# Strategy for Testing Series

1. If the series is of the form $\sum \frac{1}{n^p}$, it is a $p$**-series**, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.

2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a **geometric series**, which converges if $|r| < 1$ and diverges if $|r| \geq 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

3. If the series has a form that is similar to a $p$-series or a geometric series, then one of the **comparison tests** (Theorems 10, 11) should be considered. In particular, if $a_n$ is a rational function or an algebraic function of $n$ (involving roots of polynomials), then the series should be compared with a $p$-series (The value of $p$ should be chosen by keeping only the highest powers of $n$ in the numerator and denominator). The comparison tests apply only to series with positive terms. If $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum |a_n|$ and test for **absolute convergence**.

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the ***n*th Term Test for Divergence** (equivalent to Theorem 7) should be used.

5. If the series is of the form $\sum (-1)^{n+1} u_n$ or $\sum (-1)^n u_n$, then the **Alternating Series Test** (Theorem 14) is an obvious possibility.

6. Series that involve factorial or other products (including a constant raised to the $n$th power) are often conveniently tested using the **Ratio Test** (Theorem 12). Bear in mind that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$ for all $p$-series and therefore all rational or algebraic functions of $n$. Thus the Ratio Test should not be used for such series.

7. If $a_n$ is of the form $(b_n)^n$, then the **Root Test** (Theorem 13) may be useful.

8. If $a_n = f(n)$, where $\int_1^\infty f(x)\,\mathrm{d}x$ or $\int_N^\infty f(x)\,\mathrm{d}x$ for some $N \in \mathbb{N}$. is easily evaluated, then the **Integral Test** (Theorem 9) is effective (assuming that the hypotheses of this test are satisfied).

In the following examples, I will simply indicate which tests should be used. Please work out the details yourself.

**Example 1**

$$\sum_{n=1}^{\infty} \frac{2n-1}{3n+1}$$

Since $a_n \to \frac{2}{3} \neq 0$ as $n \to \infty$, we should use the $n$th Term Test for Divergence.

**Example 2**

$$\sum_{n=1}^{\infty} \frac{\sqrt[4]{n^5+1}}{3n^3+4n^2+2}$$

Since $a_n$ is an algebraic function of $n$, we compare the given series with a $p$-series. The comparison series $\sum b_n$ in the Limit Comparison Test has the following form:

$$b_n = \frac{\sqrt[4]{n^5}}{3n^3} = \frac{1}{3n^{7/4}}.$$

**Example 3**

$$\sum_{n=1}^{\infty} n\mathrm{e}^{-n^2}$$

Since the integral $\int_1^{\infty} xe^{-x^2}\,\mathrm{d}x$ is easily evaluated, we use the Integral Test. The Ratio Test also works.

**Example 4**

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^4}{n^5+1}$$

Since the series is alternating, we use the Alternating Series Test.

**Example 5**

$$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$

Since the series involves $n!$, we use the Ratio Test.

**Example 6**

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1}$$

Since the series is closely related to the geometric series $\sum \frac{1}{2^n}$, we use the Comparison Test.