and Orantin* [37]*. It is a framework of the recursion formula of a particular kind that is based on a* ***spectral curve*** *and two analytic functions (often with singularities) on it that immerse the curve into a complex symplectic surface as a complex Lagrangian.*
- *The passage from A-model to B-model, i.e., the mirror symmetry operation of the class of problems that we are concerned, is given by the* ***Laplace transform****. The spectral curve on the B-model side is defined as the* ***Riemann surface*** *of the Laplace transform, which means that it is the domain of holomorphy of the Laplace transformed function.*

There are many examples of mathematical problems that fall in to this principle. Among them is the theory of simple Hurwitz numbers [36, 76] that we are going to present now. Besides Hurwitz numbers, numerous mathematical results have been established. They include counting of Grothendieck's dessins d'enfants [17, 72, 79, 80, 83], higher genus Catalan numbers [31, 33], orbifold Hurwitz numbers [12], double Hurwitz numbers and higher spin structures [74], and the stationary Gromov-Witten invariants of $\mathbb{P}^1$ [34, 84]. One of the most important results in the Gromov-Witten setting is the remodeling conjecture [13] and its solution [40] mentioned above. A new direction was suggested in Kontsevich-Soibelman [56]. It is now impossible to list all research in this direction, simply called *topological recursion*, so we refer to recent papers by many authors and the papers they cite: Andersen, Borot, Bouchard, Chidambaram, Do, Dunin-Barkowski, Eynard, Garcia-Failde, Iwaki, Lewański, Norbury, Orantin, Osuga, Shadrin, ... A comprehensive and the most recent introduction is given by Bouchard [11], in which one can see how the focus of the topological recursion community has changed into new developments in the last decade.

The study of simple Hurwitz numbers is one of the earliest results in this direction and exhibits all important ingredients found in the later papers.

2.2. **Simple Hurwitz numbers** [51]**.** In this subsection, we define simple Hurwitz numbers, and a combinatorial equation that they satisfy, the *cut-and-join equation*, is proved. In the next subsection, the Laplace transform of the Hurwitz numbers is presented. The Laplace transformed holomorphic functions live on the mirror B-side of the model, according to Princeple 2.3. The Lambert curve is defined as the domain of holomorphy of these holomorphic functions. Then in the following subsections, we give the Laplace transform of the cut-and-join equation. The result is a simple **polynomial recursion formula** and is equivalent to the Eynard-Orantin topological recursion for the Lambert curve. We also give a straightforward and simple derivation [76] of the Witten-Kontsevich theorem on the ψ-class intersection numbers [23, 54, 95], and the λ_g-formula of Faber and Pandharipande [38, 39], using the recursion formula we establish in [76].

In all examples of Principle 2.3 we know so far, the A-model side always has a series of combinatorial equations that should uniquely determine the quantities in question, at least theoretically. But in practice solving these equations is quite complicated. As we develop in these lectures, the Laplace transform changes these equations to a *topological recursion* in the B-model side, which is an inductive formula based on the absolute value of the Euler characteristic of punctured surfaces.

Remark 2.4. Recently there have been totally unexpected spectacular developments in the topology of moduli spaces $\mathcal{M}_g, \mathcal{M}_{g,n}$, and $\overline{\mathcal{M}}_{g,n}$ (see for example, [16]). From the point of view of *In Search of a Hidden Curve*, and from the perspective of Norbury discovering the Norbury classes in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ using topological recursion [82], which were later identified as the Euler classes associated with a supersymmetric generalization of the work of Mirzakhani by Stanford-Witten [90], we wonder if there could be a *spectral curve* behind the recent developments.

A *simple Hurwitz number* represents the number of a particular type of meromorphic functions defined on an algebraic curve C of genus g. Let $\mu = (\mu_1, \dots, \mu_\ell) \in \mathbb{Z}_+^\ell$ be a *partition* of a positive integer d of length ℓ. This means that $|\mu| \overset{\text{def}}{=} \mu_1 + \cdots + \mu_\ell = d$. Instead of ordering parts of μ in the decreasing order, we consider them as a vector consisting of ℓ positive integers. By a *Hurwitz covering* of type (g, μ) we mean a meromorphic function $h : C \to \mathbb{C}$ that has ℓ labelled poles $\{x_i, \dots, x_\ell\}$, such that the pole order of h at x_i is μ_i for every $i = 1, \dots, \ell$, and that except for these poles, the holomorphic 1-form dh has simple zeros on $C \setminus \{x_i, \dots, x_\ell\}$ with distinct critical values of h. A meromorphic function of C is a holomorphic map of C onto $\mathbb{P}^1$. In algebraic geometry, the situation described above is summarized as follows: $h : C \to \mathbb{P}^1$ is a *ramified covering* of $\mathbb{P}^1$, simply ramified except for $\infty \in \mathbb{P}^1$. We identify two Hurwitz coverings $h_1 : C_1 \to \mathbb{P}^1$ with poles at $\{x_1, \dots, x_\ell\}$ and $h_2 : C_2 \to \mathbb{P}^1$ with poles at $\{y_1, \dots, y_\ell\}$ if there is a biholomorphic map $\phi : C_1 \overset{\sim}{\to} C_2$ such that $\phi(x_i) = y_i$, $i = 1, \dots, \ell$, and

$$\begin{array}{ccc} C_1 & \xrightarrow[\sim]{\phi} & C_2 \\ & {\scriptstyle h_1}\searrow \quad \swarrow{\scriptstyle h_2} & \\ & \mathbb{P}^1 & \end{array} .$$

When $C_1 = C_2$, $x_i = y_i$, and $h_1 = h_2 = h$, such a biholomorphic map ϕ is called an *automorphism* of a Hurwitz covering h. Since biholomorphic Hurwitz coverings are identified, following the stack theoretic principle, we count the number of Hurwitz coverings with the automorphism factor $1/|\mathrm{Aut}(h)|$. And when the above $\phi : C_1 \to C_2$ is merely a homeomorphism, we say h_1 and h_2 have the same *topological type*.

We are calling a meromorphic function a *covering*. This is because if we remove the critical values of h (including ∞) from $\mathbb{P}^1$, then on this open set h becomes a topological covering. More precisely, let $B = \{z_1, \dots, z_r, \infty\}$ denote the set of distinct critical values of h. Then

$$h : C \setminus h^{-1}(B) \longrightarrow \mathbb{P}^1 \setminus B$$

is a topological covering of degree d. Each $z_k \in B$ is a *branched point* of h, and a critical point (i.e., a zero of dh) on C is called a *ramification point* of h. Since dh has only simple zeros with distinct critical values of h, the number of ramification points of h, except for the poles, is equal to the number of branched points, which we denote by r. Therefore, $h^{-1}(z_k)$ consists of $d-1$ points. Then by comparing the Euler characteristic of the covering space and its base space, we obtain

$$d(2-r-1) = d \cdot \chi(\mathbb{P}^1 \setminus B) = \chi(C \setminus h^{-1}(B)) = 2 - 2g - r(d-1) - \ell.$$

Thus we establish the **Riemann-Hurwitz formula**

$$r = 2g - 2 + |\mu| + \ell. \tag{2.1}$$

What we wish to enumerate is:

Definition 2.5. The simple Hurwitz number of type (g, μ) for $g \geq 0$ and $\mu \in \mathbb{Z}_+^\ell$ that we consider in these lectures is

$$H_g(\mu) = \frac{1}{r(g,\mu)!} \sum_{[h] \text{ type } (g,\mu)} \frac{1}{|\mathrm{Aut}(h)|}, \tag{2.2}$$

where the sum runs all topological equivalence classes $[h]$ of Hurwitz coverings h of type (g, μ). Here,

$$r = r(g,\mu) = 2g - 2 + (\mu_1 + \cdots + \mu_\ell) + \ell$$

is the number of simple ramification points of h.

Remark 2.6. Our definition of simple Hurwitz numbers differs from the standard definition by two automorphism factors. The quantity $h_{g,\mu}$ of [35] and $H_g(\mu)$ are related by

$$H_g(\mu) = \frac{|\mathrm{Aut}(\mu)|}{r!}\, h_{g,\mu},$$

where $\mathrm{Aut}(\mu)$ is the group of permutations that permutes equal parts of μ considered as a partition. This is due to the convention that we label the poles of h and consider $\mu \in \mathbb{Z}_+^\ell$ as a vector, while we do not label simple ramification points.

Remark 2.7. Note that interchanging the entries of μ means permutation of the label of the poles $\{x_1, \dots, x_\ell\}$ of h. Thus it does not affect the count of simple Hurwitz numbers. Therefore, as a function in $\mu \in \mathbb{Z}_+^\ell$, $H_g(\mu)$ is a symmetric function.

Simple Hurwitz numbers satisfy a simple equation, known as the *cut-and-join equation* [46, 91]. Here we give it in the format used in [76].

Proposition 2.8 (Cut-and-join equation [76])**.** *Simple Hurwitz numbers satisfy*

$$\begin{aligned} r(g,\mu)H_g(\mu) &= \sum_{i<j} (\mu_i + \mu_j) H_g(\mu(\hat{i},\hat{j}), \mu_i + \mu_j) \\ &\quad + \frac{1}{2} \sum_{i=1}^{\ell} \sum_{\alpha+\beta=\mu_i} \alpha\beta \left[H_{g-1}(\mu(\hat{i}), \alpha, \beta) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \mu(\hat{i})}} H_{g_1}(I,\alpha) H_{g_2}(J,\beta) \right]. \end{aligned} \tag{2.3}$$

Here we use the following notations.

- $\mu(\hat{i})$ *is the vector of* $\ell - 1$ *entries obtained by deleting the* i*-th entry* μ_i*.*
- $(\mu(\hat{i}), \alpha, \beta)$ *is the vector of* $\ell + 1$ *entries obtained by appending two new entries* α *and* β *to* $\mu(\hat{i})$*.*
- $\mu(\hat{i}, \hat{j})$ *is the vector of* $\ell - 2$ *entries obtained by deleting the* i*-th and the* j*-th entries* μ_i *and* μ_j*.*
- $(\mu(\hat{i}, \hat{j}), \mu_i + \mu_j)$ *is the vector of* $\ell - 1$ *entries obtained by appending a new entry* $\mu_i + \mu_j$ *to* $\mu(\hat{i}, \hat{j})$*.*

The final sum is over all partitions of g *into non-negative integers* g_1 *and* g_2*, and a disjoint union decomposition (or a set partition) of entries of* $\mu(\hat{i})$ *as a set, allowing the empty set.*

Remark 2.9. Since $H_g(\mu)$ is a symmetric function, the way we append a new entry to a vector does not affect the function value.

The idea to prove the formula is reducing the number r of simple ramification points. Note that

$$h : C \setminus h^{-1}(B) \longrightarrow \mathbb{P}^1 \setminus B$$

is a topological covering of degree d. Therefore, it is obtained by a representation

$$\rho : \pi_1(\mathbb{P}^1 \setminus B) \longrightarrow S_d,$$

where S_d is the permutation group of d letters. The covering space X_ρ of $\mathbb{P}^1 \setminus B$ is obtained by the quotient construction

$$X_\rho = \widetilde{X} \times_{\pi_1(\mathbb{P}^1 \setminus B)} [d],$$

where $\widetilde{X}$ is the universal covering space of $\mathbb{P}^1 \setminus B$, and $[d] = \{1, 2, \dots, d\}$ is the index set on which $\pi_1(\mathbb{P}^1 \setminus B)$ acts via the representation ρ.

To make X_ρ a Hurwitz covering, we need to specify the monodromy of the representation at each branch point of B. Let $\{\gamma_1, \dots, \gamma_r, \gamma_\infty\}$ denote the collection of non-intersecting loops on $\mathbb{P}^1$, starting from $0 \in \mathbb{P}^1$, rotating around z_k counter-clockwise, and coming back to 0, for each $k = 1, \dots, r$. The last loop γ_∞ does the same for $\infty \in \mathbb{P}^1$. Since $\mathbb{P}^1$ is simply connected, we have

$$\pi_1(\mathbb{P}^1 \setminus B) = \langle \gamma_1, \dots, \gamma_r, \gamma_\infty | \gamma_1 \cdots \gamma_r \cdot \gamma_\infty = 1 \rangle.$$

Since X_ρ must have r simple ramification points over $\{z_1, \dots, z_r\}$, the monodromy at z_k is given by a transposition

$$\rho(\gamma_k) = (a_k b_k) \in S_d,$$

where $a_k, b_k \in [d]$ and all other indices are fixed by $\rho(\gamma_k)$. To impose the condition on poles $\{x_1, \dots, x_\ell\}$ of h, we need

$$\rho(\gamma_\infty) = c_1 c_2 \cdots c_\ell,$$

where $c_1, \dots, c_\ell$ are disjoint cycles of S_d of length $\mu_1, \dots, \mu_\ell$, respectively.

We want to reduce the number r by one. To do so, we simply merge z_r with ∞. The monodromy at ∞ then changes from $c_1 c_2 \cdots c_\ell$ to $(ab) \cdot c_1 c_2 \cdots c_\ell$, where $(ab) = (a_r b_r)$ is the transposition corresponding to γ_r. There are two cases we have now:

1. Join case: a and b belong to two disjoint cycles, say $a \in c_i$ and $b \in c_j$;
2. Cut case: both a and b belong to the same cycle, say c_i.

An elementary computation shows that for Case (1), the product $(ab)c_ic_j$ is a single cycle of length $\mu_i + \mu_j$. For the second case, the result depends on how far a and b are apart in cycle c_i. If b appears α slots after a with respect to the cyclic ordering, then

$$(ab)c_i = c_\alpha c_\beta, \tag{2.4}$$

where $\alpha + \beta = \mu_i$. The cycles c_α and c_β are disjoint of length α and β, respectively, and $a \in c_\alpha$, and $b \in c_\beta$. Note that everything is symmetric with respect to interchanging a and b. A couple of simple examples makes sense here.

$$\begin{aligned} \text{Join Case:}\quad & (12)(3517)(46829) = (468173529) \\ \text{Cut Case:}\quad & (12)(351746829) = (3529)(17468). \end{aligned}$$

With this preparation, we can now give the proof of (2.3). The right-hand side of (2.3) represents the set of all monodromy representations obtained by merging one of the branch points z_k with ∞. The factor r on the left-hand side represents the choice of z_k.

