MAT 280: Laplacian Eigenfunctions: Theory, Applications, and Computations
Lecture 11: Laplacian Eigenvalue Problems for General Domains–III. Completeness of a Set of Eigenfunctions and the Justification of the Separation of Variables

Lecturer: Naoki Saito
Scribe: Alexander Sheynis/Allen Xue

May 3, 2007

1 Completeness of a Set of Eigenfunctions

1.1 The Neumann Boundary Condition

In this lecture, we begin examining a generalized look at the Laplacian Eigenvalue Problem, particularly related to generalized domains. Our goal here is to establish the notion of the completeness of a set of eigenfunctions, which we then use to justify separation of variables, a tool we so far have taken for granted. Basic references for this lecture are [1, Sec. 11.3, 11.5], [2, Chap. 11] and [3, Sec. 3.3]. For more advanced treatments, see [4, Sec. 6.5].

In the earlier lectures, we used:

$$\begin{cases} \nu_j \text{ or } \lambda_j^{(N)} \text{ for the Neumann-Laplacian eigenvalues,} \\ \psi_j \text{ or } \varphi_j^{(N)} \text{ for the Neumann-Laplacian eigenfunctions.} \end{cases}$$

For simplicity, let us use (ν_j, ψ_j) for Neumann BC and (λ_j, φ_j) for Dirichlet BC.

And we number ν_j in ascending order: $0 = \nu_1 < \nu_2 \le \nu_3 \le \cdots$, also recall that $\psi_1(\boldsymbol{x}) = \text{const.}$

Theorem 1.1. *For the Neumann Condition, we define a "trial function" as any $w(\boldsymbol{x}) \in C^2(\Omega)$ such that $w(\boldsymbol{x}) \not\equiv 0$. Then similarly to the Dirichlet case, (MP), (MP)$_n$, RRA (see Lecture 9), and the minimax principle are all valid.*

Note: $w \in C^2(\Omega)$ means that there is no constraint at $\partial\Omega$. In particular, w may not satisfy the Neumann condition (nor the Dirichlet condition). As such, the Neumann condition is also referred to as the **free** condition and the Dirichlet condition is referred to as the **fixed** condition.

Proof. Define

$$m = \min_{\substack{w \in C^2(\Omega) \\ w \not\equiv 0}} \left\{ \frac{\|\nabla w\|^2}{\|w\|} \right\}.$$

Let $u \in C^2(\Omega)$ attain this minimum. Set $w = u + \varepsilon v$, $\forall\, v \in C^2(\Omega)$. The similar procedure leads to (as a result of variational calculus):

$$\int_\Omega (-\nabla u \cdot \nabla v + muv)\, \mathrm{d}\boldsymbol{x} = 0, \qquad \forall\, v \in C^2(\Omega) \tag{1}$$

By Green's first identity,

$$(1) \iff \int_\Omega (\Delta u + mu) v\, \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\, \mathrm{d}S. \tag{2}$$

(i) Let's choose $v(\boldsymbol{x}) = v_0(\boldsymbol{x})$, where

$$v_0(\boldsymbol{x}) \triangleq \begin{cases} \text{arbitrary } C^2 \text{ function} & \text{in } \Omega, \\ 0 & \text{on } \partial\Omega. \end{cases}$$

Then the right hand side of (2) equals 0, which implies that $\Delta u + mu = 0$ inside Ω.

(ii) Since $\Delta u + mu = 0$ in Ω for <u>any</u> $v \in C^2(\Omega)$ in (2), we have

$$(2) = 0 = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\, \mathrm{d}S \implies \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

□

1.2 Completeness (in the L^2 sense)

The notion of completeness for eigenfunctions is similar to that of a complete basis, which many of us have seen in an analysis class (or any class that significantly dealt with Fourier series), in which a set of eigenfunctions is complete if any arbitrary function in the space of concern can be exactly represented as a linear combination of the eigenfunctions (here, the exact in terms of the norm of that space). To this end, we begin with the following theorem to illustrate this point.

