MAT 271: Applied & Computational Harmonic Analysis
Supplementary Notes V by Naoki Saito

The Balian-Low Theorem

Suppose $\{g_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ *constitutes a windowed Fourier frame of* $L^2(\mathbb{R})$ *with* $\Delta x \Delta \xi = 1$ *(which includes the case of an orthonormal basis). Then, either* $\sigma_x(g) = \infty$ *or* $\sigma_\xi(g) = \infty$*.*

Proof. We only prove here the orthonormal basis case due to Battle [1]. For the general non-orthogonal case, which includes the Gabor frame, see [2].

Our strategy here is the following: Assume $\sigma_x(g) < \infty$ *and* $\sigma_\xi(g) < \infty$, then lead to contradiction. Let us consider the inner product, $\langle xg, g' \rangle$, which also appeared in the proof of the inequality of the Heisenberg uncertainty principle. Note that xg is in $L^2(\mathbb{R})$ so as g', because

$$\|xg\|^2 = \int x^2 |g(x)|^2 \, \mathrm{d}x = \sigma_x^2(g) < \infty,$$

since the mean of g is 0 and $\|g\|^2 = 1$. Recognizing that $\mathcal{F}g' = 2\pi \mathrm{i} \xi \, \widehat{g}(\xi)$ and $\sigma_\xi^2(g) < \infty$, we can show $g' \in L^2(\mathbb{R})$.

Now, we have the following:

$$\begin{aligned}
\langle xg, g' \rangle &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle xg, g_{m,n} \rangle \langle g_{m,n}, g' \rangle \\
&\overset{(a)}{=} \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle g_{-m,-n}, xg \rangle \langle -(g')_{m,n}, g \rangle \\
&\overset{(b)}{=} \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle g_{-m,-n}, xg \rangle \langle -g', g_{-m,-n} \rangle \\
&= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle -g', g_{m,n} \rangle \langle g_{m,n}, xg \rangle \\
&= -\langle g', xg \rangle . \qquad (1)
\end{aligned}$$

Here, (a) was derived by the following computations.

$$\begin{aligned}
\langle xg, g_{m,n} \rangle &= \int xg(x) \overline{g(x - m\Delta x)} \mathrm{e}^{-2\pi \mathrm{i} n \Delta \xi x} \, \mathrm{d}x \\
[\text{by change of variable } y = x - m\Delta x] &= \int (y + m\Delta x) g(y + m\Delta x) \overline{g(y)} \mathrm{e}^{-2\pi \mathrm{i} n \Delta \xi y} \, \mathrm{d}y \cdot \mathrm{e}^{-2\pi \mathrm{i} n \Delta \xi m \Delta x} \\
[\text{since } \mathrm{e}^{-2\pi \mathrm{i} n m \Delta \xi \Delta x} = \mathrm{e}^{-2\pi \mathrm{i} n m} = 1] &= \int (x + m\Delta x) g(x + m\Delta x) \overline{g(x)} \mathrm{e}^{-2\pi \mathrm{i} n \Delta \xi x} \, \mathrm{d}x \\
&= \langle g_{-m,-n}, xg \rangle + m\Delta x \langle g_{-m,-n}, g \rangle \\
&= \langle g_{-m,-n}, xg \rangle . \qquad (2)
\end{aligned}$$

The last equality holds since $g_{-m,-n}$ is orthogonal to $g = g_{0,0}$ for $(m,n) \neq (0,0)$, and if $(m,n) = (0,0)$, then $m\Delta x = 0$ at any rate.

Similarly, we can proceed:

$$\begin{aligned}\left\langle g_{m,n}, g' \right\rangle &= \int g_{m,n}(x)\overline{g'(x)}\,\mathrm{d}x \\ \text{[by integration by parts]} &= g_{m,n}(x)\overline{g(x)}\Big|_{-\infty}^{\infty} - \int (g_{m,n}(x))'\overline{g(x)}\,\mathrm{d}x \\ \text{[since } g, g' \in L^2(\mathbb{R})] &= -\int (g_{m,n}(x))'\overline{g(x)}\,\mathrm{d}x \\ &= \left\langle -(g')_{m,n}, g \right\rangle - 2\pi \mathrm{i} n\Delta\xi \left\langle g_{m,n}, g\right\rangle \\ &= \left\langle -(g')_{m,n}, g \right\rangle,\end{aligned}$$

where we used the same logic as in the last equality of (2).

As for (b) in (1), we use the same change of variable as in (2) to get $\langle -(g')_{m,n}, g\rangle = \langle -g', g_{-m,-n}\rangle$.

Now, let us consider a function $f \in C_c^\infty(\mathbb{R})$, i.e., a C^∞-function vanishing as $|x| \to \infty$. Then,

$$\begin{aligned}\left\langle xf, f' \right\rangle &= \int xf(x)\overline{f'(x)}\,\mathrm{d}x \\ \text{[by integration by parts]} &= xf(x)\overline{f(x)}\Big|_{-\infty}^{\infty} - \int \overline{f(x)}(xf'(x) + f(x))\,\mathrm{d}x \\ \text{[since } f \in C_c^\infty(\mathbb{R})] &= -\int x\overline{f(x)}f'(x)\,\mathrm{d}x - \int |f(x)|^2\,\mathrm{d}x \\ &= -\left\langle f', xf \right\rangle - \|f\|^2.\end{aligned}$$

Now, $C_c^\infty(\mathbb{R})$ is dense in $\mathcal{H} = \{f \in L^2 \,|\, xf \in L^2, f' \in L^2\}$. Hence, the function g under consideration must satisfy

$$\left\langle xg, g' \right\rangle = -\left\langle g', xg \right\rangle - \|g\|^2.$$

Combining this with (1), we conclude $\|g\| = 0$. This contradicts the condition $\|g\| = 1$. □

References

[1] G. BATTLE, *Heisenberg proof of the Balian-Low theorem*, Lett. Math. Phys., 15 (1988), pp. 175–177.

[2] I. DAUBECHIES AND A. J. E. M. JANSSEN, *Two theorems on lattice expansions*, IEEE Trans. Inform. Theory, 39 (1993), pp. 3–6.