Proof. Since we are reducing $r = 2g - 2 + d + \ell$ by one without changing d, there are three different ways of reduction:

$$(g,\ell) \longmapsto (g, \ell - 1) \tag{2.5}$$

$$(g,\ell) \longmapsto (g-1, \ell + 1) \tag{2.6}$$

$$(g,\ell) \longmapsto (g_1, \ell_1 + 1) + (g_2, \ell_2 + 1), \text{ where } \begin{cases} g_1 + g_2 = g \\ \ell_1 + \ell_2 = \ell - 1. \end{cases} \tag{2.7}$$

- The first reduction (2.5) is exactly the first line of the right-hand side of (2.3), which corresponds to the join case. Two cycles of length μ_i and μ_j are joined to form a longer cycle of length $\mu_i + \mu_j$. Note that the number a has to be recorded somewhere in this long cycle. This explains the factor $\mu_i + \mu_j$. Then the number b is automatically recorded, because it is the entry appearing exactly μ_i slots after a in this long cycle.
- The second line of the right-hand side of (2.3) represents the cut cases. In (2.4), we have α choices for a and β choices for b. The symmetry of interchanging a and b explains the factor $\frac{1}{2}$. The first term of the second line of (2.3) corresponds to (2.6).
- The second term of the second line corresponde to (2.7). Note that in this situation, merging a branched point with ∞ breaks the connectivity of the Hurwitz covering. We have two ramified coverings $h_1 : C_1 \to \mathbb{P}^1$ of degree d_1 and genus g_1 with $\ell_1 + 1$ poles, and $h_2 : C_2 \to \mathbb{P}^1$ of degree d_2 and genus g_2 with $\ell_2 + 1$ poles. If we denote by r_i the number of simple ramifications points of h_i for $i = 1, 2$, then we have

$$\begin{array}{rcl} r_1 & = & 2g_1 - 2 + d_1 + \ell_1 + 1 \\ +)\quad r_2 & = & 2g_2 - 2 + d_2 + \ell_2 + 1 \\ \hline r - 1 & = & 2g - 2 + d + \ell - 1. \end{array}$$

This completes the proof of (2.3). □

Remark 2.10. The reduction of the number r of simple ramification points by one is exactly reflecting the reduction of the Euler characteristic of the punctured surface $X_\rho = C \setminus h^{-1}(B)$ appearing in our consideration by one. Since we do not change the degree d of the covering, the reduction of r is simply reducing $2g - 2 + \ell$ by one.

2.3. **Fugue in two themes: The Laplace transform of the simple Hurwitz numbers.** Let us now compute the Laplace transform of the simple Hurwitz number $H_g(\mu)$, considered as a function in $\mu \in \mathbb{Z}_+^\ell$. According to Principle 2.3, the result should give us the mirror dual of simple Hurwitz numbers. We explain where the **Lambert curve** $x = ye^{1-y}$ comes from, and its essential role in computing the Laplace transform. The most surprising feature is that the result of **the Laplace transform of** $H_g(\mu)$ **is a polynomial** if $2g - 2 + \ell > 0$. This polynomiality produces powerful consequences, which is the main subject of the following lectures.

The most important reason why we are interested in Hurwitz numbers in this summer school on complex Lagrangians is because of the theorem due to Ekedahl, Lando, Shapiro and Vainshtein [35] that relates the simple Hurwitz numbers with the intersection numbers of tautological classes on the moduli spaces of curves. From analyzing their formula, we see the emergence of the spectral curve, which we consider as a complex Lagrangian in the holomorphic symplectic surface.

Let us recall the necessary notations here. Our main object is the moduli stack $\overline{\mathcal{M}}_{g,\ell}$ consisting of stable algebraic curves of genus $g \geq 0$ with $\ell \geq 1$ distinct smooth labeled points. Forgetting the last labeled point on a curve gives a canonical projection

$$\pi : \overline{\mathcal{M}}_{g,\ell+1} \longrightarrow \overline{\mathcal{M}}_{g,\ell}.$$

Since the last labeled point moves on the curve, the projection π can be considered as a **universal family** of ℓ-pointed curves. This is because for each point $[C, (x_1, \ldots, x_\ell)] \in \overline{\mathcal{M}}_{g,\ell}$, the fiber of π is indeed C itself.

If $x_{\ell+1} \in C$ is a smooth point of C other than $\{x_1, \ldots, x_\ell\}$, then this point represents an element $[C, (x_1, \ldots, x_{\ell+1})] \in \overline{\mathcal{M}}_{g,\ell+1}$. If $x_{\ell+1} = x_i$ for some $i = 1, \ldots, \ell$, then this point represents a stable curve obtained by attaching a rational curve $\mathbb{P}^1$ to C at the original location of x_i, while carrying three special points on it. One is the singular point at which C and $\mathbb{P}^1$ intersects. The other two points are labeled as x_i and $x_{\ell+1}$. And if $x_{\ell+1} \in C$ coincides with one of the nodal points of C, say $x \in C$, then this point represents another stable curve. This time, consider the local normalization $\widetilde{C} \to C$ about the singular point $x \in C$, and let x_+ and x_- be the two points in the fiber. The stable curve we have is the curve $\widetilde{C} \cup \mathbb{P}^1$, where the two curves intersect at x_+ and x_-. The labeled point $x_{\ell+1}$ is placed on the attached $\mathbb{P}^1$ different from these two singular points. Because $\mathbb{P}^1$ with three distinct labeled points form a moduli space consisting of a single point, the processes above (called *stabilization*) leave no ambiguity.

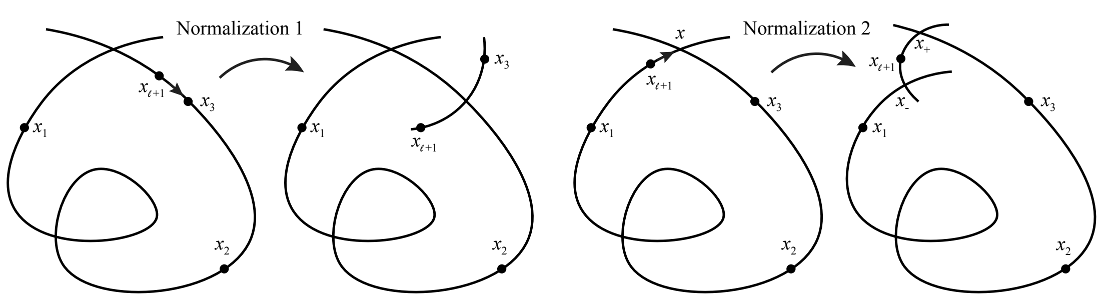

FIGURE 2.1. A new point $x_{\ell+1}$ is attached on top of x_3 (left), and at nodal point x (right).

A universal family produces what we call the *tautological* bundles on the moduli space. The cotangent sheaf $T^*C = \omega_C$ of each fiber of π is glued together to form a *relative* *dualizing sheaf* ω on $\overline{\mathcal{M}}_{g,\ell+1}$. The push-forward

$$\mathbb{E} = \pi_* \omega$$

is such a tautological vector bundle on $\overline{\mathcal{M}}_{g,\ell}$ of fiber dimension

$$\dim H^0(C, \omega_C) = g,$$

and is called the *Hodge bundle* on $\overline{\mathcal{M}}_{g,\ell}$. By assigning $x_{\ell+1} = x_i$ to each $[C, (x_1, \dots, x_\ell)] \in \overline{\mathcal{M}}_{g,\ell}$, we construct a section

$$\sigma_i : \overline{\mathcal{M}}_{g,\ell} \longrightarrow \overline{\mathcal{M}}_{g,\ell+1}.$$

It defines another tautological bundle

$$\mathbb{L}_i = \sigma^*(\omega) \tag{2.8}$$

on $\overline{\mathcal{M}}_{g,\ell}$. The fiber of $\mathbb{L}_i$ at $[C, (x_1, \dots, x_\ell)]$ is identified with the cotangent line $T^*_{x_i} C$.

The *tautological classes* of $\overline{\mathcal{M}}_{g,\ell}$ are rational cohomology classes including

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,\ell}, \mathbb{Q}) \quad \text{and} \quad \lambda_j = c_j(\mathbb{E}) \in H^{2j}(\overline{\mathcal{M}}_{g,\ell}, \mathbb{Q}).$$

In these lectures we do not consider the other classes, such as the κ-classes.

With these notational preparations, we can now state an amazing theorem.

Theorem 2.11 (The ELSV formula [35])**.** *The simple Hurwitz numbers are expressible as the intersection numbers of tautological classes on the moduli space $\overline{\mathcal{M}}_{g,\ell}$ as follows. Let $\mu \in \mathbb{Z}_+^\ell$ be a positive integer vector. Then we have*

$$\begin{aligned} H_g(\mu) &= \prod_{i=1}^{\ell} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,\ell}} \frac{\sum_{j=0}^{g} (-1)^j \lambda_j}{\prod_{i=1}^{\ell} (1 - \mu_i \psi_i)} \\ &= \sum_{n_1, \dots, n_\ell \geq 0} \sum_{j=0}^{g} (-1)^j \langle \tau_{n_1} \cdots \tau_{n_\ell} \lambda_j \rangle_{g,\ell} \prod_{i=1}^{\ell} \frac{\mu_i^{\mu_i + n_i}}{\mu_i!}. \end{aligned} \tag{2.9}$$

Here we use Witten's symbol

$$\langle \tau_{n_1} \cdots \tau_{n_\ell} \lambda_j \rangle_{g,\ell} = \int_{\overline{\mathcal{M}}_{g,\ell}} c_1(\mathbb{L}_1)^{n_1} \cdots c_1(\mathbb{L}_\ell)^{n_\ell} \cdot c_j(\mathbb{E}).$$

It is 0 *unless* $n_1 + \cdots + n_\ell + j = 3g - 3 + \ell$.

Remark 2.12. It is not our purpose to give a proof of the ELSV formula in these lectures. There are excellent articles by now about this remarkable formula. We refer to [60, 85].

To explore the mirror partner to simple Hurwitz numbers, we wish to compute the Laplace transform of the ELSV formula. Let us recall Stirling's formula

$$\frac{k^{k+n}}{k!} e^{-k} \sim \frac{1}{\sqrt{2\pi}} k^{n-\frac{1}{2}}, \qquad k \gg 0 \tag{2.10}$$

for a fixed n.

Definition 2.13. For a complex parameter w with $Re(w) > 0$, we define

$$\xi_n(w) = \sum_{k=1}^{\infty} \frac{k^{k+n}}{k!} e^{-k} e^{-kw}. \tag{2.11}$$

Because of Stirling's formula (2.10), we expect that asymptotically near $w \sim 0$,

$$\xi_n(w) \sim \int_0^\infty \frac{1}{\sqrt{2\pi}} x^{n-\frac{1}{2}} e^{-xw} dx.$$

To illustrate our strategy of computing the Laplace transform, let us first compute

$$f_n(w) = \int_0^\infty x^n e^{-xw} dx.$$

We notice that

$$-\frac{d}{dw} f_n(w) = f_{n+1}(w). \tag{2.12}$$

Therefore, if we know $f_0(w)$, then we can calculate all $f_n(w)$ for $n > 0$. Of course we have

$$f_0(w) = \frac{1}{w}.$$

Therefore, we immediately conclude that

$$f_n(w) = \frac{\Gamma(n+1)}{w^{n+1}}, \tag{2.13}$$

which satisfies the initial condition and the differential recursion formula (2.12). The important fact in complex analysis is that when we derive a formula like (2.13), it holds for an arbitrary n, not necessarily a positive integer. In particular, we have

$$\xi_n(w) \sim \int_0^\infty \frac{1}{\sqrt{2\pi}} x^{n-\frac{1}{2}} e^{-xw} dx = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{2\pi}\; w^{n+\frac{1}{2}}}.$$

From this asymptotic expression, we learn that ξ_n has an expansion in $w^{-\frac{1}{2}}$. Thus to identify the domain of holomorphy, we wish to find a natural coordinate that behaves like $w^{-\frac{1}{2}}$.

Note that for every $n > 0$, the summation in the definition of $\xi_n(w)$ in (2.11) can be taken from $k = 0$ to ∞. For $n = 0$, the $k = 0$ term contributes $0^0 = 1$ in the summation. So let us define

$$t - 1 = \xi_0(w) = \sum_{k=1}^\infty \frac{k^k}{k!} e^{-k} e^{-kw}. \tag{2.14}$$

Then the computation of the Laplace transform $\xi_n(w)$ is reduced to finding the inverse function $w = w(t)$ of (2.14), because all we need after identifying the inverse is to differentiate $\xi_0(w)$ n-times.

Here we utilize the *Lagrange Inversion Formula.*

Theorem 2.14 (The Lagrange Inversion Formula)**.** *Let $f(y)$ be a holomorphic function defined near $y = 0$ such that $f(0) \neq 0$. Then the inverse of the function*

$$x = \frac{y}{f(y)}$$

is given by

$$y = \sum_{k=1}^\infty \left[\frac{d^{k-1}}{dy^{k-1}} (f(y))^k\right]_{y=0} \frac{x^k}{k!}.$$

We give a proof of this formula in Appendix. For our purpose, let us consider the case $f(y) = e^{y-1}$. The function

$$x = ye^{1-y} \tag{2.15}$$

is called the **Lambert function**.

This is our spectral curve, the hidden curve for Hurwitz numbers! The difficulty is in the calculation of (2.11). Since we do not know how to calculate it, we just introduce the symbol y for (2.16) below, *pretending* that we know what it is. Then we can calculate *everything* in terms of what we do not know, i.e., (2.16). Indeed, the Lagrange Inversion Formula immediately tells us that its inverse function is given by

$$y = \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} e^{-k} x^k. \tag{2.16}$$

So if we substitute

$$x = e^{-w}, \tag{2.17}$$

then we have

$$y = \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} e^{-k} e^{-kw} = \xi_{-1}(w). \tag{2.18}$$

The differential of the Lambert function gives

$$dx = (1-y)e^{1-y} dy.$$

Therefore, we have

$$-\frac{d}{dw} = x\frac{d}{dx} = \frac{y}{1-y}\frac{d}{dy}. \tag{2.19}$$

Since

$$t - 1 = \xi_0(w) = -\frac{d}{dw}\xi_{-1}(w) = \frac{y}{1-y},$$

we conclude that

$$y = \frac{t-1}{t}. \tag{2.20}$$

As a consequence, we complete the calculation:

$$-\frac{d}{dw} = x\frac{d}{dx} = \frac{y}{1-y}\frac{d}{dy} = t^2(t-1)\frac{d}{dt}. \tag{2.21}$$

We also obtain a formula for w in terms of t, since $e^{-w} = ye^{1-y}$.

$$w = -\frac{1}{t} - \log\left(1 - \frac{1}{t}\right) = \sum_{m=2}^{\infty} \frac{1}{m}\,\frac{1}{t^m}. \tag{2.22}$$

Notice that near $w = 0$, we have $t \sim \sqrt{2w}$, as we wished! Now we can calculate $\xi_n(w)$ in terms of t for every $n \geq 0$.