Theorem 1.2. *Both the Dirichlet-Laplacian (DL) and the Neumann-Laplacian (NL) eigenfunctions are* **complete** *in the L^2 sense, i.e., $\forall\, f \in L^2(\Omega)$,*

$$\begin{aligned} &\left\| f - \textstyle\sum_{n=1}^{N} c_n\varphi_n \right\|^2 \to 0 \text{ as } N \to \infty, \quad \text{where } c_n = \frac{\langle f, \varphi_n\rangle}{\langle \varphi_n, \varphi_n\rangle}. \\ &\left\| f - \textstyle\sum_{n=1}^{N} d_n\psi_n \right\|^2 \to 0 \text{ as } N \to \infty, \quad \text{where } d_n = \frac{\langle f, \psi_n\rangle}{\langle \psi_n, \psi_n\rangle}. \end{aligned} \tag{3}$$

or $\langle f, \varphi_n\rangle = 0$, $\forall n \in \mathbb{N} \quad \Leftrightarrow \quad f \equiv 0$, a.e., and $\langle f, \psi_n\rangle = 0$, $\forall n \in \mathbb{N} \quad \Leftrightarrow$ $f \equiv 0$, a.e..

Remark: This is important since $\{\varphi_n\}$, $\{\psi_n\}$ are not useful if they are not complete. In other words, if $\exists f \in L^2(\Omega)$, such that $\|f - \sum_{n=1}^{\infty} c_n\varphi_n\| > 0$, then $\{\varphi_n\}$ spans only a subspace of $L^2(\Omega)$.

Proof. The proof depends on the following two facts:

(i) The existence of the minima of the Rayleigh quotient, and

(ii) $\lambda_n \uparrow \infty$, $\nu_n \uparrow \infty$, as $n \to \infty$. (will be proved in the lecture of "eigenvalue asymptotics")

We'll prove (3) for all $f \in C_0^2(\Omega)$ for (DL) and for all $f \in C^2(\Omega)$ for (NL), respectively. For an arbitrary $f \in L^2(\Omega)$, see examples in [4, Sec. 6.5] and [2, Chap. 11].

Let's start with the (DL) case. Given an arbitrary function $f \in C_0^2(\Omega)$, set

$$r_N(\boldsymbol{x}) \triangleq f(\boldsymbol{x}) - \sum_{n=1}^{N} c_n\varphi_n(\boldsymbol{x}) \in C_0^2(\Omega).$$

By the orthogonality, for $j = 1, ..., n$,

$$\langle r_N, \varphi_j \rangle = \left\langle f - \sum_{n=1}^{N} c_n \varphi_n, \varphi_j \right\rangle = \langle f, \varphi_j \rangle - c_j \langle \varphi_j, \varphi_j \rangle = 0.$$

So, r_N satisfies the conditions of (MP)$_{N+1}$. Thus, we have

$$\lambda_{N+1} = \min_{\substack{w \in C_0^2(\Omega),\ w \not\equiv 0 \\ \langle w, \varphi_j \rangle = 0,\ j=1,...,N}} \frac{\|\nabla w\|^2}{\|w\|^2} \leq \frac{\|\nabla r_N\|^2}{\|r_N\|^2}.$$

Now we have

$$\begin{aligned} \|\nabla r_N\|^2 &= \int_\Omega |\nabla(f - \textstyle\sum_{n=1}^N c_n \varphi_n)|^2 \, \mathrm{d}\boldsymbol{x} \\ &= \int_\Omega \left(|\nabla f|^2 - 2 \sum_n c_n \nabla f \cdot \nabla \varphi_n + \sum_m \sum_n c_m c_n \nabla \varphi_n \cdot \nabla \varphi_m \right) \, \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{4}$$

By Green's first identity, we have

$$\int_\Omega \nabla f \cdot \nabla \varphi_n \, \mathrm{d}\boldsymbol{x} = -\int_\Omega f \Delta \varphi_n \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} \frac{\partial \varphi_n}{\partial \nu} f \, \mathrm{d}S = \lambda_n \langle f, \varphi_n \rangle .$$

$$\text{and similarly} \quad \int_\Omega \nabla \varphi_n \cdot \nabla \varphi_n = \delta_{mn} \lambda_n \|\varphi_n\|^2.$$

So, (4) will give us

$$\|\nabla r_N\|^2 = \int_\Omega |\nabla f|^2 \, \mathrm{d}\boldsymbol{x} - \sum_{n=1}^{N} c_n^2 \lambda_n \|\varphi_n\|^2.$$

Since $\lambda_n > 0$, we have

$$\|\nabla r_N\|^2 < \int_\Omega |\nabla f|^2 \, \mathrm{d}\boldsymbol{x} = \|\nabla f\|^2.$$

Therefore,

$$\lambda_{N+1} \leq \frac{\|\nabla r_N\|^2}{\|r_N\|^2} \implies \|r_N\|^2 \leq \frac{\|\nabla f\|^2}{\lambda_{N+1}}.$$

As we will show later $\lambda_{N+1} \uparrow \infty$ as $N \to \infty$, we have that $\|r_N\|^2 \downarrow 0$ as $N \to \infty$. The Neumann case can be derived similarly.

