Contemporary Mathematics

Metric graph theory and geometry: a survey

Hans-Jürgen Bandelt and Victor Chepoi

Abstract. The article surveys structural characterizations of several graph classes defined by distance properties, which have in part a general algebraic flavor and can be interpreted as subdirect decomposition. The graphs we feature in the first place are the median graphs and their various kinds of generalizations, e.g., weakly modular graphs, or fiber-complemented graphs, or l_1-graphs. Several kinds of l_1-graphs admit natural geometric realizations as polyhedral complexes. Particular instances of these graphs also occur in other geometric contexts, for example, as dual polar graphs, basis graphs of (even Δ-)matroids, tope graphs, lopsided sets, or plane graphs with vertex degrees and face sizes bounded from below. Several other classes of graphs, e.g., Helly graphs (as injective objects), or bridged graphs (generalizing chordal graphs), or tree-like graphs such as distance-hereditary graphs occur in the investigation of graphs satisfying some basic properties of the distance function, such as the Helly property for balls, or the convexity of balls or of the neighborhoods of convex sets, etc. Operators between graphs or complexes relate some of the graph classes reported in this survey.

0. Introduction

Discrete geometry involves finite configurations of points, lines, planes or other geometric objects, with the emphasis on combinatorial properties (Matoušek [**139**]). This leads to a number of intriguing problems - indeed, "the subject of combinatorics is devoted to the study of structures on a finite set; many of the most interesting of these structures arise from elimination of continuous parameters in problems from other mathematical disciplines" (Borovik et al. [**58**]). Pure graph theory (Diestel [**96**]) may then offer the appropriate language, but some extra structure is needed: in order to express the combinatorial features of incidence geometries and certain buildings, geodesic (graph) distance plays a key role [157, §5], and the graphs under investigation should possess rather strong distance properties.

The objects of departure are very simple models: n-cubes (with the l_1-metric), n-dimensional grids endowed with the l_∞-metric, trees, and certain plane graphs. The structural theories initially developed for the classes of median graphs and Helly graphs, respectively, serve as prototypes for more general and complex theories. Helly property, geodesic convexity, gated sets, isometric embedding, and

1991 *Mathematics Subject Classification.* Primary 05C12, 52-XX; Secondary 51KXX.

decomposition (mimicking subdirect factorization) are then recurrent themes. The choice of themes, of course, reflects our personal research interests in this field, with the emphasis on structure theory. Certainly, the books and papers by Boltyanskii and Soltan [**57**], Dress [**99**], Isbell [**127**], Mulder [**142**], and Soltan et al. [**163**] had significant influence on our work over the past 20 years.

It is clear that a short survey cannot cover all aspects of metric graph theory that are related to geometric questions. We have thus to be short about or even to dismiss entirely the following topics. A classical subject is that of distance-regular graphs, which are intimately related with combinatorial designs and finite geometries; the book by Brouwer et al. [**68**] gives a detailed account of this. There exists a vast literature on graphs that are hypermetric (or l_1-embeddable, in particular) and can be realized as 1-skeletons of certain polytopes (only some of which we will mention below); see the book by Deza and Laurent [**94**]. The study of low-distortion embeddings of graphs and finite metric spaces into l_2- or l_1-spaces, with numerous applications in the design of approximation algorithms, was initiated by Linial et al. [**135**] and is the subject of several surveys or book chapters, see for example [**139**].

1. Basic notions

All graphs $G = (V, E)$ occurring here are simple, connected, without loops or multiple edges, but not necessarily finite.

Convex and isometric subgraphs. The *distance* $d(u, v) := d_G(u, v)$ between two vertices u and v is the length of a shortest (u, v)-path, and the *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v)–paths, that is, of all vertices (metrically) *between* u and v:

$$I(u, v) := \{x \in V : d(u, x) + d(x, v) = d(u, v)\}.$$

An induced subgraph of G (or the corresponding vertex set A) is called *convex* if it includes the interval of G between any of its vertices. An induced subgraph H of G is *isometric* if the distance between any pair of vertices in H is the same as that in G. In particular, convex subgraphs are isometric.