Definition 2.15. As a function in t, we denote

$$\hat{\xi}_n(t) = \xi_n(w(t)). \tag{2.23}$$

Theorem 2.16 (Polynomiality)**.** *For every $n \geq 0$, $\hat{\xi}_n(t)$ is a **polynomial** in t of degree $2n+1$. For $n > 0$ it has an expansion*

$$\hat{\xi}_n(t) = (2n-1)!!t^{2n+1} - \frac{(2n+1)!!}{3}\, t^{2n} + \cdots + a_n t^{n+2} + (-1)^n n!\, t^{n+1}, \tag{2.24}$$

where a_n is defined by

$$a_n = -\big[(n+1)a_{n-1} + (-1)^n n!\big]$$

and is identified as the sequence A001705 or A081047 of the On-Line Encyclopedia of Integer Sequences*.*

Proof. It is a straightforward calculation of

$$\hat{\xi}_n(t) = t^2(t-1)\frac{d}{dt}\hat{\xi}_{n-1}(t) = \left(t^2(t-1)\frac{d}{dt}\right)^n (t-1).$$

□

Remark 2.17. J. Zhow informed the author that all other coefficients of $\hat{\xi}_n(t)$ had been identified.

Theorem 2.18 (Laplace transform of simple Hurwitz numbers)**.** *The Laplace transform of simple Hurwitz numbers is given by*

$$\begin{aligned}\mathbf{H}_{g,\ell}(t) = \mathbf{H}_{g,\ell}(t_1, t_2, \ldots, t_\ell) &= \sum_{\mu \in \mathbb{Z}_+^\ell} H_g(\mu) e^{-|\mu|} e^{-(\mu_1 w_1 + \cdots + \mu_\ell w_\ell)} \\ &= \sum_{n_1,\ldots,n_\ell \geq 0} \sum_{j=0}^{g} (-1)^j \langle \tau_{n_1} \cdots \tau_{n_\ell} \lambda_j \rangle_{g,\ell} \prod_{i=1}^{\ell} \hat{\xi}_{n_i}(t_i).\end{aligned} \tag{2.25}$$

This is a polynomial of degree $3(2g-2+\ell)$. Its highest degree terms form a homogeneous polynomial

$$\mathbf{H}^{top}_{g,\ell}(t) = \sum_{n_1+\cdots+n_\ell = 3g-3+\ell} \langle \tau_{n_1} \cdots \tau_{n_\ell} \rangle_{g,\ell} \prod_{i=1}^{\ell} (2n_i - 1)!!\, t_i^{2n_i+1}, \tag{2.26}$$

and the lowest degree terms also form a homogeneous polynomial

$$\mathbf{H}^{lowest}_{g,\ell}(t) = \sum_{n_1+\cdots+n_\ell = 2g-3+\ell} (-1)^{3g-3+\ell} \langle \tau_{n_1} \cdots \tau_{n_\ell} \lambda_g \rangle_{g,\ell} \prod_{i=1}^{\ell} n_i!\, t_i^{n_i+1}. \tag{2.27}$$

Remark 2.19. We note that there is no a priori reason for the Laplace transform of $H_g(\mu)$ to be a polynomial. Because it is a polynomial, we obtain a polynomial generating function of linear Hodge integrals $\langle \tau_{n_1} \cdots \tau_{n_\ell} \lambda_j \rangle_{g,\ell}$. We utilize this polynomiality in the concluding Fuge below.

Remark 2.20. The motivation of the author to work on Catalan numbers [33, 31] was to find an analytically simpler example of enumeration problem for which a similar polynomiality holds. The coordinate change for the Catalan case was obtained from looking for an analogy of (2.20).

Remark 2.21. The existence of the polynomials $\hat{\xi}_n(t)$ in (2.25) is significant, because it reflects the ELSV formula (2.9). Indeed, Eynard predicts that this is the general structure of the Eynard-Orantin formalism.

2.4. **Fugue in three themes: Two quantum curves for the Lambert curve.** How about the quantum curve associated with the Lambert curve? Since $x = ye^{1-y}$ does not give an algebraic curve, the argument we used for the case of Catalan generating function, (1.1) and (1.3), does not apply here. Yet the counter part of the *quantum curve* (1.5) exists for Hurwitz numbers. Indeed, we discover *two* differential equations that recover the Lambert curve via the semi-classical limit.

Our particular choice of the Lambert function (2.15) is for the purpose to make the subsequent calculations less cumbersome, by reducing the appearance of powers of e. More traditional choice is

$$x = ye^{-y}. \tag{2.28}$$

The difference is only in the constant multiplication $x \longmapsto ex$. For the calculation of quantum curves, we use the classical one. This works better for quantum curves, because the differential operator we need is $D := x\frac{d}{dx}$, which is invariant under the $\mathbb{C}^*$-action.

Theorem 2.22 ([12, 74, 75])**.** *As a straightforeward analogy of the formula* (1.7) *for Catalan numbers, let us define*

$$Z(t,\hbar) := \exp\left(\sum_{g\geq 0, n>0} \frac{1}{n!}\hbar^{2g-2+n}\,\mathbf{H}_{g,n}(t,t,\dots,t)\right). \tag{2.29}$$

Then it satisfies the following two equations, one is a partial differential equation, and the other a difference-differential equation:

$$\left(\frac{\hbar}{2}D^2 - \left(1+\frac{\hbar}{2}\right)D - \hbar\frac{\partial}{\partial\hbar}\right)Z\big(t(x),\hbar\big) = 0, \tag{2.30}$$

$$\left(\hbar D - xe^{\hbar D}\right)Z\big(t(x),\hbar\big) = 0. \tag{2.31}$$

Here, $D = x\frac{d}{dx}$, and the variable t in (2.29) *is considered to be a function in x by the relations*

$$y = \frac{t-1}{t} \quad \textit{and} \quad x = ye^{-y}.$$

The semi-classical limit calculations can be applied to these partial differential and difference-differential equations. The result is the same as replacing $\begin{cases}\hbar D \longmapsto y\\ x\longmapsto x\end{cases}$ for (2.31), which recovers the Lambert curve. For the first equation (2.30), we need to perform the actual WKB analysis to obtain the semi-classical limit, as done in [75]. The quantum curve (2.31) was also independently obtained in [97].

An important aspect arising from (2.29) is the emergence of the KP τ-function right in this formula. Indeed, $Z(t,\hbar)$ is a principal specialization of the KP τ-function. We refer to [75] for more detail.

We are now ready to compute the Laplace transform of the cut-and-join equation itself. The result turns out to be a simple polynomial recursion formula. Here again there is no a priori reason for the result to be a polynomial relation, because the cut-and-join equation (2.3) contains unstable geometries, and they contribute non-polynomial terms after the Laplace transform.

Remark 2.23. We remark here that the Laplace transform of the cut-and-join equation is equivalent to the Eynard-Orantin topological recursion formula [37] based on the Lambert curve (2.15) as the spectral curve of the theory. This fact solves the Bouchard-Mariño conjecture [14] of Hurwitz numbers [36, 76], and establishes the Lambert curve as the remodeled B-model corresponding to simple Hurwitz numbers through mirror symmetry.

The unexpected power [76] of the topological recursion type formula appearing in our context is the following.

(1) It restricts to the top degree terms, and recovers the Dijkgraaf-Verlinde-Verlinde formula, or the Virasoro constraint condition, for the ψ-class intersection numbers on $\overline{\mathcal{M}}_{g,\ell}$ [95].
(2) It also restricts to the lowest degree terms, and recovers the λ_g-conjecture of Faber that was proved in [38, 39] in a totally different method. Our proof is straightforward.

In other words, we obtain a straightforward, simple proofs of the Witten conjecture and Faber's λ_g-conjecture from the Laplace transform of the cut-and-join equation. We note that the Laplace transform contains the information of the large μ asymptotics. Therefore, our proof [76] of the Witten conjecture uses the same idea of Okounkov and Pandharipande [85], yet it is much simpler because we do not have to use any of the asymptotic analyses of matrix integrals, Hurwitz numbers, and graph enumeration.

The proof of the λ_g-conjecture using the topological recursion is still somewhat mysterious. Here again the complicated combinatorics is wiped out and we have a transparent proof.

Let us now state the Laplace transform of the cut-and-join equation.

Theorem 2.24 ([76])**.** *The polynomial generating functions of the linear Hodge integrals* $\mathbf{H}_{g,\ell}(t)$ *satisfy the following topological recursion type formula*

$$\left(2g-2+\ell+\sum_{i=1}^{\ell}\frac{1}{t_i}D_i\right)\mathbf{H}_{g,\ell}(t_1,t_2,\dots,t_\ell) \tag{2.32}$$
$$=\sum_{i<j}\frac{t_i^2(t_j-1)D_i\mathbf{H}_{g,\ell-1}\big(t_{[\ell;\hat{j}]}\big)-t_j^2(t_i-1)D_j\mathbf{H}_{g,\ell-1}\big(t_{[\ell;\hat{i}]}\big)}{t_i-t_j}$$
$$+\sum_{i=1}^{\ell}\Big[D_{u_1}D_{u_2}\mathbf{H}_{g-1,\ell+1}\big(u_1,u_2,t_{[\ell;\hat{i}]}\big)\Big]_{u_1=u_2=t_i}$$
$$+\frac{1}{2}\sum_{i=1}^{\ell}\sum_{\substack{g_1+g_2=g\\ J\sqcup K=[\ell;\hat{i}]}}^{\text{stable}} D_i\mathbf{H}_{g_1,|J|+1}(t_i,t_J)\cdot D_i\mathbf{H}_{g_2,|K|+1}(t_i,t_K),$$

where $D_i=t_i^2(t_i-1)\frac{\partial}{\partial t_i}$. *As before,* $[\ell]=\{1,\dots,\ell\}$ *is the index set, and* $[\ell;\hat{i}]$ *is the index set obtained by deleting* i *from* $[\ell]$. *The last summation is taken over all partitions* $g=g_1+g_2$ *of the genus* g *and disjoint union decompositions* $J\sqcup K=[\ell;\hat{i}]$ *satisfying the stability conditions* $2g_1-1+|J|>0$ *and* $2g_2-1+|K|>0$. *For a subset* $I\subset[\ell]$ *we write* $t_I=(t_i)_{i\in I}$.

The biggest difference between the cut-and-join equation (2.3) and the Laplace transformed formula (2.32) is the restriction to stable geometries in the latter. In the case of the cut-and-join equation, the cut case contains $g_1=0$ and $I=\emptyset$. Then $H_{g_2}(J,\beta)$ has the same complexity of $H_g(\mu)$. Thus the cut-and-join equation is simply a relation among Hurwitz numbers, not a *recursive* formula.

The new feature of our (2.32) is that it is a genuine recursion formula about linear Hodge integrals. Indeed, we can re-write the formula as follows.

$$
(2.33)\quad \sum_{n_{[\ell]}} \langle \tau_{n_{[\ell]}} \Lambda_g^\vee(1)\rangle_{g,\ell} \left((2g-2+\ell)\hat{\xi}_{n_{[\ell]}}(t_{[\ell]}) + \sum_{i=1}^{\ell} \frac{1}{t_i}\,\hat{\xi}_{n_i+1}(t_i)\hat{\xi}_{[\ell;\hat{i}]}(t_{[\ell;\hat{i}]}) \right)
$$

$$
= \sum_{i<j} \sum_{m,n_{[\ell;\hat{i}\hat{j}]}} \langle \tau_m \tau_{n_{[\ell;\hat{i}\hat{j}]}} \Lambda_g^\vee(1)\rangle_{g,\ell-1} \hat{\xi}_{n_{[\ell;\hat{i}\hat{j}]}}(t_{[\ell;\hat{i}\hat{j}]}) \frac{\hat{\xi}_{m+1}(t_i)\hat{\xi}_0(t_j)t_i^2 - \hat{\xi}_{m+1}(t_j)\hat{\xi}_0(t_i)t_j^2}{t_i - t_j}
$$

$$
+ \frac{1}{2}\sum_{i=1}^{\ell} \sum_{n_{[\ell;\hat{i}]}} \sum_{a,b} \Bigg(\langle \tau_a\tau_b\tau_{n_{[\ell;\hat{i}]}} \Lambda_{g-1}^\vee(1)\rangle_{g-1,\ell+1}
$$

$$
+ \sum_{\substack{g_1+g_2=g \\ I\coprod J=[\ell;\hat{i}]}}^{\text{stable}} \langle \tau_a \tau_{n_I} \Lambda_{g_1}^\vee(1)\rangle_{g_1,|I|+1} \langle \tau_b \tau_{n_J} \Lambda_{g_2}^\vee(1)\rangle_{g_2,|J|+1} \Bigg) \hat{\xi}_{a+1}(t_i)\hat{\xi}_{b+1}(t_i)\hat{\xi}_{n_{[\ell;\hat{i}]}}(t_{[\ell;\hat{i}]}),
$$

where $[\ell] = \{1, 2 \dots, \ell\}$ is the index set, and for a subset $I \subset [\ell]$, we denote

$$
t_I = (t_i)_{i\in I}, \quad n_I = \{\, n_i \mid i \in I\,\}, \quad \tau_{n_I} = \prod_{i\in I} \tau_{n_i}, \quad \hat{\xi}_{n_I}(t_I) = \prod_{i\in I} \hat{\xi}_{n_i}(t_i).
$$

We also use a convenient notation

$$
\Lambda_g^\vee(1) = 1 - \lambda_1 + \lambda_2 - \cdots + (-1)^g \lambda_g.
$$

It is now obvious that in (2.33), the complexity $2g-2+\ell$ is reduced exactly by 1 on the right-hand side. Thus we can compute linear Hodge integrals one by one using this formula.

The Deligne-Mumford stack $\overline{\mathcal{M}}_{g,\ell}$ is defined as the moduli space of *stable* curves satisfying the stability condition $2-2g-\ell<0$. However, Hurwitz numbers are well defined for *unstable* geometries $(g,\ell) = (0,1)$ and $(0,2)$. It is an elementary exercise (of tree counting, see [62]) to show that

$$
H_0\big((d)\big) = \frac{d^{d-1}}{d!}. \tag{2.34}
$$

We note that **this is the type $(0,1)$-Hurwitz number of degree d, and the new variable $y = y(x)$ of (2.16), or the spectral curve, is its generating function.**

The ELSV formula remains true for unstable cases by *defining*

$$
\int_{\overline{\mathcal{M}}_{0,1}} \frac{1}{1-k\psi} = \frac{1}{k^2}, \tag{2.35}
$$

$$
\int_{\overline{\mathcal{M}}_{0,2}} \frac{1}{(1-\mu_1\psi_1)(1-\mu_2\psi_2)} = \frac{1}{\mu_1+\mu_2}. \tag{2.36}
$$

In terms of simple Hurwitz numbers, we have

$$
H_0\big((\mu_1,\mu_2)\big) = \frac{\mu_1^{\mu_1}}{\mu_1!}\cdot\frac{\mu_2^{\mu_2}}{\mu_2!}\cdot\frac{1}{\mu_1+\mu_2}.
$$

From these expressions we can actually compute $\mathbf{H}_{0,1}(t)$ and $\mathbf{H}_{0,2}(t_1,t_2)$. Since these computations are quite involved, we refer to [36, 76]. What happens often in mathematics is what we call a *miraculous cancellation.* In our situation, when we honestly compute all terms appearing in the Laplace transform in the cut-and-join equation (2.3), somewhat miraculously, all non-polynomial terms cancel out, and the rest becomes an effective recursion formula (2.33).