□

2 Justification of the Separation of Variables via Completeness

Prior to all this, we heuristically used the separation of variables technique to solve certain partial differential equations. With the idea of completeness, we can justify its use if we have that $\{\varphi_n\}_{n\in\mathbb{N}}$ and $\{\psi_n\}_{n\in\mathbb{N}}$ are a complete orthogonal basis.

2.1 Separation of the Time Variable

To justify the separation of the time variable, we examine the heat equation for a more practical treatment of the justification. One can adjust this argument to validate the separation of the time variable in a more general context, but for our purposes it is not necessary. The heat conduction is modeled by:

$$\begin{cases} u_t = k\Delta u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(\boldsymbol{x}, 0) = f(\boldsymbol{x}) & \text{in } \Omega. \end{cases} \tag{5}$$

A similar model can be formed for the Neumann and Robin conditions.

As we know, the separation of variables leads to

$$u(\boldsymbol{x}, t) = \sum_{n=1}^{\infty} A_n \mathrm{e}^{-\lambda_n k t} \varphi_n(\boldsymbol{x}).$$

But we will prove this by assuming certain differentiability on u and $u \in H_0^1(\Omega)$ i.e., $\|u\|^2 + \|\nabla u\|^2 < \infty$, instead of the separation of variables. For the details, see [4, Chap. 7].

Proof. Let $u(\boldsymbol{x}, t)$ be a solution to (5). For each t, $u(\cdot, t) \in L^2(\Omega)$ and satisfies the Dirichlet condition. Since $\{\varphi_n\}_{n\in\mathbb{N}}$ is complete in $L^2(\Omega)$, we can expand $u(\cdot, t)$ as

$$u(\boldsymbol{x},t) = \sum_{n=1}^{\infty} c_n(t)\varphi_n(\boldsymbol{x}). \tag{6}$$

Note that $c_n(t)$ is unknown at this point. Plugging (6) into (5) (term-by-term differentiation of the series is assumed; derivatives taken in this manner fall into the "weak derivative" sense), we get

$$\sum_{n=1}^{\infty} c_n'(t)\varphi_n(\boldsymbol{x}) = k\sum_{n=1}^{\infty} c_n(t)\Delta\varphi_n(\boldsymbol{x}) = \sum_{n=1}^{\infty} -k\lambda_n c_n(t)\varphi_n(\boldsymbol{x}).$$

Thanks to the completeness of $\{\varphi_n\}_{n\in\mathbb{N}}$, we can deduce $c_n'(t) = -k\lambda_n c_n(t)$. Hence we have

$$c_n(t) = A_n \mathrm{e}^{-k\lambda_n t}, \quad A_n : \text{ arbitrary const.} \tag{7}$$

□

2.2 Separation of the Spatial Variables

For simplicity, we will only justify separation of the spatial variables for the 2D rectangle $\Omega = I_1 \times I_2 \subset \mathbb{R}^2$ with I_1, $I_2 \subset \mathbb{R}$ as shown in Figure 1.

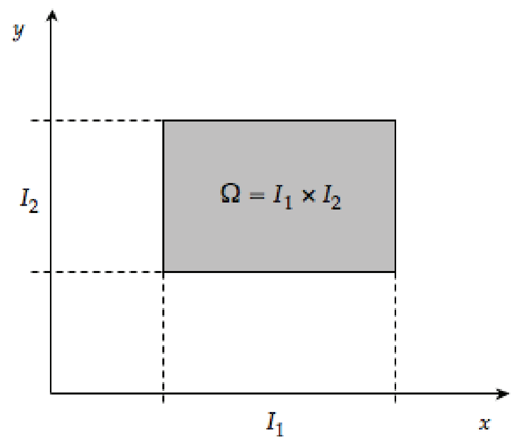

Figure 1: $\Omega \subset \mathbb{R}^2$, with I_1, $I_2 \subset \mathbb{R}$.

With $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} = \partial_{xx} + \partial_{yy}$, let $-\partial_{xx}\varphi_n^1(x) = \alpha_n\varphi_n^1(x)$ and $-\partial_{yy}\varphi_m^2(y) = \beta_n\varphi_n^2(y)$ with appropriate boundary conditions for the (D), (N), or (R) cases.