Balls. The *ball* (or disk) $N_r(x)$ of center x and radius $r \geq 0$ consists of all vertices of G at distance at most r from x. In particular, the unit ball $N_1(x)$ comprises x and the neighborhood $N(x)$. The ball $N_r(S)$ centered at a convex set S is the union of all balls $N_r(x)$ with centers x from S. The smallest number r for which some ball (centered at a vertex) with radius r covers the whole graph G is then called the *radius* of G. The radius of G is at least one half of the *diameter* of G, the largest distance in G. Any two vertices at diameter distance are said to form a *diametrical pair.*

Metric triangles and rectangles. Three vertices v_1, v_2, and v_3 form a *metric triangle* $v_1v_2v_3$ if the intervals $I(v_1, v_2), I(v_2, v_3)$ and $I(v_3, v_1)$ pairwise intersect only in the common end vertices. If $d(v_1, v_2) = d(v_2, v_3) = d(v_3, v_1) = k$, then this metric triangle is called *equilateral* of *size* k. For example, the three vertices of degree 2 in the 6-vertex sun (which is the graph obtained by gluing three triangles to the three edges of another triangle) form a metric triangle of size 2. Four vertices v_1, v_2, v_3, v_4 form a *metric rectangle* if $v_1, v_3 \in I(v_2, v_4)$ and $v_2, v_4 \in I(v_1, v_3)$. Notice that in a metric rectangle opposite sides have the same length: $d(v_1, v_2) = d(v_3, v_4)$ and $d(v_1, v_4) = d(v_2, v_3)$.

Gated sets and Helly property. A subset W of V or the subgraph H of G induced by W is called *gated* (in G) if for every vertex x outside H there exists a vertex x' (the *gate* of x) in H such that each vertex y of H is connected with x by a shortest path passing through the gate x'; cf. [**102**]. Gated sets emerged first in location theory [**116**] and independently in the theory of buildings [**167**]. Gated sets enjoy the finite *Helly property,* that is, every finite family of gated sets that pairwise intersect has a nonempty intersection. Since the intersection of gated subgraphs is gated, for every subset $S \subseteq V$ there exists the smallest gated set $\langle\langle S\rangle\rangle$ containing S, referred to as the *gated hull* of S. A graph G is a *gated amalgam* of two graphs G_1 and G_2 if G_1 and G_2 constitute two intersecting gated subgraphs of G whose union is all of G.

Isometric embeddings and retractions. A graph $G = (V, E)$ is *isometrically embeddable* into a graph $H = (W, F)$ if there exists a mapping $\varphi : V \to W$ such that $d_H(\varphi(u), \varphi(v)) = d_G(u, v)$ for all vertices $u, v \in V$. A *retraction* φ of H is an idempotent nonexpansive mapping of H into itself, that is, $\varphi^2 = \varphi : W \to W$ with $d(\varphi(x), \varphi(y)) \leq d(x, y)$ for all $x, y \in W$. The subgraph of H induced by the image of H under φ is referred to as a *retract* of H. Retracts are isometric subgraphs, but the converse is not true in general: the 6-cycle C_6 is an isometric subgraph but not a retract of the 3-cube H_3. Note that nonexpansive mappings are just the morphisms in the category of the so-called reflexive graphs (i.e., the graphs with loops at all vertices); cf. [**39, 144**].

Cartesian products. The *Cartesian product* $G = G_1\Box \ldots \Box G_n$ has the n-tuples $(x_1, \ldots, x_n)$ as its vertices (with vertex x_i from G_i) and an edge between two vertices $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ if and only if, for some i, the vertices x_i and y_i are adjacent in G_i, and $x_j = y_j$ for the remaining $j \neq i$. Obviously, $d_G(u, v) = \sum_{i=1}^{n} d_{G_i}(u_i, v_i)$ for any two vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ of G. Infinitary Cartesian multiplication does not yield connected graphs, and therefore one resorts to connected components (referred to as "weak Cartesian products"). In particular, *hypercubes* are the weak Cartesian powers of K_2, the complete graph on two vertices. Then the *n-cube* H_n, the Cartesian product of n copies of K_2, encodes all subsets of an n-set $X = \{1, 2, \ldots, n\}$, with two subsets A, B being adjacent if and only if the symmetric difference $A\Delta B$ is a singleton; in other words, H_n is the underlying graph of the Boolean algebra 2^X, that is, the unoriented version of the (Hasse) diagram of the Boolean lattice, which is also referred to as the covering graph of 2^X (see [164, p.189]). More generally, a weak Cartesian product of complete graphs is a *Hamming graph;* in particular, $H_{m_1,\ldots,m_s}$ denotes the Cartesian product of the complete graphs $K_{m_1}, \ldots, K_{m_s}$.

Half-cubes, Johnson graphs, and hyperoctahedra. Some further graphs occur as hosts for isometric embeddings of graphs. The *half-cube* $\frac{1}{2}H_n$ is the graph whose vertex set is the collection of all subsets of X which have the same cardinality modulo 2, and two vertices A, B are adjacent in $\frac{1}{2}H_n$ exactly when $A\Delta B$ is a doubleton. The *Johnson graph* $J_{n,k}$ is the (isometric) subgraph of $\frac{1}{2}H_n$ induced by the family of all subsets of cardinality k. The *m-octahedron* (alias hyperoctahedron or Cocktail-party graph) $K_{m\times 2}$ is the complete multipartite graph with m parts, each of size 2. Thus $K_{m\times 2}$ is obtained from the complete graph K_{2m} by deleting a perfect matching. It is no accident that these and other distance-regular graphs (such as dual polar graphs, the Schläfli and Gosset graphs; see Sections 6.3 and 8.2 below) occur here, because they are all connected with certain finite geometries.