2.5. **Fugue in four themes: New proofs of Witten-Kontsevich and the λ_g formulas.** Now let us move to proving the Witten conjecture and the λ_g-formula using our recursion (2.33). Although these important formulas have been proved a long time ago, we present a new proofs here which are much simpler, just to illustrate the power of the topological recursion type formula.

The DVV formula for the Virasoro constraint condition on the ψ-class intersections reads

$$\begin{aligned}
(2.37)\quad \langle \tau_{n_{[\ell]}}\rangle_{g,\ell} &= \sum_{j\geq 2} \frac{(2n_1+2n_j-1)!!}{(2n_1+1)!!(2n_j-1)!!}\langle \tau_{n_1+n_j-1}\tau_{n_{[\ell;\hat{1}\hat{j}]}}\rangle_{g,\ell-1}\\
&+\frac{1}{2}\sum_{a+b=n_1-2}\left(\langle \tau_a\tau_b\tau_{n_{[\ell;\hat{1}]}}\rangle_{g-1,\ell+1} + \sum^{\text{stable}}_{\substack{g_1+g_2=g\\ J\sqcup K=[\ell;\hat{1}]}} \langle \tau_a\tau_{n_J}\rangle_{g_1,|J|+1}\cdot\langle \tau_b\tau_{n_K}\rangle_{g_2,|K|+1}\right)\\
&\qquad\qquad\times\frac{(2a+1)!!(2b+1)!!}{(2n_1+1)!!}.
\end{aligned}$$

Here $[\ell;\hat{1}\hat{j}] = \{2,3,\dots,\hat{j},\dots,\ell\}$, and for a subset $I\subset[\ell]$ we write

$$n_I = (n_i)_{i\in I} \qquad \text{and} \qquad \tau_{n_I} = \prod_{i\in I}\tau_{n_i}.$$

Proposition 2.25. *The DVV formula* (2.37) *is exactly the relation among the top degree coefficients of the recursion* (2.32).

Proof. Choose $n_{[\ell]}$ so that $|n_{[\ell]}| = n_1+n_2+\dots+n_\ell = 3g-3+\ell$. The degree of the left-hand side of (2.32) is $3(2g-2+\ell)+1$. So we compare the coefficients of $t_1^{2n_1+2}\prod_{j\geq 2} t_j^{2n_j+1}$ in the recursion formula. The contribution from the left-hand side of (2.32) is

$$\langle \tau_{n_{[\ell]}}\rangle_{g,\ell}(2n_1+1)!!\prod_{j\geq 2}(2n_j-1)!!.$$

The contribution from the first line of the right-hand side comes from

$$\begin{aligned}
&\sum_{j\geq 2}\langle \tau_m\tau_{n_{[\ell;\hat{1}\hat{j}]}}\rangle_{g,\ell-1}(2m+1)!!\frac{t_1^2t_jt_1^{2m+3}-t_j^2t_1t_j^{2m+3}}{t_1-t_j}\\
&\qquad = \sum_{j\geq 2}\langle \tau_m\tau_{n_{[\ell;\hat{1}\hat{j}]}}\rangle_{g,\ell-1}(2m+1)!!t_1t_j\frac{t_1^{2m+4}-t_j^{2m+4}}{t_1-t_j}\\
&\qquad\qquad = \sum_{j\geq 2}\langle \tau_m\tau_{n_{[\ell;\hat{1}\hat{j}]}}\rangle_{g,\ell-1}(2m+1)!!\sum_{a+b=2m+3}t_1^{a+1}t_j^{b+1},
\end{aligned}$$

where $m = n_1+n_j-1$. The matching term in this formula is $a = 2n_1+1$ and $b = 2n_j$. Thus we extract as the coefficient of $t_1^{2n_1+2}\prod_{j\geq 2} t_j^{2n_j+1}$

$$\sum_{j\geq 2}\langle \tau_{n_1+n_j-1}\tau_{n_{[\ell;\hat{1}\hat{j}]}}\rangle_{g,\ell-1}(2n_1+2n_j-1)!!\prod_{k\neq 1,j}(2n_k-1)!!.$$

The contributions of the second and the third lines of the right-hand side of (2.32) are

$$\frac{1}{2}\sum_{a+b=n_1-2}\left(\langle\tau_a\tau_b\tau_{L\setminus\{1\}}\rangle_{g-1,\ell+1}+\frac{1}{2}\sum_{\substack{g_1+g_2=g\\ J\sqcup K=[\ell;\hat{1}]}}^{\text{stable}}\langle\tau_a\tau_{n_J}\rangle_{g_1,|J|+1}\cdot\langle\tau_b\tau_{n_K}\rangle_{g_2,|K|+1}\right)$$
$$\times(2a+1)!!(2b+1)!!\prod_{j\geq 2}(2n_j-1)!!.$$

We have thus recovered the Witten-Kontsevich theorem [23, 54, 95]. □

The λ_g formula [38, 39] is

$$\langle\tau_{n_{[\ell]}}\lambda_g\rangle_{g,\ell}=\binom{2g-3+\ell}{n_{[\ell]}}b_g, \tag{2.38}$$

where

$$\binom{2g-3+\ell}{n_{[\ell]}}=\binom{2g-3+\ell}{n_1,\dots,n_\ell} \tag{2.39}$$

is the multinomial coefficient, and

$$b_g=\frac{2^{2g-1}-1}{2^{2g-1}}\;\frac{|B_{2g}|}{(2g)!}$$

is a coefficient of the series

$$\sum_{j=0}^{\infty}b_js^{2j}=\frac{s/2}{\sin(s/2)}.$$

Proposition 2.26. *The lowest degree terms of the topological recursion* (2.32) *proves the combinatorial factor of the λ_g formula*

$$\langle\tau_{n_{[\ell]}}\lambda_g\rangle_{g,\ell}=\binom{2g-3+\ell}{n_{[\ell]}}\langle\tau_{2g-1}\lambda_g\rangle_{g,1}. \tag{2.40}$$

Proof. Choose $n_{[\ell]}$ subject to $|n_{[\ell]}|=2g-3+\ell$. We compare the coefficient of the terms of $\prod_{i\geq 1}t_i^{n_i+1}$ in (2.32), which has degree $|n_{[\ell]}|+\ell=2g-3+2\ell$. The left-hand side contributes

$$(-1)^{2g-3+\ell}(-1)^g\langle\tau_{n_{[\ell]}}\lambda_g\rangle_{g,\ell}\prod_{i\geq 1}n_i!\left(2g-2+\ell-\sum_{i=1}^{\ell}(n_i+1)\right)$$
$$=(-1)^{\ell}(-1)^g\langle\tau_{n_{[\ell]}}\lambda_g\rangle_{g,\ell}(\ell-1)\prod_{i\geq 1}n_i!.$$

The lowest degree terms of the first line of the right-hand side are

$$(-1)^g\sum_{i<j}\sum_m\langle\tau_m\tau_{n_{[\ell;\hat{i}\hat{j}]}}\lambda_g\rangle_{g,\ell-1}(-1)^m(m+1)!\frac{t_i^{m+4}-t_j^{m+4}}{t_i-t_j}(-1)^{2g-3+\ell-n_i-n_j}\prod_{k\neq i,j}n_k!t_k^{n_k+1}.$$

Since $m=n_i+n_j-1$, the coefficient of $\prod_{i\geq 1}t_i^{n_i+1}$ is

$$-(-1)^g(-1)^{2g-3+\ell}\sum_{i<j}\langle\tau_{n_i+n_j-1}\tau_{n_{[\ell;\hat{i}\hat{j}]}}\lambda_g\rangle_{g,\ell-1}\binom{n_i+n_j}{n_i}\prod_{i\geq 1}n_i!.$$

Note that the lowest degree coming from the second and the third lines of the right-hand side of (2.32) is $|n_{[\ell]}| + \ell + 2$, which is higher than the lowest degree of the left-hand side. Therefore, we have obtained a recursion equation with respect to ℓ

$$(\ell - 1)\langle \tau_{n_{[\ell]}} \lambda_g \rangle_{g,\ell} = \sum_{i<j} \langle \tau_{n_i+n_j-1} \tau_{n_{[\ell;\hat{i}\hat{j}]}} \lambda_g \rangle_{g,\ell-1} \binom{n_i + n_j}{n_i}. \tag{2.41}$$

The solution of the recursion equation (2.41) is the multinomial coefficient (2.39). □

Remark 2.27. Although the polynomial recursion (2.32) determines all linear Hodge integrals, the closed formula

$$b_g = \langle \tau_{2g-2} \lambda_g \rangle_{g,1} \qquad g \geq 1$$

does not directly follow from it.

2.6. **Coda: From a tree counting to the Lambert curve.** As in Section 2.4, but different from (2.16), let us use the classical convention

$$y(x) = \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} x^k \tag{2.42}$$

for the function $y = y(x)$. Then the Lagrange Inversion Theorem gives us the classical Lambert curve $x = ye^{-y}$, as pointed out in (2.28). We take the exterior derivative of this expression $dx = (1-y)e^{-y}dy$. Then we obtain a differential equation for $y(x)$:

$$Dy = \frac{y}{1-y}, \qquad D = x\,\frac{d}{dx}. \tag{2.43}$$

Since it is a nonlinear differential equation, it is not a Picard-Fuchs type equation such as the one (1.3) for the Catalan case. However, as the Picard-Fuchs equation (1.3) leads to the spectral curve $x = z + 1/z$, the differential equation (2.43) actually determines the Lambert curve, as we see below.

Although there is a difference between linear equation (1.3) and nonlinear equation (2.43), we consider them to be of the same nature, because they both lead to the spectral curve.

In this Coda, we deduce Differential Equation 2.43 from a purely combinatorial nature of the tree counting, and solve it to identify the Lambert curve, without ever appealing to the analysis of the Lagrange theorem.

We learn from the excellent book of Lovász et al. [62] that the $(0,1)$ Hurwitz number of degree d, $H_0\big((d)\big) = \frac{d^{d-1}}{d!}$ of (2.34), is the number of *rooted trees* on d nodes, counted in the stack sense. It means the reciprocal of the order of the automorphism group of each tree is used as a weight in counting. Cayley's Theorem says that the number of all *node-labeled* trees on d nodes is d^{d-2}, which is of course a positive integer. Since d nodes are labeled in $d!$ different ways, the ratio $\frac{d^{d-2}}{d!}$ is the "number" of *unlabeled* trees. Since it is not an integer for $d \geq 2$, this "counting" is not the count of elements of a set. We are counting *objects* in a category, and automorphisms are taken into account.

Question 2.28. *If you have two isomorphic objects, then you count it as one. For example, there are two groups of order* 6*, the cyclic group C_6 and the permutation group S_3. The dihedral group of a triangle D_3 is not counted because it is isomorphic to S_3. Now a question: Suppose we have an object that has a non-trivial automorphism group of order* 2*. Do we count it as one, or a* ***half****?*

Remark 2.29 (On categorical counting).

- The book [62] mentioned above also talks about the set-theoretical count of the number of trees. There is no exact formula for that number. What becomes an important question in set-theoretical count is the *asymptotic behavior* of the number.

- When we count, it is always the best practice to count labeled objects first, because then we have a clear definition of the objects we are counting. For example, in (1.10), $C_{g,n}(\mu_1, \dots, \mu_n)$ is the number of *cell graphs* with *labeled* vertices and no local rotation symmetries around each vertex are allowed. We then obtain $C_{g,n}(\mu_1, \dots, \mu_n)$ as an integer. But in (1.10), we have the denominator $\mu_1 \cdots \mu_n$, which is exactly the order of the product group of the rotation group at each vertex. This ratio is therefore *not* equal to the "number" of actual isomorphism classes of a cell graph.
- Identifying the automorphism group of a large cell graph, even a tree, is a computationally complex task. Often the *categorical count* leads to a beautiful formula. We employ this point of view everywhere in these lectures.
- But if your interest is really the actual set-theoretical count, then what do you do? Since there is no exact formula expected, the question is how you obtain the asymptotic analysis of the formula.
- **Bingo!** Yes, you go it. **This is why we are calculating the Laplace transform!** In both Catalan and Hurwitz cases, the Laplace transform led to the topology of the moduli spaces $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$. The idea here is similar to the Ehrhart polynomials and the Weil conjecture.

So we have $\frac{d^{d-2}}{d!}$ *unlabeled* trees on d nodes in our categorical count. To make a tree a *rooted* tree, we need to pick a node and declare that it is the root. We have d such choices. Therefore, the number of rooted trees on d nodes is

$$d\,\frac{d^{d-2}}{d!} = \frac{d^{d-1}}{d!}.$$

Suppose now we have two rooted trees. We can joint these two roots with a new edge. The result is a new tree, where there is no root any more. But we have a particular edge, which did not exist in any of the two original trees. Thus we obtain a **based tree**, i.e., a tree in which a particular edge is chosen and declared it to be the *base* of the tree.

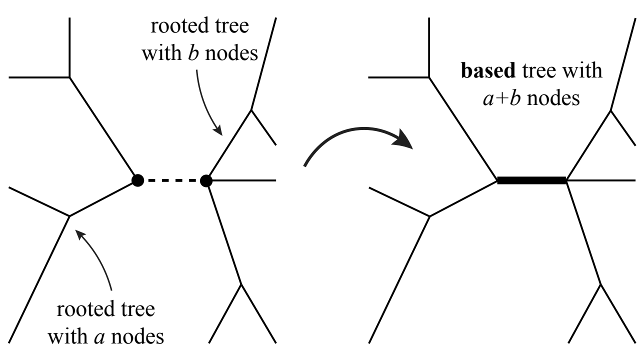

FIGURE 2.2. Construction of a **based** tree from two **rooted** trees.