Theorem 2.1. *The set of products $\{\varphi_n^1(x)\varphi_m^2(y)\}_{(n,m)\in\mathbb{N}^2}$ is a* **complete** *set of eigenfunctions for $-\Delta$ in Ω with the given boundary conditions.*

Proof. We have

$$\begin{aligned} -\Delta\left(\varphi_n^1(x)\varphi_m^2(y)\right) &= \left(-\partial_{xx}\varphi_n^1(x)\right)\varphi_m^2(y) + \varphi_n^1(x)\left(-\partial_{yy}\varphi_m^2(y)\right) \\ &= \left(\alpha_n + \beta_m\right)\varphi_n^1(x)\varphi_m^2(y). \end{aligned}$$

Hence, $\{(\alpha_n + \beta_m, \varphi_n^1(x)\varphi_m^2(y))\}_{(n,m)\in\mathbb{N}^2}$ are eigenpairs, and we clearly see that $\langle\varphi_n^1\varphi_m^2, \varphi_{n'}^1\varphi_{m'}^2\rangle = \text{const}\cdot\delta_{nn'}\delta_{mm'}$.

We know that $\{\varphi_n^j\}_{n\in\mathbb{N}}$ is complete in $L^2(I_j)$, $j = 1, 2$, so the question is whether this 2D eigenvalue problem has eigenfunctions other than the product form $\{\varphi_n^1\varphi_m^2\}$.

Suppose u is such an eigenfunction (non-product form) so that u satisfies $-\Delta u = \lambda u$, $u|_{\partial\Omega} = 0$.

(i) Suppose $\lambda \notin \{\alpha_n + \beta_m\}_{(n,m)\in\mathbb{N}^2}$. Then $u \perp \varphi_n^1\varphi_m^2$ via the Fundamental Theorem of orthogonality of Laplacian eigenfunctions (see Lecture 4). Thus, we have

$$0 = \langle u, \varphi_n^1\varphi_m^2\rangle = \int_{I_2}\left(\int_{I_1} u(x,y)\varphi_n^1(x)\,\mathrm{d}x\right)\varphi_m^2(y)\,\mathrm{d}y.$$

By the completeness of $\{\varphi_m^2(y)\}_{m\in\mathbb{N}}$ in $L^2(I_2)$, we must have $\int_{I_1} u(x,y)\varphi_n^1(x)\,\mathrm{d}x = 0$ for $y \in I_2$, a.e.. Similarly by the completeness of $\{\varphi_n^1(x)\}_{n\in\mathbb{N}}$ in $L^2(I_1)$, $u(x,y) = 0$, a.e..

So such a u is not an eigenfunction, which means that we must have $\lambda \in \{\alpha_n + \beta_m\}_{(n,m)\in\mathbb{N}^2}$ i.e., $\lambda = \alpha_n + \beta_m$ for some $(n,m) \in \mathbb{N}^2$. It is possible that $\alpha_n + \beta_m$ may have multipilicity greater than 1. So let $\{\varphi_n^1\varphi_m^2\}_{(n,m)\in\Lambda}$ be the corresponding eigenfunctions where $\Lambda \subsetneq \mathbb{N}^2$. Then set

$$r(x,y) \triangleq u(x,y) - \sum_{(n,m)\in\Lambda} c_{nm}\varphi_n^1(x)\varphi_m^2(y), \text{ with } c_{nm} = \frac{\langle u, \varphi_n^1\varphi_m^2\rangle}{\|\varphi_n^1\varphi_m^2\|^2}.$$

Now $r(x,y) \perp \{\varphi_n^1 \varphi_m^2\}_{(n,m)\in\Lambda}$ by construction. So, $r(x,y) \equiv 0$ (a.e.) via a similar argument as before. Thus,

$$u(x,y) = \sum_{(n,m)\in\Lambda} c_{nm}\varphi_n^1(x)\varphi_m^2(y) \qquad \text{a.e.}$$

Therefore, $\{\varphi_n^1 \varphi_m^2\}_{(n,m)\in\mathbb{N}^2}$ is complete in $L^2(\Omega)$. □

References

[1] W. A. STRAUSS, *Partial Differential Equations: An Introduction*, Brooks/Cole Publishing Company, 1992.

[2] R. YOUNG, *An Introduction to Hilbert Space*, Cambridge Univ. Press, 1988.

[3] G. B. FOLLAND, *Fourier Analysis and Its Applications*, Brooks/Cole Publishing Company, 1992.

[4] L. C. EVANS, *Partial Differential Equations*, AMS, 1998.