Simplicial and cubical complexes. A number of combinatorial and geometric structures are related with graphs. An abstract *simplicial complex* $\mathcal{X}$ is a collection of sets (called *simplices*) such that $\sigma \in \mathcal{X}$ and $\sigma' \subseteq \sigma$ implies $\sigma' \in \mathcal{X}$. A *cubical complex* $\mathcal{C}$ is a set of (graph) cubes of any dimensions which is closed under taking subcubes and nonempty intersections. Simplices or cubes of the respective complex are called *faces*. For a complex $\mathcal{K}$ denote by $V(\mathcal{K})$ and $E(\mathcal{K})$ the *vertex set* and the *edge set* of $\mathcal{K}$, namely, the set of all 0-dimensional and 1-dimensional faces of $\mathcal{K}$. The pair $(V(\mathcal{K}), E(\mathcal{K}))$ is called the *(underlying) graph* or the *1-skeleton* of $\mathcal{K}$ and is denoted by $G(\mathcal{K})$. Conversely, for a graph G one can derive a simplicial complex $\mathcal{X}(G)$ and a cubical complex $\mathcal{C}(G)$ by taking all complete subgraphs (simplices) or all induced subhypercubes, respectively, as faces of the complexes. A simplicial complex $\mathcal{X}$ is a *flag complex* (or a *clique complex*) if any set of vertices is included in a face of $\mathcal{X}$ whenever each pair of its vertices is contained in a face of $\mathcal{X}$. (In the theory of hypergraphs this condition is called conformality.) A flag complex can therefore be recovered by its underlying graph $G(\mathcal{X})$: the complete subgraphs of $G(\mathcal{X})$ are exactly the simplices of $\mathcal{X}$.

The *geometric realization* $|\mathcal{K}|$ of a simplicial or cubical complex $\mathcal{K}$ is the polyhedral complex obtained by replacing every face σ by a "solid" regular simplex or "solid" unit cube $|\sigma|$ of the same dimension such that realization commutes with intersection, that is, $|\sigma'| \cap |\sigma''| = |\sigma' \cap \sigma''|$ for any two faces σ' and σ''. Then $|\mathcal{K}| = \bigcup\{|\sigma| : \sigma \in \mathcal{K}\}$. Analogously, for a plane graph G (that is, a planar graph embedded in the plane such that no edges cross) one can define a polygonal complex $|G|$ by replacing each inner face with k sides of G by a regular k-gon with side length 1 in the Euclidean plane. $\mathcal{K}$ is called *simply connected* if it is connected and if every continuous mapping of the 1-dimensional sphere S^1 into $|\mathcal{K}|$ can be extended to a continuous mapping of the disk D^2 with boundary S^1 into $|\mathcal{K}|$.

Intrinsic metrics. The polyhedron $|\mathcal{K}|$ of a cubical complex $\mathcal{K}$ can be endowed with an intrinsic l_p-metric in the following way (similarly, one can define the intrinsic l_2-metric on the polyhedron of a simplicial or polygonal complex). Assume that inside every maximal face $|\sigma|$ of $|\mathcal{K}|$ the distance is measured by an l_1, l_2, or l_∞ metric. The *intrinsic* l_1-, l_2-, or l_∞*-metric* of $|\mathcal{K}|$ is defined by letting the distance between two points $x, y \in |\mathcal{K}|$ be equal to the greatest lower bound on the length of the paths joining them; here a *path* in $|\mathcal{K}|$ from x to y is a sequence $x = x_0, x_1, \ldots, x_m = y$ of points in $|\mathcal{K}|$ such that for each $i = 0, \ldots, m-1$ there exists a face $|F_i|$ containing x_i and x_{i+1}, and the *length* of the path equals $\sum_{i=0}^{m-1} d(x_i, x_{i+1})$, where $d(x_i, x_{i+1})$ is computed inside $|F_i|$ according to the respective metric. The resulting metric space is *geodesic*, i.e., every pair of points in $|\mathcal{K}|$ can be joined by a geodesic; see [**67**]. A *geodesic* joining two points x and y from $|\mathcal{K}|$ is a map γ from the segment $[a, b]$ of length $|a-b| = d(x, y)$ to $|\mathcal{K}|$ such that $\gamma(a) = x, \gamma(b) = y$, and $d(\gamma(s), \gamma(t)) = |s-t|$ for all $s, t \in [a, b]$. A complete metric space is geodesic exactly when it is *Menger-convex*, that is, for any two distinct points x and y there exists another point z between x and y.