The process if reversible. When you have a tree, and an edge has a name "base" on it, then we can simply remove it. Being a tree, the process produces two disjoint trees. The two ends of the removed edge will become the roots of the two trees. A tree on n nodes has

$n-1$ edges. So we obtain a bijective counting formula

$$\frac{1}{2}\sum_{\substack{a+b=n\\a\geq 1,b\geq 1}}\frac{a^{a-1}}{a!}\cdot\frac{b^{b-1}}{b!}=(n-1)\,\frac{n^{n-2}}{n!},\qquad n\geq 2. \tag{2.44}$$

The half on the left hand side compensates the double count of interchanging the two parts. In terms of the generating function, or the Laplace transform $y(x)$ of (2.42), the tree counting formula (2.44), after multiplying nx^n, produces the following differential equation with $D = x(d/dx)$:

$$D\left(\frac{1}{2}y^2\right)=xD\left(\frac{y}{x}\right)\quad\Longleftrightarrow\quad yy'=\frac{xy'-y}{x}\quad\Longleftrightarrow\quad Dy=\frac{y}{1-y}.$$

Thus we recover (2.43), and this is exactly what we need throughout Section 2. We can also directly solve this differential equation and find the classical Lambert curve, avoiding Lagrange's Inversion Formula all together:

$$(1-y)\,\frac{y'}{y}=\frac{1}{x}\quad\Longrightarrow\quad x=ye^{-y}.$$

Here, we use the initial values $y(0) = 0$ and $y'(0) = 1$ from (2.42).

3. Variations of spectral curves in Higgs bundles and opers

The goal of this section is to review the biholomorphic correspondence between the moduli space of spectral curves and that of opers for any given smooth projective algebraic curve C of genus $g(C) > 1$, mentioned in Introduction. Since the passage goes through $\hbar$-family of deformations of a differential operator, it has a counter intuitive feature.

3.1. **Variation 1: Moduli spaces of Higgs bundles and character varieties.**

Hitchin's introduction [49] of spectral curves in the study of the cotangent bundle of the moduli spaces of vector bundles on a smooth base curve has considerably expanded the world of *spectral curves*. Spectral curves have appeared independently in integrable systems and random matrix theory. With Hitchin's work, they appear at the heart of algebraic geometry [5].

Looking at the formalism of Lax (1.17), we immediately see the following: deformations that the system of PDEs are making preserve eigenvalues of the Lax operator. This is due to the right-hand side of the equation, which is a commutator. Because of this feature, the soliton equation type integrable systems are studied through the unified idea of *isospectral deformation theory*. The spectral curve $\mathrm{Spec}(A_L)$ of (1.21) is therefore the fundamental invariant of the evolution equation.

Hitchin's perspective is to deform these spectral curves. For a pair of a vector bundle and a curve (E, C), one can construct a pair of a spectral curve and a line bundle on it, $(\pi : \Sigma \to C, \mathcal{L})$, such that $E \cong \pi_*\mathcal{L}$. When we move the line bundle on the Jacobian, the construction "covers" the moduli space of vector bundles on C of a fixed topology, by "recovering" the vector bundle as $\pi_*\mathcal{L}$. But there is no canonical choice of Σ. Then, why don't we consider all possibilities?

From the point of view of (1.12) and the goal of constructing differential operators, we now place our spectral curve in the cotangent bundle T^*C of C as

$$\begin{array}{ccc} \Sigma & \xrightarrow{\ i\ } & T^*C \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle\pi} \\ & & C. \end{array} \tag{3.1}$$

Such a curve arrises as the *spectrum* of a matrix $\phi : E \longrightarrow E \otimes K_C$, which Hitchin named a *Higgs field*. It is a matrix of 1-forms, $\phi \in H^0(C, \mathrm{End}(E) \otimes K_C)$. The spectral curve is the set of eigenvalues of ϕ, i.e.,

$$\Sigma = \left\{(z,\lambda) \in T^*C \;\middle|\; z \in C,\ \lambda \in T^*_zC,\ \det\big(\lambda Id - \phi(z)\big) = 0\right\}.$$

Notice that the infinitesimal deformation space of E is $H^1(C, \mathrm{End}(E))$, which is dual to the space of Higgs fields:

$$H^1(C, \mathrm{End}(E)) \cong H^0(C, \mathrm{End}(E) \otimes K_C)^*. \tag{3.2}$$

Hitchin was naturally led to considering the moduli space of pairs (E, ϕ), or the *Higgs bundles*, to simultaneously deal with all possible deformations of E, Σ, and $\mathcal{L}$ over a fixed base curve C.

Since vector bundles on curves of genus 0 and 1 behave differently from the curves of *general type* $g = g(C) > 1$, let us restrict ourselves to the latter case for now. We also restrict our attention to vector bundles E of rank n with the fixed trivial determinant $\det(E) \cong \mathcal{O}_C$. It is natural to restrict the endomorphism sheaf for such a bundle to be *traceless*, because of the same reason of $\exp : sl_n(\mathbb{C}) \longrightarrow SL_n(\mathbb{C})$. We denote the sheaf of traceless endomorphism by $\mathrm{End}_0(E)$, which is a rank $n^2 - 1$ locally free module over $\mathcal{O}_C$.

The Riemann-Roch formula

$$\dim H^0(C, \mathrm{End}_0(E)) - \dim H^1(C, \mathrm{End}_0(E)) = -(g-1)(n^2-1) \tag{3.3}$$

tells us that the *expected* dimension of the moduli space of vector bundles of the trivial determinant is $(g-1)(n^2-1)$. Although the identity map of E into itself is a non-trivial endomorphism, it has a non-zero trace, hence it is not a section of $\mathrm{End}_0(E)$. When do we have a vector bundle with non-trivial endomorphisms? For example, $E = \mathcal{O}_C(m) \oplus \mathcal{O}_C(-m)$ for a large $m > 0$ has endomorphisms because

$$\dim H^0(C, \mathrm{End}_0(E)) = \dim \mathrm{Hom}(\mathcal{O}_C(-m), \mathcal{O}_C(m)) + 1 = 2m - g + 2.$$

It then contributes to the dimension of the space of infinitesimal deformations of E. Although these vector bundles appear as *objects* of the moduli **stack** of vector bundles with trivial determinants, they need to be excluded from the moduli **space**. The *slope stability condition*

$$\frac{\deg(E)}{\mathrm{rank}(E)} > \frac{\deg(F)}{\mathrm{rank}(F)} \tag{3.4}$$

for every proper vector subbundle $F \subset E$ is imposed for the construction of moduli spaces to avoid such appearances of large endomorphisms. For our case, since E has $\deg(E) = 0$, it cannot have any subbundles of non-negative degrees. And the moduli space $\mathcal{SU}(C, n)$ is smooth at a stable vector bundle E with the tangent and cotangent spaces given by

$$\begin{cases} T_E\mathcal{SU}(C,n) = H^1(C, \mathrm{End}_0(E)) \\ T^*_E\mathcal{SU}(C,n) = H^0(C, \mathrm{End}_0(E) \otimes K_C). \end{cases}$$

The Higgs field $\phi \in H^0(C, \mathrm{End}_0(E) \otimes K_C)$ functions as a marking of the vector bundle E. We define an automorphism of (E, ϕ) to be an automorphism of E that fixes ϕ. Thus a Higgs field makes the pair (E, ϕ) more stable. So the slope stability of E is modified for a Higgs bundle (E, ϕ), requiring that (3.4) holds only for *ϕ-invariant* subbundles F, meaning that ϕ maps F to $F \otimes K_C$.

A concrete example better explains this effect. Since $\deg(K_C) = 2g - 2$, the canonical bundle has a square root $K_C^{\frac{1}{2}}$. Actually, the $\pm$ sign choice redundancy exists here, so there are $\left|H^0\big(C,\mathbb{Z}/2\mathbb{Z}\big)\right| = 2^{2g}$ different choices of square roots. Any one of them is called a *spin structure* of C, and it appears in Riemann's work as a *θ-characteristic*. Choose a spin structure $K_C^{\frac{1}{2}}$ on C and define $E = K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}$. Clearly E is not a stable vector bundle. Since $\mathrm{End}(E) = K_C \oplus \mathcal{O}_C \oplus \mathcal{O}_C \oplus K_C^{-1}$, a non-zero quadratic differential

$$q \in H^0(C, K_C^{\otimes 2}) = \mathrm{Hom}\big(K_C^{-\frac{1}{2}}, K_C^{\frac{1}{2}} \otimes K_C\big)$$

defines a non-trivial traceless Higgs field of E. Similarly, the identity map

$$1 \in H^0(C, K_C^{-1} \otimes K_C) = \mathrm{Hom}\big(K_C^{\frac{1}{2}}, K_C^{-\frac{1}{2}} \otimes K_C\big)$$

is also a traceless Higgs field of E. Adding together, we have a non-trivial traceless Higgs field of E constructed by

$$\phi = \begin{bmatrix} 0 & q \\ 1 & 0 \end{bmatrix} : \begin{pmatrix} K_C^{\frac{1}{2}} \\ \oplus \\ K_C^{-\frac{1}{2}} \end{pmatrix} \longrightarrow \begin{pmatrix} K_C^{\frac{1}{2}} \\ \oplus \\ K_C^{-\frac{1}{2}} \end{pmatrix} \otimes K_C. \tag{3.5}$$

None of the subbundles $K_C^{\pm\frac{1}{2}}$ of E are invariant under this ϕ. Hence (E,ϕ) is stable. Being a split bundle, E has a non-trivial automorphism for every choice of 1-form $\omega \in H^0(C, K_C)$,

$$u = \sqrt{-1}\begin{bmatrix} 1 & \omega \\ 0 & -1 \end{bmatrix} : \begin{pmatrix} K_C^{\frac{1}{2}} \\ \oplus \\ K_C^{-\frac{1}{2}} \end{pmatrix} \longrightarrow \begin{pmatrix} K_C^{\frac{1}{2}} \\ \oplus \\ K_C^{-\frac{1}{2}} \end{pmatrix}, \qquad \det u = 1.$$

But u does not fix ϕ, because $u^*(\phi) := u^{-1} \circ \phi \circ u = \begin{bmatrix} \omega & \omega^2 - q \\ -1 & -\omega \end{bmatrix} \neq \phi$. This means u is not an automorphism of the Higgs bundle (E,ϕ). We rather consider (E,ϕ) and $(E, u^*(\phi))$ are *isomorphic* as Higgs bundles.

Let us denote by $\mathcal{M}(C,G)$ the moduli space of isomorphism classes of stable $SL_n(\mathbb{C})$-Higgs bundles we have been discussing. Here, G is meant to be $G = SL_n(\mathbb{C})$, which can be generalized to other Lie groups. If E is itself a stable vector bundle, then (E,ϕ) is stable as a Higgs bundle for any Higgs field ϕ. So we see that the cotangent bundle of the moduli of stable bundles is included, as an open dense subset, in the moduli space of stable Higgs bundles:

$$T^*\mathcal{SU}(C,n) \subset \mathcal{M}(C,G).$$

The *complex symplectic structure* on $\mathcal{M}(C,G)$ is constructed from this embedding by analyzing the codimension of the complement of $T^*\mathcal{SU}(C,n)$.

Here, we note the dimension, $\dim_{\mathbb{C}} \mathcal{M}\big(C, SL_n(\mathbb{C})\big) = 2(g-1)(n^2-1)$. There is another complex symplectic manifold of the same dimension, which appears to be very different from the context of Higgs bundles. Let

$$\mathrm{Hom}\big(\pi_1(C), SL_n(\mathbb{C})\big)$$

denote the space of representations of the fundamental group of C into $SL_n(\mathbb{C})$. Slightly modifying the homology bases (1.15), we can choose a homotopy basis for $\pi_1(C)$ and give a presentation

$$\pi_1(C) = \big\langle \alpha_1, \alpha_2, \dots, \alpha_g, \beta_1, \beta_2, \dots, \beta_g \;\big|\; [\alpha_1,\beta_1]\cdot[\alpha_2,\beta_2]\cdots[\alpha_g,\beta_g] = 1 \big\rangle, \tag{3.6}$$

where $[\alpha,\beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ is the multiplicative commutator.

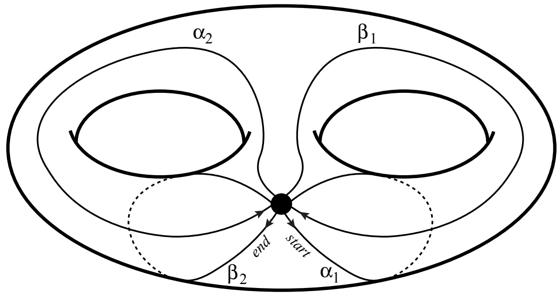

FIGURE 3.1. A homotopy basis for $\pi_1(C)$ of a curve of genus 2. $[\alpha_1,\beta_1]\cdot[\alpha_2,\beta_2]=1$.

Since every representation of the fundamental group $\rho:\pi_1(C)\longrightarrow SL_n(\mathbb{C})$ is determined by its values $a_i=\rho(\alpha_i)\in SL_n(\mathbb{C})$ and $b_i=\rho(\beta_i)\in SL_n(\mathbb{C})$, and since the commutator relation

$$[a_1,b_1]\cdot[a_2,b_2]\cdots[a_g,b_g]=1 \tag{3.7}$$

is a polynomial equation with respect to the entries of the standard $n\times n$ matrix representation of $SL_n(\mathbb{C})$, the representation space is realized as an affine algebraic variety

$$\mathrm{Hom}\big(\pi_1(C),SL_n(\mathbb{C})\big)=\big\{(a_1,b_1),\dots,(a_g,b_g)\in SL_n(\mathbb{C})^{2g}\;\big|\;[a_1,b_1]\cdots[a_g,b_g]=1\big\}.$$

For many purposes, we identify two representations when they have the same *character*. In our geometric situation, two representations ρ_1 and ρ_2 are identified if there is a group element $h\in SL_n(\mathbb{C})$ so that $\rho_1=h^{-1}\rho_2 h$. This identification induces a conjugation action of $SL_n(\mathbb{C})$ on the space of representations. Thus we define the **character variety** of the fundamental group of $\pi_1(C)$ into a complex algebraic group G as the categorical quotient

$$\chi\big(\pi_1(C),G\big)=\mathrm{Hom}\big(\pi_1(C),G\big)/\!\!/G, \tag{3.8}$$

understanding that the G-action on the representation space is by conjugation. The *Geometric Invariant Theory* construction [77] of this quotient makes the character variety a scheme of dimension $2(g-1)\dim_{\mathbb{C}}G$, where $\dim_{\mathbb{C}}G$ is subtracted twice because of the commutator relation (3.7) and the quotient of the conjugation action. This quotient is also identified as the **symplectic quotient** (see the added chapter in the 3rd edition of [77] by Kirwan), which makes the character variety a symplectic variety. For our case of $G=SL_n(\mathbb{C})$, we have

$$\dim_{\mathbb{C}}\chi\big(\pi_1(C),SL_n(\mathbb{C})\big)=2(g-1)(n^2-1).$$

It was Narasimhan and Seshadri [78] who noticed the homeomorphism

$$\chi\big(\pi_1(C),SU(n)\big)\xrightarrow{\sim}\mathcal{SU}(C,n), \tag{3.9}$$

where the character variety is a *real* variety of dimension $2(g-1)(n^2-1)$, and $\mathcal{SU}(C,n)$ is a complex variety of dimension $(g-1)(n^2-1)$. Yet another space of this dimension is the *base*, whose dimension is calculated via Riemann-Roch again:

$$B:=\bigoplus_{\ell=2}^{n}H^0\big(V,K_C^{\otimes\ell}\big),\quad \dim_{\mathbb{C}}\left(\sum_{\ell=2}^{n}\big(\ell(2g-2)-(g-1)\big)\right)=(g-1)(n^2-1). \tag{3.10}$$

This is the space we mentioned in Section 1.2, (1.14). Now our *numerology* is complete:

$$\dim_{\mathbb{R}}\chi\big(\pi_1(C),SU(n)\big)=2\dim_{\mathbb{C}}\mathcal{SU}(C,n)=2\dim_{\mathbb{C}}B$$

$$= \dim_{\mathbb{C}} \mathcal{M}\big(C, SL_n(\mathbb{C})\big) = \dim_{\mathbb{C}} \chi\big(\pi_1(C), SL_n(\mathbb{C})\big) = 2(g-1)(n^2-1).$$

The complexification of Narasimhan-Seshadri (3.9) has been established in its vast generality ([21, 24, 49, 86, 87, 88, 89], see also [66]). In our context, it is a curve case of the homeomorphism of Simpson, or *non-Abelian Hodge correspondence*

$$\chi\big(\pi_1(C), G\big) \overset{\sim}{\longrightarrow} \mathcal{M}(C, G) \tag{3.11}$$

for a complex Lie group G. These are hyperkähler manifolds and their complex structures differ by a hyperkähler rotation. The complex structure of the character variety has nothing to do with the complex structure of the curve C.