Hyperconvexity. A metric space (X, d) is called *hyperconvex* (or *injective*) [**3, 126**] provided that any family of closed balls $N_{r_i}(x_i)$ with centers x_i and radii r_i, $i \in I$, satisfying $d(x_i, x_j) \leq r_i + r_j$ for all $i, j \in I$ has a nonempty intersection, that is, (X, d) is a Menger-convex space such that the closed balls have the (infinite) Helly property. It is well known that (X, d) is hyperconvex if and only if it is an absolute retract, that is, (X, d) is a retract of every metric space into which it embeds isometrically. For every metric space (X, d) there exists the smallest injective space extending (X, d), referred to as the *injective hull* [**126**] or *tight span* [**99, 101**] of (X, d), which has most recently appeared in the context of tropical geometry [**91**].

2. Median graphs

Median graphs and related median structures (median algebras and median complexes) have many nice properties and admit numerous characterizations. These structures have been investigated in several contexts by quite a number of authors for more than half a century. Median structures are still being rediscovered in various disguises. We present here only a brief account of the characteristic properties of median structures; for more detailed information, the interested reader can consult the books [**110, 125, 142, 170**] and the paper [**29**].

A graph G is called *median* if the interval intersection $I(x, y) \cap I(y, z) \cap I(z, x)$ is a singleton, comprising the *median* $m(x, y, z)$ for each triplet x, y, z of vertices. Basic examples of median graphs are trees (which are successive point amalgams of K_2) and hypercubes (which are Cartesian powers of K_2). The intervals of arbitrary median graphs, being themselves median subgraphs, are precisely the covering graphs of finite distributive lattices. All median structures are intimately related to hypercubes: median graphs are isometric subgraphs of hypercubes; in fact, they are even retracts of those hypercubes into which they embed isometrically.

Theorem 2.1. [**8, 127, 169**] *Median graphs are exactly the retracts of hypercubes. Every median graph with more than two vertices is either a Cartesian product or a gated amalgam of proper median subgraphs.*

In particular, every finite median graph G can be obtained by successive applications of gated amalgamations from hypercubes. A related construction of median graphs via convex expansions is given in [**141, 142**]; this characterization leads to efficient algorithms for recognizing median graphs [**124, 125**]. Gated sets play a fundamental role in the investigation of median graphs (and more generally, metric median spaces). For example, all convex sets of median graphs are gated and as such they satisfy the finite Helly property. Notice also that for every edge ab of a median graph $G = (V, E)$ the sets

$$a/b := \{v \in V : d(v, a) < d(v, b)\},$$

$$a \backslash b := \{v \in V : d(v, b) < d(v, a)\}$$

induce complementary gated subgraphs (in fact, this property characterizes median graphs). Finally note that the median graphs are exactly the graphs in which all intervals are gated; cf. [**125**].

There is a canonical construction of median graphs departing from arbitrary graphs: namely, for a graph G the *simplex graph* $\kappa(G)$ has the simplices (the complete subgraphs) of G as its vertices and pairs of (comparable) simplices differing in exactly one vertex as its edges. Since $\mathcal{X}(G)$ is a flag complex, $\kappa(G)$ is a median graph [**48**].

While the dimension n of the smallest hypercube into which the median graph G embeds is easy to determine, the problem of determining the smallest number of tree factors necessary for an embedding into a Cartesian product of trees is hard.

PROPOSITION 2.2. [**44**] *The simplex graph $\kappa(G)$ of a graph G can be isometrically embedded into the Cartesian product of at most k trees if and only if the chromatic number of G is at most k.*

In particular, it is NP-complete to decide whether a 3-cube-free median graph embeds into the Cartesian product of three trees [**44**]. In contrast, the graphs isometrically embeddable into the product of two trees can be recognized in polynomial time: one can show that they are exactly the 3-cube-free median graphs without odd bipartite wheels (an *odd bipartite wheel* is the simplex graph of a k-cycle with $k > 3$ odd).

The fact that median graphs are built up from hypercubes by (successive) gated amalgamation is also reflected in the behavior of nonexpansive mappings. This is made precise in the following fixed-cube theorem, which can be seen as a kind of discrete analogue of classical fixed-point theorems.

THEOREM 2.3. [**43**] *Every nonexpansive map f of a finite median graph G into itself has a fixed k-cube ($k \geq 0$), i.e., a k-cube of G which is mapped isomorphically onto itself by f.*

A polynomial time algorithm for computing minimal fixed cubes of nonexpansive mappings of a hypercube into itself was presented in [**110**].

2.1. Median algebra. Median graphs have a remarkable algebraic structure, which is induced by the ternary operation on the vertex set that assigns to each triplet of vertices the median vertex. This operation can be studied in an axiomatic way as follows: an *abstract median operator* on a (not necessarily finite) set X is a function $m : X^3 \to X$ satisfying the following four conditions, where the short-hand $(uvw) := m(u, v, w)$ is used:

(M1) $(uvv) = v$ (right absorption);

(M2) $(uvw) = (uwv)$ (right symmetry);

(M3) $(uvw) = (vuw)$ (left symmetry);

(M4) $(uv(uwx)) = (u(uvw)x)$ (transitivity).

This system of identities is equivalent to the one from [170, p.8]. The resulting pair (X, m) is called a *median algebra*. All median algebras are subdirect products of the two-element algebra $\{0, 1\}$. Median algebras have a rich theory going back to Birkhoff, Kiss, Sholander and Isbell; for an extensive survey, see [**29, 127, 170**]. The median operator of a median graph trivially satisfies the first three axioms, and some computation shows that the fourth axiom also holds. This axiom expresses associativity of the derived binary operation $v, w \mapsto (uvw) := v \wedge w$, or equally, transitivity of the relation $v \leq w \Leftrightarrow (uvw) = v$ for fixed u. In fact, it was one of the starting points for the development of the theory of median algebras that under a few axioms (equivalent to (M1)-(M4)) certain semilattices $(X, \wedge)$ arise, which were then called *median semilattices*. They are characterized by the property that all principal ideals $\{x \in X : x \leq u\}$ are distributive lattices and three elements have an upper bound whenever each pair of them does. Therefore the median operator can be retrieved from the median semilattice as

$$(uvw) = (u \wedge v) \vee (u \wedge w) \vee (v \wedge w),$$

just as in the classical case of distributive lattices.

Axioms (M1)-(M4) are evidently independent, but one can reduce the number of axioms by integrating (M2) into a twisted variant of (M4). For instance, if one substitutes in

(M5) $(uv(uwx)) = (ux(uvw))$

x by either (i) (uvw), or (ii) w, or (iii) v, or substitutes (iv) x, v, w by v, w, v, respectively, then (by employing right absorption wherever possible) one derives right symmetry by applying (i)-(iv) in this order: $(uvw) = (uv(uw(uvw))) = (uv(uvw)) = (uv(uwv)) = (uwv)$. In a similar fashion, one can dispense with left symmetry by introducing a further twist into (M5):

(M6) $(uv(uwx)) = (ux(wuv))$.

Then one arrives at the most compact axiom system for median algebras, (M1) plus (M6) [**133**]. There are many other ways to describe median algebras by identities. For example, one system comprises (M1),(M2),(M3), and the axiom

(M7) $((vwx)u(uwx)) = (uwx)$,

which then replaces (M4); see [127, 6.6].

There is a bijection between discrete median algebras and median graphs: discreteness refers to the non-existence of bounded infinite chains in the ternary algebra; then with any discrete median algebra (X, m) one can associate a connected graph by taking X as the vertex set and the pairs xy such that $m(x, y, z) \in \{x, y\}$ for all $z \in X$ as edges.

PROPOSITION 2.4. [**6**] *Median graphs and discrete median algebras constitute the same objects.*

A subset Y of a median algebra is called *median-stable* (or a subalgebra) if $m(x, y, z) \in Y$ for all $x, y, z \in Y$. For any subset M there exists the smallest median-stable set containing M (the *median closure* of M). It was shown in [**127, 170**] that the median closure of a finite set is finite and therefore constitutes a median graph in its own right. In other words, the free median algebra with finitely many generators is finite; for more information about the free median algebras, see [**45**]. Every median subalgebra of a hypercube generated by a subset X (not necessarily freely) is determined by the splits (i.e., bipartitions) of X induced by the projections to the K_2 factors. Analyzing these splits in pairs then provides the necessary information that enters into simple counting formulae for the total number of k-cubes [**103**] in those median subalgebras, which in turn lead to Euler-type formulae via binomial inversion [**37**].

Median-stable subsets of Boolean algebras naturally arise as solution sets of certain Boolean expressions, namely of 2-SAT instances [**156**]; for example, the 2-SAT instance $(\overline{x_1} \vee \overline{x_3})(\overline{x_1} \vee \overline{x_4})(\overline{x_2} \vee \overline{x_3})(x_2 \vee \overline{x_4})(\overline{x_3} \vee \overline{x_4})$ is the median-stable set $\{(0,0,0,0), (1,0,0,0), (1,1,0,0), (0,1,0,0), (0,0,1,0), (0,1,0,1)\}$. This fact, which can also be found in [**89**] and be derived from [**127**] and [**143**], was the main tool (along with Theorems 2.1 and 2.3) in investigating the stability of certain logical networks [**110**]:

PROPOSITION 2.5. [**156**] *Median-stable subsets of Boolean algebras are exactly the solution sets of instances of the 2-SAT problem.*

Indeed, a median-stable subset S of 2^X is uniquely determined by its 2-fold projections, that is, the projections to pairs of factors. Since each 2-fold projection can trivially be realized as the solution set of some 2-SAT instance in the corresponding 2 variables, the conjunction of all those instances has S as its solution set. Conversely, assume that u, v, w are solutions of a 2-SAT instance. Then each of these solutions assigns value 1 to at least one letter from each clause (a disjunction of two literals, i.e., variables or their negations), whence there exists one letter from each clause for which two solutions assign value 1. Therefore the majority assignment generated from u, v, w also constitutes a solution.

2.2. Geometry of median graphs. The cubical complex $\mathcal{C}(G)$ associated with a median graph G is called a *median cubical complex* and its geometric realization $|G| := |\mathcal{C}(G)|$ is then referred to as a *median polyhedral complex* [**170**]. Such polyhedral complexes endowed with any of the intrinsic l_1, l_2, or l_∞-metrics constitute geodesic (isometric) subspaces of normed spaces carrying the same type of norm. For example, $|G|$ equipped with the l_1-metric is a median space (i.e., every triplet of points has a unique median) and therefore is an l_1-subspace [**170**]. If $|G|$ carries the intrinsic l_∞-metric instead, then the resulting metric space is injective [**138, 171**]. Finally, if we impose the intrinsic l_2-metric on $|G|$, we obtain a metric space with global non-positive curvature. To be more precise, some further notions have to be introduced. First, note that all three resulting metric spaces are geodesic.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of Δ) and a geodesic between each pair of vertices (the edges of Δ). A *comparison triangle* for $\Delta(x_1, x_2, x_3)$ is a triangle $\Delta(x_1', x_2', x_3')$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(x_i', x_j') = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space (X, d) is defined to be a *CAT(0) space* [**118**] if all geodesic triangles $\Delta(x_1, x_2, x_3)$ of X satisfy the comparison axiom of Cartan–Alexandrov–Toponogov:

If y is a point on the side of $\Delta(x_1, x_2, x_3)$ with vertices x_1 and x_2 and y' is the unique point on the line segment $[x_1', x_2']$ of the comparison triangle $\Delta(x_1', x_2', x_3')$ such that $d_{\mathbb{E}^2}(x_i', y') = d(x_i, y)$ for $i = 1, 2$, then $d(x_3, y) \leq d_{\mathbb{E}^2}(x_3', y')$.

This simple axiom turned out to be very powerful, because CAT(0) spaces can be characterized in several different natural ways (for a full account of this theory consult the book [**67**]). CAT(0) spaces play a vital role in modern combinatorial group theory, where various versions of hyperbolicity are related to group-theoretic properties [**107, 113, 114, 118**]; many arguments in this area have a strong metric graph-theoretic flavor. A geodesic metric space (X, d) is CAT(0) if and only if any two points of this space can be joined by a unique geodesic. CAT(0) is also equivalent to convexity of the function $f : [0, 1] \to X$ given by $f(t) = d(\alpha(t), \beta(t))$, for any geodesics α and β (which is further equivalent to convexity of the neighborhoods of convex sets). This implies that CAT(0) spaces are contractible. Several classes of CAT(0) complexes (mostly 2-dimensional) can be characterized combinatorially, and the characterization of cubical CAT(0) complexes given by M. Gromov is especially nice:

THEOREM 2.6. [**118**] *A cubical polyhedral complex $|\mathcal{C}|$ with the intrinsic l_2-metric is CAT(0) if and only if $|\mathcal{C}|$ is simply connected and satisfies the following condition: whenever three $(k+2)$-cubes of $|\mathcal{C}|$ share a common k-cube and pairwise share common $(k+1)$-cubes, they are contained in a $(k+3)$–cube of $\mathcal{C}$.*

In some recent papers, CAT(0) cubical polyhedral complexes were called *cubings.* The following relationship holds between cubings and median polyhedral complexes.