Hitchin's discovery [49] includes the algebraically completely integrable system on the moduli space $\mathcal{M}\big(C, SL_n(\mathbb{C})\big)$ of Higgs bundles through the characteristic polynomial map to the base space B,

$$\mu_H : \mathcal{M}\big(C, SL_n(\mathbb{C})\big) \ni (E, \phi) \longmapsto \det(\lambda I - \phi) \in B. \tag{3.12}$$

It is a generically Abelian variety fibration, and each fiber is a complex Lagrangian of the complex symplectic manifold $\mathcal{M}\big(C, SL_n(\mathbb{C})\big)$. Since $T^*\mathcal{SU}(C,n) \subset \mathcal{M}\big(C, SL_n(\mathbb{C})\big)$, each cotangent space $T^*_E\mathcal{SU}(C,n)$ at a stable vector bundle E is also a complex Lagrangian of $\mathcal{M}\big(C, SL_n(\mathbb{C})\big)$.

There is another complex Lagrangian, which plays a critically important role in our current investigation. It is obtained by a particular section $\kappa : B \hookrightarrow \mathcal{M}\big(C, SL_n(\mathbb{C})\big)$ of the map μ_H. Let us review the construction of this section, called the *Hitchin section*, utilizing Kostant's *principal three-dimensional subgroup* (TDS) [58]. We first choose a spin structure $K_C^{\frac{1}{2}}$, and take an arbitrary point $\mathbf{q} = (q_2, q_3, \dots, q_n) \in B$ of the base B. Define three $n \times n$ matrices in $sl(n, \mathbb{C})$ by

$$\begin{aligned} &X_- := \left[\sqrt{s_{i-1}}\delta_{i-1,j}\right]_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \sqrt{s_1} & & & & 0 \\ 0 & \sqrt{s_2} & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & \sqrt{s_{n-1}} & 0 \end{bmatrix}, \qquad X_+ := X_-^t, \\ &H := [X_+, X_-], \end{aligned} \tag{3.13}$$

where $s_i := i(n-i)$. H is a diagonal matrix whose (i,i)-entry is $s_i - s_{i-1} = n - 2i + 1$, with $s_0 = s_r = 0$. The Lie subalgebra generated by X_+, X_-, H is isomorphic to $sl(2, \mathbb{C})$, and is called the *principal TDS* of $SL(r, \mathbb{C})$.

Definition 3.1. The *Hitchin section* is defined as the set $\big\{\big(E_0, \phi(\mathbf{q})\big) \bigm| \mathbf{q} \in B\big\}$ consisting of Higgs bundles, with the fixed vector bundle

$$E_0 := \left(K_C^{\frac{1}{2}}\right)^{\otimes H} = \bigoplus_{i=1}^{n} \left(K_C^{\frac{1}{2}}\right)^{\otimes(n-2i+1)} = \bigoplus_{\ell=0}^{n-1} K_C^{-\frac{n-1}{2}} \otimes K_C^{\otimes \ell}, \quad (\ell = n-i) \tag{3.14}$$

and varying Higgs fields with parameter $\mathbf{q}$:

$$\phi(\mathbf{q}) := X_- + \sum_{\ell=2}^{n} q_\ell X_+^{\ell-1}. \tag{3.15}$$

Each Higgs bundle $\big(E_0, \phi(\mathbf{q})\big)$ as constructed is stable. The Hitchin section

$$\kappa : B \ni \mathbf{q} \longmapsto \big(E_0, \phi(\mathbf{q})\big) \in \mathcal{M}\big(C, SL_n(\mathbb{C})\big)$$

is a biholomorphic map from B to $\kappa(B) \subset \mathcal{M}\big(C, SL_n(\mathbb{C})\big)$. The example $\left(K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}, \phi\right)$ of (3.5) is the rank 2 case of the Hitchin section.

3.2. **Variation 2: Higher order linear differential operators on a curve C.**

Question 3.2. *What is a linear differential operator of order $n \geq 2$ globally defined on a compact smooth algebraic curve C?*

If ω is a global holomorphic 1-form on C, then $(d+\omega)\psi = 0$ is a first order linear differential equation globally defined on C. But we have no globally defined second order differential operator, since $d^2 = 0$. Once again let us go back to (0.4), and consider the second order case first:

$$\left(\frac{d^2}{dz^2} - q(z)\right)\psi(z) = 0. \tag{3.16}$$

We need the following:

- To find special coordinate system on the curve C that allows the expression (3.16) globally stays the same; and
- To show that every complex structure of C admits such a coordinate system.

We consider $\psi(z)$ of (3.16) to be a section of some unknown line bundle L with a transition function $e^{g(w)}$ such that $e^{-g(w)}\psi(w) = \psi(z)$. The second order differential operator acts on L and change its section into a section of $K_C^{\otimes 2} \otimes L$. Therefore, we wish to solve the equation

$$\left(\left(\frac{d}{dz}\right)^2 - q(z)\right)\psi(z) = \left(\frac{dw}{dz}\right)^2 e^{-g(w)} \left(\left(\frac{d}{dw}\right)^2 - q(w)\right)\psi(w). \tag{3.17}$$

Here, the unknowns are the coordinate change $w = w(z)$ as a function in z, and the function $g(w)$. It is natural to assume that the 0-th order term q of the differential operator satisfies $q(z)dz^2 = q(w)dw^2$, i.e., it is a quadratic differential $q \in K_C^{\otimes 2}$. With these considerations, we can simplify (3.17) and obtain

$$\begin{aligned}
&\left(\frac{d}{dz}\right)^2 = \left(\frac{dw}{dz}\right)^2 e^{-g(w)} \left(\frac{d}{dw}\right)^2 e^{g(w)} \\
\iff\ & \frac{d}{dz}\left(\frac{dw}{dz}\frac{d}{dw}\right) = \left(\frac{dw}{dz}\right)^2 \left(\left(\frac{d}{dw}\right)^2 + 2g_w(w)\frac{d}{dw} + g_{ww}(w) + g_w(w)^2\right) \\
\iff\ & w''\frac{d}{dw} = 2(w')^2 g_w \frac{d}{dw} + g_{ww} + g_w^2 \\
\iff\ & \begin{cases} 2g_w w' = \frac{w''}{w'} \\ g_{ww} + g_w^2 = 0 \end{cases} \iff \begin{cases} g(w)' = \frac{1}{2}(\log w')' \\ g_{ww} + g_w^2 = 0\ , \end{cases}
\end{aligned} \tag{3.18}$$

where $w' = \frac{dw}{dz}$ and $g_w = \frac{dg}{dw}$. The first line of the final equations implies

$$e^{-g(w)} = \sqrt{\frac{dz}{dw}}$$

because $g(w) = 0$ when $w(z) = z$. Therefore, the line bundle is $L \cong K_C^{-\frac{1}{2}}$. And the second line, $g_{ww} + g_w^2 = 0$, after substituting $g_w = \frac{1}{2}\frac{w''}{(w')^2}$ into it, gives us exactly $s_z(w) = 0$, where

$$s_z(w) := \left(\frac{w''(z)}{w'(z)}\right)' - \frac{1}{2}\left(\frac{w''(z)}{w'(z)}\right)^2 \tag{3.19}$$

is the **Schwarzian derivative**. We learn from Gunning [47] that every complex structure of C admits a *projective coordinate system*, and that the transition function of this coordinate system

$$w(z) = \frac{az+b}{cz+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}), \tag{3.20}$$

satisfies the equation $s_z(w) = 0$.

The calculation of (3.18) can be extended to an operator P of order n. The comparison of the $(n-1)$-order terms in

$$\left(\frac{d}{dz}\right)^n = \left(\frac{dw}{dz}\right)^n e^{-g(w)} \left(\frac{d}{dw}\right)^n e^{g(w)} \tag{3.21}$$

gives

$$\binom{n}{2}(w')^{n-2}w'' \left(\frac{d}{dw}\right)^{n-1} = n(w')^n g_w \left(\frac{d}{dw}\right)^{n-1} \iff \quad g(w)' = \frac{n-1}{2}(\log w')'.$$

Hence the line bundle on which P acts is $K_C^{-\frac{n-1}{2}}$. And when we use the projective coordinate system, from (3.20) we find $\frac{dw}{dz} = \frac{1}{(cz+d)^2}$. Therefore, the transition function for the canonical sheaf is $(cz+d)^2$, and with a choice of a spin structure on C, we can use $cz+d$ as the transition function of $K_C^{\frac{1}{2}}$. Then (3.21) should give

$$\begin{aligned} &\left(\frac{d}{dz}\right)^n = \left(\frac{dw}{dz}\right)^n e^{-g(w)} \left(\frac{d}{dw}\right)^n e^{g(w)} \\ \iff \quad &\left(\frac{d}{dz}\right)^n = (cz+d)^{-n-1} \left(\frac{d}{dw}\right)^n (cz+d)^{-n+1} \\ \iff \quad &\left(\frac{d}{dw}\right)^n = (cz+d)^{n+1} \left(\frac{d}{dz}\right)^n (cz+d)^{n-1}, \end{aligned} \tag{3.22}$$

which is easy to check by induction on n.

The differential equation $P\psi = 0$ makes sense globally on C if ψ is a section of $K_C^{-\frac{n-1}{2}}$, and P changes it to a section of $K_C^{\otimes n} \otimes K_C^{-\frac{n-1}{2}} = K_C^{\frac{n+1}{2}}$. In terms of the projective coordinates, we start with a section $\psi \in \otimes K_C^{-\frac{n-1}{2}}$, which satisfies $\psi(w) = (cz+d)^{-(n-1)}\psi(z)$. Then the third line of (3.22) gives

$$\begin{aligned} \psi^{(n)}(w) := \left(\frac{d}{dw}\right)^n \psi(w) &= (cz+d)^{n+1} \left(\frac{d}{dz}\right)^n (cz+d)^{n-1}(cz+d)^{-(n-1)}\psi(z) \\ &= (cz+d)^{n+1} \left(\frac{d}{dz}\right)^n \psi(z) = (cz+d)^{n+1}\psi^{(n)}(z). \end{aligned}$$

Thus the n-derivative $\psi^{(n)}$ globally makes sense as a section of $K_C^{\frac{n+1}{2}}$ in this projective coordinate system.

In conclusion, we have verified that a locally given n-th order differential operator P of (0.4) can globally be extended on the curve C if we use a projective coordinate system on C, and if P is considered to be a $\mathbb{C}$-linear map $P : K_C^{-\frac{n-1}{2}} \longrightarrow K_C^{\frac{n+1}{2}}$.

3.3. **Variation 3: Spectral curves and opers.**

Question 3.3. *An n-th oder globally defined linear differential equation $P\psi = 0$ on C is equivalent to a system of n first order differential equations on every local neighborhood $U \subset C$. Does this equivalence globally hold? And if so, what is the condition for the system of first order differential equations?*

The *opers* [6, 7] give an answer. Our starting point is the local connection ∇_z of (0.5). Since $\psi \in K_C^{-\frac{n-1}{2}}$, and its n-th derivative $\psi^{(n)} \in K_C^{\frac{n+1}{2}}$, the vector Ψ of (0.5) on which ∇_z acts is a local section of

$$\bigoplus_{\ell=0}^{n-1} K_C^{-\frac{n-1}{2}} \otimes K_C^{\otimes \ell},$$

which is the same as E_0 of (3.14). And the matrix expression of ∇_z of (0.5) exhibits a similarity with the Higgs field $\phi(\mathbf{q})$ of (3.15). Here, *similarity* means that the connection matrix of (0.5) and the matrix of (3.15) have the same characteristic polynomial, which is a point of the *base* B. However, we learn from Atiyah [4] that there is no holomorphic connection on this particular vector bundle E_0. We need to identify the vector bundle E such that the differential operator $\left(P, K_C^{-\frac{n-1}{2}}\right)$ is equivalent to a connection (E, ∇), globally on C. The *equivalence* we want to achieve here is the obvious local equivalence between (0.4) and (0.5). In the local expression

$$P = \left(\frac{d}{dz}\right)^n + \sum_{k=2}^{n} a_k \left(\frac{d}{dz}\right)^{n-k}, \tag{3.23}$$

the coefficients are $a_k \in H^0\big(C, K_C^{\otimes k}\big)$. Therefore, the dimension of the space of globally defined differential operator of order n of the shape (3.23) is equal to $\dim B = (g-1)(n^2-1)$.

For the purpose of analyzing the situation, let us go back to the consideration of the second order case (3.16) again. The operator $P = (d/dz)^2 + q(z)$ depends only on $q \in H^0\big(C, K_C^{\otimes 2}\big) = B$ in this case. As in Section 3.2, we use local coordinates w and z connected by a fractional linear transformation (3.20). The spin structure $K_C^{\frac{1}{2}}$ we choose corresponds to the transition function $\xi = \xi_{wz} = cz + d$.