THEOREM 2.7. [**84, 153**] *Median polyhedral complexes and cubings (both equipped with the l_2-metric) constitute the same objects.*

Examples of cubings and simplex graphs come up in the context of taxonomic models. The space $\mathcal{T}^0_{n-1}$ of all ultrametrics $\delta \leq \mathbf{1}$ on a finite set $\{1,\ldots,n-1\}$ $(n \geq 3)$ is a cubing [**53**]. The corresponding cone of all ultrametrics has been dubbed the "Bergman fan" of the graphical matroid of the complete graph K_{n-1} [**2**]. In this context, an ultrametric δ is defined as a particular pseudometric (where δ may be zero for a pair of distinct points), that is, δ is a nonnegative symmetric function on $\{1,\ldots,n-1\}^2$ with zero diagonal satisfying the ultrametric inequality $\delta(i,j) \leq \max\{\delta(i,k),\delta(j,k)\}$ for all $1 \leq i,j,k \leq n-1$. It is well known that ultrametrics on $\{1,\ldots,n-1\}$ correspond to rooted trees with $n-1$ leaves (end vertices) that are labeled from 1 to $n-1$, where all edges have nonnegative lengths such that the path lengths from the root to the leaves are all equal. These rooted trees can be reorganized by subtracting the largest number from the lengths of all terminal links (maintaining nonnegativity of lengths) and taking this number as the length of a new edge joining the old root with a new root vertex 0. When one now adds arbitrary nonnegative values to the length of the terminal edges incident with the leaves $1,\ldots,n-1$, then one arrives at the usual tree representation of a tree (pseudo)metric d on $\{0,1,\ldots,n-1\}$ (which is characterized by the 4-point condition; see Section 5 below); cf. [**11**]. The space $\mathcal{T}_n$ of all tree metrics $d \leq \mathbf{1}$ on $\{0,1,\ldots,n-1\}$, the tropical Grassmannian of lines $\mathcal{G}_{2,n}$ [**160**], then includes the space $\mathcal{T}^0_{n-1}$ as a factor. The combinatorial structure of $\mathcal{T}_n$ (or $\mathcal{T}^0_{n-1}$ accordingly) is fully defined by the finite median polyhedral complex consisting of the unit cubes of $\mathcal{T}_n$ (or $\mathcal{T}^0_{n-1}$) sharing the origin of all orthants, which is derived from a simplex graph, as will be detailed next.

The space $\mathcal{T}_n$ is composed in the following way. For every trivalent tree T (i.e., a tree in which all interior vertices have degree 3) with n leaves labeled by the elements of the set $X = \{0,1,\ldots,n-1\}$ and with edges $e_1,\ldots,e_{2n-3}$ of lengths $0 < l_1,\ldots,l_{2n-3} \leq 1$, the vector $(l_1,\ldots,l_{2n-3})$ specifies a point in the positive unit orthant $(0,1]^{2n-3}$. To each other point from $(0,1]^{2n-3}$, one can associate a unique tree of this kind which has the same topology as T but different edge lengths, specified by the coordinates of that point. Points on the boundary of the unit orthant correspond to trees which are obtained from T by contracting some edges of T: these contracted trees are referred to as the *X-trees* [**158**]. The cubing $\mathcal{T}_n$ is obtained by taking one $(2n-3)$-dimensional unit orthant for each trivalent tree and gluing these unit orthants together along their common faces.

The combinatorial structure of $\mathcal{T}_n$ is fully defined by the finite median polyhedral complex consisting of the unit cubes of $\mathcal{T}_n$ sharing the origin of all orthants. This complex is determined by the compatibility graph S_n of the splits of X via the simplex graph and cubical complex operators. S_n has the splits (i.e., bipartitions) $\{A, B\}$ of the set X as its vertices such that two vertices are connected by an edge in S_n if and only if the corresponding splits are *compatible,* i.e., some part of one split is contained in one part of the other split. The simplices of S_n then correspond to the X-trees [47,48,70,158]; in particular, the maximal simplices of S_n are in one-to-one correspondence with the trivalent trees having their n leaves labeled by the elements of X. The flag complex $\mathcal{X}(\kappa(S_n))$, derived from the (median) simplex graph of S_n, is known under various names (e.g., the Grassmannian $\mathcal{G}'''_{2,n}$ [**160**]); it encodes the tree space $\mathcal{T}_n$, which in turn is the geometric realization $|\mathcal{C}(\kappa(S_n))|$ of the cubical complex $\mathcal{C}(\kappa(S_n))$.

In applications to phylogenetic trees [**158**], the spaces $\mathcal{T}^0_{n-1}$ and $\mathcal{T}_n$ would mainly serve as conceptual models for dealing with optimisation problems or stochastic questions [**53**]. Although the total spaces are too large for immediate visualization (when $n > 4$), some low-dimensional projections may very well turn up, e.g., when displaying the median spaces generated from competing phylogenetic trees or (non-majority) threshold consensus of a collection of trees [**13**].