Proposition 3.4 (Section 3.1 of [32])**.** *Choose an element $\hbar \in H^1(C, K_C) \cong \mathbb{C}$. Then the Higgs bundle on the Hitchin section*

$$(E_0, \phi) = \left(K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}, \begin{bmatrix} & q \\ 1 & \end{bmatrix}\right) \tag{3.24}$$

automatically selects a unique $\hbar$-connection $\big(E^\hbar, \nabla^\hbar\big)$ on the extension bundle

$$0 \longrightarrow K_C^{\frac{1}{2}} \longrightarrow E^\hbar \longrightarrow K_C^{-\frac{1}{2}} \longrightarrow 0 \tag{3.25}$$

determined by $\hbar \in \mathrm{Ext}^1\big(K_C^{-\frac{1}{2}}, K_C^{\frac{1}{2}}\big)$. Here, the $\hbar$-connection is given by

$$\nabla^\hbar = \hbar\, d + \begin{bmatrix} & q \\ 1 & \end{bmatrix}. \tag{3.26}$$

Remark 3.5. The surprise here is in the connection matrix (3.26). It is identical to the Higgs field ϕ of (3.24)! But this is exactly what we wanted.

Proof. The transition function for the vector bundle E_0 is $\begin{bmatrix}\xi_{wz} & \\ & \xi_{wz}^{-1}\end{bmatrix}$, and the Higgs field ϕ transforms

$$\begin{bmatrix} & q(w)\\ 1 & \end{bmatrix} dw = \begin{bmatrix}\xi_{wz} & \\ & \xi_{wz}^{-1}\end{bmatrix}\begin{bmatrix} & q(z)\\ 1 & \end{bmatrix} dz \begin{bmatrix}\xi_{wz} & \\ & \xi_{wz}^{-1}\end{bmatrix}^{-1},$$

because ϕ is a matrix valued 1-form. We note that $dw/dz = \xi_{wz}^{-2} = 1/(cz+d)^2$. As calculated in [32, Section 3.1], the extension group $\mathrm{Ext}^1\big(K_C^{-\frac{1}{2}}, K_C^{\frac{1}{2}}\big)$ is generated by the cohomology class $\frac{d}{dz}\xi_{wz} = c$, which is the image of the line bundle $K_C^{\frac{1}{2}}$ under the cohomology map

$$\cdots \longrightarrow H^1\big(C, \mathcal{O}_C^*\big) \longrightarrow H^1\big(C, K_C\big) \longrightarrow \cdots$$

associated with the short exact sequence $0 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{O}_C^* \longrightarrow K_C \longrightarrow 0$. The extension (3.25) is then defined by a matrix $\begin{bmatrix}\xi_{wz} & \hbar\, c\\ & \xi_{wz}^{-1}\end{bmatrix}$, and the $\hbar$-connection $\nabla^\hbar$ satisfies the desired transition relation

$$\hbar\, d + \begin{bmatrix} & q(w)\\ 1 & \end{bmatrix} dw = \begin{bmatrix}\xi_{wz} & \hbar\, c\\ & \xi_{wz}^{-1}\end{bmatrix}\left(\hbar\, d + \begin{bmatrix} & q(z)\\ 1 & \end{bmatrix} dz\right)\begin{bmatrix}\xi_{wz} & \hbar\, c\\ & \xi_{wz}^{-1}\end{bmatrix}^{-1}.$$

The precise cancellation here makes us feel miraculous. □

Definitions of a few terminologies are due. Deligne's $\hbar$-connection $\nabla^\hbar$ is a linear differential operator $\nabla^\hbar : E \longrightarrow E \otimes K_C[[\hbar]]$ on a vector bundle E that satisfies

$$\nabla^\hbar(fs) = s \otimes \hbar\, df + f\nabla^\hbar(s).$$

The restriction $\nabla^\hbar\big|_{\hbar=0}$ is a Higgs field, and $\nabla^\hbar\big|_{\hbar=1}$ is a connection in the usual sense. The extension $E^\hbar$ of (3.25) comes with a natural 3-term *filtration* $0 \subset \mathcal{F}^1 \subset E^\hbar$ that satisfies a condition $\mathcal{F}^1 \cong \big(E^\hbar/\mathcal{F}^1\big) \otimes K_C$. More generally, a connection (∇, E) on a vector bundle E of rank n is an **oper** if there is a full length filtration

$$0 = \mathcal{F}^n \subset \mathcal{F}^{n-1} \subset \mathcal{F}^{n-2} \subset \cdots \subset \mathcal{F}^0 = E$$

that satisfies the **Griffiths transversality**

$$\nabla\big|_{\mathcal{F}^i} : \mathcal{F}^i \longrightarrow \mathcal{F}^{i-1} \otimes K_C, \quad i = 1, 2, \dots, n,$$

and the $\mathcal{O}_C$-module isomorphism of graded objects

$$\overline{\nabla} : \mathcal{F}^i/\mathcal{F}^{i+1} \xrightarrow{\sim} \big(\mathcal{F}^{i-1}/\mathcal{F}^i\big) \otimes K_C, \quad i = 1, 2, \dots, n-1,$$

induced by the connection ∇. The concept can be extended to the case of $\hbar$-connections.

The discovery of [32, Section 3.1] is a full generalization of Proposition 3.4 to the case of the Hitchin section on $\mathcal{M}\big(C, SL_n(\mathbb{C})\big)$.

Theorem 3.6 ([32], Theorem 3.8, Theorem 3.10, Theorem 3.11)**.** *Let C be a smooth projective algebraic curve of genus $g(C) > 1$. Choose a projective coordinate system on C, and a spin structure $K_C^{\frac{1}{2}}$. Consider the characteristic polynomial map (3.12) on $\mathcal{M}\big(C, SL_n(\mathbb{C})\big)$. Recall that every point $p \in B$ of the base B represents a spectral curve $\Sigma \subset T^*C$. Then every spectral curve uniquely determines an $\hbar$-family of opers $\big(E^\hbar, \nabla^\hbar\big)$, which corresponds to an $\hbar$-family of globally defined differential operators $P^\hbar$ of order n. The semi-classical limit of $P^\hbar$ recovers the spectral curve Σ. Moreover, the map from $p \in B$ to the corresponding connection $\big(E^{\hbar=1}, \nabla^{\hbar=1}\big)$ is a biholomorphic map between B and the moduli space of opers in the character variety $\chi\big(\pi_1(C), SL_n(\mathbb{C})\big)$, which is a complex Lagrangian.*

4. Finale Incompiuto: The Geometry that knows the irrationality of $\zeta(3)$

4.1. The Mystery Formula.

$$(-1)^n(n!)^2\sum_{\substack{\ell+m=n\\ \ell,m\ge 0}}\frac{(2\ell+m)!(\ell+2m)!}{(\ell!)^5(m!)^5}\Big(1+(m-\ell)\big(H_{2\ell+m}+2H_{\ell+2m}-5H_m\big)\Big) = \sum_{\ell=0}^{n}\binom{n}{\ell}^2\binom{n+\ell}{\ell}^2, \qquad n\ge 0. \tag{4.1}$$

Apéry's astonishing discovery of [3] announced in 1978, that proves the irrationality of a special value of the Riemann zeta function, $\zeta(3)$, caused a huge sensation (see [92]). It was followed by an almost immediate, and unexpected, geometric surprise [8, 9] that the generating function of the key integer sequence A_n of Apéry, the second line of (4.1), solves a Picard-Fuchs differential equation associated with a particular family of K3 surfaces. Since last decade, there has been a renewed sensation [43, 44, 45, 64, 96] on this topic. This time it is due to the identification that the Borel-Laplace transform of the same generating function is **identical** to the generating function of the genus 0, 1-marked point, degree $d\ge 2$ Gromov-Witten invariants of a Fano 3-fold described below, after adjusting the exponential factor e^{5t}. This is the content of Mystery Formula (4.1).

Following [18], let us introduce the Fano 3-fold V_{12}:

Definition 4.1 ([18], p.139. Fano 3-fold V_{12})**.** We consider a rank 3 vector bundle $\mathcal{E} = (\mathcal{U}^*\otimes\det\mathcal{U}^*)\oplus\det\mathcal{U}^*$ defined on the 6-simensional Grassmannian $Gr(2,5)$, where $\mathcal{U}$ is the universal bundle. Take a generic section $\sigma\in H^0\big(Gr(2,5),\mathcal{E}\big)$ of $\mathcal{E}$. The Fano 3-fold V_{12} is the zero locus $[\sigma]_0$ of this section. The genericness condition assures smoothness of the zero locus.

Theorem 4.2 ([18], p.141)**.** *Let X denote a non-singular model for the Fano 3-fold V_{12}. The degree $d\ge 2$, type $(g,n)=(0,1)$ Gromov-Witten invariant of X is defined by*

$$\gamma_d := \int_{\overline{\mathcal{M}}_{0,1}(X)} ev^*(pt)\psi^{d-2}, \tag{4.2}$$

where

- $\overline{\mathcal{M}}_{0,1}(X)$ *is the moduli stack of holomorphic maps* $f:(\mathbb{P}^1,\infty)\longrightarrow X$,
- $ev:\overline{\mathcal{M}}_{0,1}(X)\ni f\longmapsto f(\infty)\in X$ *is the evaluation map,*
- $ev^*: H^\bullet(X,\mathbb{Q})\longrightarrow H^\bullet\big(\overline{\mathcal{M}}_{0,1}(X),\mathbb{Q}\big)$ *is the induced cohomology map, and*
- $\psi=c_1(\mathbb{L})$ *is the first Chern class of the tautological line bundle* $\pi:\mathbb{L}\longrightarrow\overline{\mathcal{M}}_{0,1}(X)$ *whose fiber at an object* $[f]\in\overline{\mathcal{M}}_{0,1}(X)$ *is* $T^*_{f(\infty)}f(\mathbb{P}^1)$.

To make the formula simpler, we choose $\gamma_0=1,\gamma_1=0$. Then the generating function of these type $(0,1)$ Gromov-Witten invariants is given by

$$GW_{0,1}(X) := \sum_{d=0}^{\infty}\gamma_d\, t^d = e^{-5t}\sum_{n=0}^{\infty}(-1)^n n!\sum_{\substack{\ell+m=n\\ \ell,m\ge 0}}\frac{(2\ell+m)!(\ell+2m)!}{(\ell!)^5(m!)^5} \times\Big(1+(m-\ell)\big(H_{2\ell+m}+2H_{\ell+2m}-5H_m\big)\Big)\, t^n. \tag{4.3}$$

We recall that topological recursion for simple Hurwitz numbers and higher genus Catalan numbers, as we have seen in earlier sections, are uniformly formulated on the corresponding spectral curves, and the spectral curves are exactly the generating functions of the $(g,n) = (0,1)$-invariants in each case. The role of topological recursion is to produce all (g,n) invariants from the spectral curve through a universal recursive procedure.

From the definition (see below), the generating function of the Apéry sequence

$$A(t) = \sum_{n=0}^{\infty} A_n t^n$$

automatically satisfies a linear ODE with regular singular points. There is a 3-dimensional integral expression of $\zeta(3)$ making it a **period** in the sense of Kontsevich-Zagier [57], which provides an illuminating geometric understanding of Apéry's proof. And this integration formlula, modified with Apéry sequences, makes the direct passage toward the Gauss-Manin connection on the K3 fibration. Right after Apéry's discovery, it was pointed out that the differential equation for $A(t)$ is identified to the Picard-Fuchs differential equation associated with a 1-dimensional family of K3 surfaces [8, 9], as mentioned above.

Now, researchers are wondering (e.g., [96]), isn't this pair, a Fano 3-fold and a family of K3 surfaces, a *mirror symmetric pair*? Our interest is in the following question, hoping that something in the line of Section 1.1 may come up:

Question 4.3.
- *Does Apéry's $A(t)$ define a* ***spectral curve****?*
- *If so, does the topological recursion formulated on this spectral curve generate all Gromov-Witten invariants of V_{12} for arbitrary (g,n)?*
- *What is its quantum curve?*
- *What geometry does this quantum curve tell us?*
- *Is the asymptotic solution to the quantum curve, considered as a Schrödinger equation, at the irregular singular point a τ-function of an integrable system, such as the KP equations?*

Nothing precise is known at this moment. The *Quantum Differential Equation* (see [52]) for the Fano variety V_{12} is identified in [18] (see below, Theorem 4.17). The quantum curve in question is an $\hbar$-deformation family of QDE. This point relates to the author's work [27] on Gaiotto conjecture [41] discussed in Section 3.

4.2. **A quick review of Apéry's ideas.** Apéry proposed a recursion formula

$$(n+1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5)u_n + n^3 u_{n-1} = 0, \qquad n \geq 1. \tag{4.4}$$

Following Apéry [3], let us define two solutions of (4.4) and their generating functions:

$$\{A_n\}_{n=0}^{\infty}, \qquad A_0 = 1, \quad A_1 = 5, \qquad A(t) := \sum_{n=0}^{\infty} A_n t^n, \qquad \alpha(t) := \sum_{n=0}^{\infty} \frac{A_n}{n!} t^n, \tag{4.5}$$

$$\{B_n\}_{n=0}^{\infty}, \qquad B_0 = 0, \quad B_1 = 6, \qquad B(t) := \sum_{n=0}^{\infty} B_n t^n, \qquad \beta(t) := \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n . \tag{4.6}$$

The space of solutions of (4.4) has dimension 2, spanned by $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$. First few terms of B_n are: $0, 6, \frac{351}{4}, \frac{62531}{36}, \frac{11424695}{288}, \frac{35441662103}{36000}, \frac{20637706271}{800}, \frac{963652602684713}{1372000}, \dots$. Closed formulas for Apéry's sequences are established:

$$A_n = \sum_{\ell=0}^{n} \binom{n}{\ell}^2 \binom{n+\ell}{\ell}^2, \quad n \geq 0, \tag{4.7}$$

$$B_n = \sum_{\ell=0}^{n} \binom{n}{\ell}^2 \binom{n+\ell}{\ell}^2 \left(\sum_{m=1}^{n} \frac{1}{m^3} - \sum_{m=1}^{\ell} (-1)^m \frac{1}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right), \quad n \geq 1. \tag{4.8}$$

Partial sums of $\zeta(3) = \sum_{m=1}^{\infty} \frac{1}{m^3}$ are appearing here.

Re-interpreting the result of [18, 45, 96], the Mystery Formula (4.1) has a geometric expression:

$$\alpha(t) = e^{5t} \cdot GW_{0,1}(X). \tag{4.9}$$

Notice that $GW_{0,1}(X)$ does not have a linear term in t, while $A_1 = 5$. This requires the adjustment of e^{5t} in (4.9). Some of these amazing formulas are proved (relatively) easily through differential equations established in [8, 9, 18, 45]. Direct calculations remain to be very hard.

To see how fast the sequence determined by (4.4) grows or decreases, put $u_n \sim n^\alpha C^n$ with constants C and α. For large n, we then have

$$\frac{(n+1)^{3+\alpha}}{n^{3+\alpha}} C^2 - \left(34 + \frac{51}{n}\right) C + \frac{(n-1)^\alpha}{n^\alpha} \sim 0.$$

At $n \to \infty$ we find $C^2 - 34C + 1 = 0$. Define $C = 17 + 12\sqrt{2}$. Then the other solution of this quadratic equation is C^{-1}. Since $C^2 = 34C - 1$, the sub order terms give us

$$\frac{3+\alpha}{n} C^2 - \frac{51}{n} C - \frac{\alpha}{n} \sim 0 \Longrightarrow \alpha = -\frac{51C - 3}{34C - 2} = -\frac{3}{2}.$$

Therefore, every solution of (4.4) has a growth behavior

$$u_n \sim \text{const } n^{-\frac{3}{2}} \, C^{\pm n}. \tag{4.10}$$

In particular, both A_n and B_n grow exponentially fast, $\sim n^{-\frac{3}{2}} C^n$, as $n \to \infty$.

The other surprise, the relation to the Gauss-Manin connection of [9], starts with the integral expression

$$\zeta(3) = \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 \frac{dxdydz}{1 - z + xyz}.$$

Then it is proved that

$$A_n\zeta(3) - B_n = \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 \left(\frac{xyz(1-x)(1-y)(1-z)}{1 - z + xyz} \right)^n \frac{dxdydz}{1 - z + xyz}. \tag{4.11}$$

These integral expressions show that $A_n\zeta(3) - B_n$ is a *period* of [57] for every $n \geq 0$, that includes $\zeta(3)$. Since

$$0 < \left(\frac{xyz(1-x)(1-y)(1-z)}{1 - z + xyz} \right)^n < 1$$

in the domain of integration $(x, y, z) \in (0, 1)^3$, we immediately see that

$$\lim_{n\to\infty} \big(A_n\zeta(3) - B_n\big) = 0.$$

We know A_n and B_n grow exponentially fast, while the linear combination $A_n\zeta(3) - B_n$ converges to 0. It is also a solution of the recursion (4.4). Therefore, it decreases exponentially fast!

$$A_n\zeta(3) - B_n \sim \text{const } n^{-\frac{3}{2}} C^{-n} \Longrightarrow \left| \zeta(3) - \frac{B_n}{A_n} \right| \sim \text{const } C^{-2n}, \quad n \to \infty. \tag{4.12}$$

Let us recall a criterion for irrationality of a real number.

Lemma 4.4. *A positive real number $r > 0$ is irrational if there is an $\epsilon > 0$ such that infinitely many relatively prime positive pairs of integers p, q satisfy*

$$\left| r - \frac{p}{q} \right| < \frac{1}{q^{1+\epsilon}}.$$

This means if a real number is approximated too well by a sequence of rationals, then it cannot be rational.

Proof. Suppose $r = a/b$ with integers $a, b > 0$. Then for any pair of integers $p, q > 0$ such that $r \neq p/q$, we have

$$\frac{1}{q^{1+\epsilon}} > \left| r - \frac{p}{q} \right| = \left| \frac{a}{b} - \frac{p}{q} \right| = \left| \frac{aq - bp}{bq} \right| \geq \frac{1}{bq}, \quad \text{hence} \quad 0 < q < b^{1/\epsilon}.$$

There are only a finite number of such integers. □

To appeal to this Lemma, Apéry estimated the denominator of B_n. Let

$$d_n := LCM[1, 2, 3, \ldots, n].$$

He found that $d_n^3 B_n \in \mathbb{Z}$, which follows from (4.6). The recursion (4.4) suggests that the denominator of a solution u_n starting with integral initial values $u_0, u_1 \in \mathbb{Z}$ is about $(n!)^3$. The denominator of B_{30} is $\sim 1.9 \times 10^{33}$, while $LCM[1, \ldots, 30]^3$ divided by this denominator is 6630. So the above estimate is quite accurate.

Remark 4.5. Of course what is truly astonishing is that the sequence A_n is an **integer** sequence. Zagier [96] (see also [57]) discusses how special and rare such a phenomenon is.

Now we know that the denominator of the rational number B_n/A_n is estimated to be

$$d_n^3 A_n \sim n^{-\frac{3}{2}} d_n^3 C^n.$$

Does $\zeta(3)$ satisfy the criterion of Lemma 4.4? The Prime Number Theorem tells us the following:

$$\begin{aligned} d_n = LCM[1, 2, \ldots, n] &= \prod_{\substack{p:\text{ prime} \\ 2 \leq p \leq n}} p^{\left\lfloor \frac{\log(n)}{\log(p)} \right\rfloor} \lesssim \prod_{\substack{p:\text{ prime} \\ 2 \leq p \leq n}} p^{\frac{\log(n)}{\log(p)}} = \prod_{\substack{p:\text{ prime} \\ 2 \leq p \leq n}} e^{\log(n)} \\ &= n^{\pi(n)} \gtrsim n^{\frac{n}{\log(n)}} = e^n. \end{aligned}$$

Since the asymptotic inequalities are in the opposite directions, the above estimate is crude. However, since $C = 17 + 12\sqrt{2} = 33.9705\ldots$ is large enough compaired to $e^3 = 20.0855\ldots$, we have $C^n \gg n^{-\frac{3}{2}} d_n^3$ for very large n. In particular, there is an $\epsilon > 0$ such that

$$\left| \zeta(3) - \frac{B_n}{A_n} \right| \sim \text{const } C^{-2n} < \frac{1}{\left(n^{-\frac{3}{2}} d_n^3 C^n \right)^{1+\epsilon}}, \qquad n \gg 1.$$

This completes the proof of the irrationality of $\zeta(3)$. □

4.3. **The differential equations behind the scene.** We start with two linear differential equations,

$$(4.13) \qquad Pu = 0, \qquad P = D^3 - t(34D^3 + 51D^2 + 27D + 5) + t^2(D+1)^3$$

$$(4.14) \qquad Qu = 0, \qquad Q = D^4 - t(34D^3 + 51D^2 + 27D + 5) + t^2(D+1)^2,$$

where $D = t\frac{d}{dt}$, and $t \in \mathbb{P}^1$. To analyze these operators, let us first find solutions to the equation $Pu(t) = 0$. Put $u(t) = \sum_{n=0}^{\infty} u_n t^n$. Since $Dt^n = nt^n$, comparing the terms of t^{n+1}, we find the same 3-term recursion equation of Apéry (4.4)

$$(n+1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5)u_n + n^3 u_{n-1} = 0, \qquad n \geq 0.$$

We notice here that $Pu = 0$ for a formal power series implies (4.4) for *all* $n \geq 0$, not just $n \geq 1$ as in (4.4). Therefore, a solution to (4.4) *does not* always give a solution of $Pu = 0$. The recursion relation has two linearly independent solutions determined by the initial values u_0 and u_1. The differential equation $Pu = 0$ imposes another condition $u_1 = 5u_0$, hence the space of solutions of (4.13) analytic at the origin is 1-dimensional.

Thus we find that the generating functions of (4.5) and (4.6) satisfy

$$\begin{cases} PA(t) = 0 \\ Q\alpha(t) = 0, \end{cases} \qquad \begin{cases} PB(t) = 6t \\ Q\beta(t) = 6t. \end{cases} \tag{4.15}$$

We are considering this pair of differential equations to correspond to the two differential equations (1.3) and (1.5), since the first equation has only regular singular points, while the second one has only one irregular singular point at infinity and another regular singularity. For these singularity analysis, it is useful to rewrite the operators in terms of the usual expression:

$$P = \left(t^5 - 34t^4 + t^3\right)\left(\frac{d}{dt}\right)^3 + \left(6t^4 - 153t^3 + 3t^2\right)\left(\frac{d}{dt}\right)^2 + \left(7t^3 - 112t^2 + t\right)\frac{d}{dt} + t^2 - 5t,$$

$$Q = t^4\left(\frac{d}{dt}\right)^4 + \left(-34t^4 + 6t^3\right)\left(\frac{d}{dt}\right)^3 + \left(t^4 - 153t^3 + 7t^2\right)\left(\frac{d}{dt}\right)^2 + \left(3t^3 - 112t^2 + t\right)\frac{d}{dt} + t^2 - 5t.$$

It is easy to see that Eqn.(4.14) has the unique regular singular point at $t = 0$, and the unique irregular singular point at $t = \infty$. Its solution $\alpha(t)$ is the unique entire solution up to a constant factor. With a little calculation we find that Eqn.(4.13) has regular singular points at $\{0, C^{-1}, C, \infty\}$, and no other singular points.

Problem 4.6. *Find the $\hbar$-deformation family of Q, and identify the spectral curve through the semi-classical limit of this family. This spectral curve should be our* ***hidden curve****.*

The growth order (4.10) determines the radius of convergence of $A(t)$ and $B(t)$, which is $C^{-1} = 17 - 12\sqrt{2}$. Both $A(t), B(t)$ and the linear combination $A(t)\zeta(3) - B(t)$ solve

$$(D-1)Pu(t) = 0, \tag{4.16}$$

and the radius of convergence of $A(t)\zeta(3) - B(t)$ is C.

Theorem 4.7 ([18], p.141. Quantum Differential Equation)**.** *The generating function $GW_{0,1}(X)$ of the type $(0,1)$ Gromov-Witten invariants of X, a smooth model of Fano 3-fold V_{12}, atisfies a* ***Quantum Differential Equation****.*

The state-of-the-art result [45, 96] tells us that

$$GW_{0,1}(X) = e^{-5t}\,\alpha(t),$$

and hence the Quantum Differential Equation is

$$\left(e^{-5t} \circ Q \circ e^{5t}\right) GW_{0,1}(X) = 0. \tag{4.17}$$

If the correspondence between opers and spectral curves work for irregular singular differential operators, then $e^{-5t}\circ Q\circ e^{5t}$ should have a canonical $\hbar$-deformation family, and its semi-classical limit should be the spectral curve corresponding to $A(t)$.

Summing up the integral expression (4.11) with t^n for all $n\geq 0$, we obtain

$$\begin{aligned} A(t)\zeta(3)-B(t) &= \frac{1}{2}\int_0^1\int_0^1\int_0^1\sum_{n=0}^{\infty}\left(\frac{xyz(1-x)(1-y)(1-z)}{1-z+xyz}\right)^n t^n\frac{dxdydz}{1-z+xyz} \\ &= \frac{1}{2}\int_0^1\int_0^1\int_0^1\frac{dxdydz}{1-z+xyz-txyz(1-x)(1-y)(1-z)}. \end{aligned} \tag{4.18}$$

Beukers and Peters [9] considered the family of algebraic varieties defined in $\mathbb{C}^3$ by the equation

$$1-z+xyz-txyz(1-x)(1-y)(1-z)=0$$

as a parameter $t\in\mathbb{P}^1$ moves. They noticed that this is a family of K3 surfaces in $\mathbb{P}^3$ after projectivize the formula and blow-up all singularities. The family degenerates at $t\in\{0,C,C^{-1},\infty\}$.

Appendix A. The Lagrange Inversion Formula

In Appendix we give a brief proof of the Lagrange Inversion Formula. For more detail, we refer to [94].

Theorem A.1. *Let $x=f(y)$ be a holomorphic function in y defined on a neighborhood of $y=b$. Let $f(b)=a$, and suppose $f'(b)\neq 0$. Then the inverse function $y=y(x)$ is given by the following expansion near $x=a$:*

$$y-b=\sum_{k=1}^{\infty}\frac{d^{k-1}}{dy^{k-1}}\left(\frac{y-b}{f(y)-a}\right)^k\Bigg|_{y=b}\frac{(x-a)^k}{k!}. \tag{A.1}$$

Proof. Let us recall the Cauchy integration formula

$$\phi(s)=\frac{1}{2\pi i}\oint\frac{\phi(t)dt}{t-s},$$

where $\phi(t)$ is a holomorphic function defined on a neighborhood of $t=s$, and the integration contour is a small simple loop inside this neighborhood counterclockwisely rotating around the point s. Since $x=f(y)$ is one-to-one near $y=b$, for a point s close to b, we have

$$\begin{aligned} \frac{1}{f'(s)} &= \frac{1}{f'\big(f^{-1}(f(s))\big)} \\ &= \frac{1}{2\pi i}\oint\frac{df(t)}{f'\big(f^{-1}(f(t))\big)\big(f(t)-f(s)\big)} \\ &= \frac{1}{2\pi i}\oint\frac{f'(t)dt}{f'(t)\big(f(t)-f(s)\big)} \\ &= \frac{1}{2\pi i}\oint\frac{dt}{f(t)-f(s)}. \end{aligned}$$

Therefore, assuming that s is close enough to b, we compute

$$y-b=\int_b^y 1\cdot ds=\int_b^y\left(\frac{1}{2\pi i}\oint\frac{f'(s)dt}{f(t)-f(s)}\right)ds$$

$$
\begin{aligned}
&= \int_b^y \left(\frac{1}{2\pi i} \oint \frac{f'(s)dt}{\big(f(t)-a\big) - \big(f(s)-a\big)} \right) ds \\
&= \frac{1}{2\pi i} \int_b^y \oint \frac{\frac{f'(s)}{f(t)-a}}{1 - \frac{f(s)-a}{f(t)-a}} dtds \\
&= \frac{1}{2\pi i} \int_b^y \sum_{n=0}^{\infty} \oint \frac{f'(s)}{f(t)-a} \left(\frac{f(s)-a}{f(t)-a} \right)^n dsdt \\
&= \frac{1}{2\pi i} \int_{f(b)}^{f(y)} \sum_{n=0}^{\infty} \oint \frac{1}{f(t)-a} \left(\frac{f(s)-a}{f(t)-a} \right)^n df(s)dt \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint \frac{1}{\big(f(t)-a\big)^{n+1}} \cdot \frac{\big(f(y)-a\big)^{n+1}}{n+1} dt \\
&= \frac{1}{2\pi i} \sum_{k=1}^{\infty} \oint \frac{dt}{\big(f(t)-a\big)^k} \cdot \frac{(x-a)^k}{k} \\
&= \frac{1}{2\pi i} \sum_{k=1}^{\infty} \oint \frac{1}{(y-b)^k} \left(\frac{y-b}{f(y)-a} \right)^k dy \cdot \frac{(x-a)^k}{k} \\
&= \sum_{k=1}^{\infty} \frac{d^{k-1}}{dy^{k-1}} \left(\frac{y-b}{f(y)-a} \right)^k \Bigg|_{y=b} \cdot \frac{(x-a)^k}{k(k-1)!}.
\end{aligned}
$$

□

The following formula is a straightforward application of the above Lagrange Inversion Theorem.

Corollary A.2. *Let $f(y)$ be a holomorphic function defined in a neighborhood of $y=0$. If $f(0) \neq 0$, then the inverse function of*

$$x = \frac{y}{f(y)}$$

is given by

$$y = \sum_{k=1}^{\infty} \frac{d^{k-1}}{dy^{k-1}} \big(f(y)\big)^k \Bigg|_{y=0} \frac{x^k}{k!}. \tag{A.2}$$

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