Another particular instance of median cubical complexes is that of acyclic cubical complexes: a cubical complex $\mathcal{C}$ is *acyclic* if the incidence graph of vertices and maximal cubes does not contain any induced cycle of length > 4. A maximal cube Q of $\mathcal{C}$ is called *pendant* if there exists another cube R of $\mathcal{C}$ properly intersecting Q such that every proper intersection of Q with any other cube of $\mathcal{C}$ is included in $Q \cap R$. Eliminating Q and all its subcubes not contained in $Q \cap R$ from $\mathcal{C}$ then yields a subcomplex $\mathcal{C}'$. If $\mathcal{C}$ can eventually be transformed into a complex consisting of a single cube and all its subcubes by successively eliminating pendant cubes, then we say that $\mathcal{C}$ has a *cube elimination scheme.*

Theorem 2.8. [**16**] *The following statements are equivalent for a finite cubical complex $\mathcal{C}$:*

(a) *$\mathcal{C}$ is acyclic;*
(b) *$\mathcal{C}$ has a cube elimination scheme;*
(c) *the skeleton of $\mathcal{C}$ is a median graph not containing any convex bipartite wheels.*

Simplicial elimination schemes for chordal graphs translate into cube elimination schemes for the associated simplex graphs (and vice versa):

Corollary 1. *A graph G is chordal if and only if the cubical complex $\mathcal{C}(\kappa(G))$ derived from the simplex graph of G is acyclic.*

3. Helly graphs and absolute retracts

Helly graphs are the discrete analogues of hyperconvex spaces: namely, the (implicit) requirement that radii of balls are from the nonnegative reals is modified by replacing the reals by the integers. Then the discrete analogue of Menger-convexity is trivially satisfied for the metric spaces that are graphs. A graph G is thus called a *Helly graph* if the family of balls of G has the *Helly property*, that is, every collection of pairwise intersecting balls of G has a nonempty intersection. In perfect analogy with hyperconvexity, there is a close relationship between Helly graphs and absolute retracts. An object such as a graph is an absolute retract exactly when it is a retract of any super-object into which it embeds (isometrically, in this case). Then absolute retracts and Helly graphs are the same [**121**]. In particular, for any graph G there exists a smallest Helly graph comprising G as an isometric subgraph [**129, 148**].

In the preceding definition, we have referred to the metric definition of retraction as an idempotent nonexpansive mapping. There is, however, the purely graph-theoretic notion of retraction that requires idempotency and preservation of edges. When the graphs are assumed to have loops at all vertices, then the two notions coincide. This is why one would then speak of absolute retracts of reflexive graphs. If the graphs are ordinary, i.e., without loops ("irreflexive"), then idempotent edge-preserving mappings are particular retractions that preserve the chromatic number of the graph. Namely, for every retract F of this kind, any n-coloring of F can be extended to an n-coloring of the whole graph. For the corresponding concept of absolute retract, there is thus no loss of generality in assuming that the retractions (as idempotent nonexpansive mappings) are color-preserving. In the case of (connected) bipartite graphs, the coloring is unique (up to permutation of the two colors) and therefore does not have to be specified explicitly. A bipartite (irreflexive) graph G is thus a *bipartite absolute retract* (of bipartite graphs) if G is a retract of every bipartite (irreflexive) graph into which G embeds isometrically [**39, 149**].

3.1. Reflexive case: Helly graphs. In Helly graphs, the Helly property is manifest in several collections of subgraphs related to balls. Properties related to the Helly property are often preserved under the strong product rather than the Cartesian product; in the strong product (which is the ordinary product in the category of reflexive graphs) two distinct vertices are adjacent exactly when they are equal or adjacent at each coordinate. A *clique-Helly graph* is a graph in which the collection of cliques (maximal simplices) has the Helly property. A graph G is called *pseudo-modular* if any three pairwise intersecting balls of G have a nonempty intersection [**32**]. A vertex x of a graph G is *dominated* by another vertex y if the unit ball $N_1(y)$ includes $N_1(x)$. A finite graph G is *dismantlable* if its vertices can be linearly ordered, $v_1, v_2, \dots, v_n$, so that, for each v_i, $i > 1$, there is a neighbor v_j, $j < i$, of v_i dominating the vertex v_i in the subgraph G_i of G induced by the vertices $v_1, \dots, v_i$. In this case, every G_i is a retract of G. It is known [**145, 155**] that dismantlable graphs are precisely the "cop-win graphs", i.e., the graphs in which the cop has a winning strategy in the pursuit game of a cop and a robber.

THEOREM 3.1. *For a finite graph $G = (V, E)$, the following statements are equivalent:*

(a) *G is a Helly graph;*
(b) [**144**] *G is a retract of a strong product of paths;*
(c) [**38**] *G is a dismantlable clique-Helly graph;*
(d) [**39**] *G is a pseudo-modular graph in which the family of unit balls has the Helly property;*
(e) [**39**] *for every vertex v in a diametrical pair, there exists a vertex w dominating v and the vertex-deleted subgraph $G - \{v\}$ is an absolute retract;*
(f) [**98**] *every eccentricity function $e_\pi(x) = max\{\pi(v)d(x, v) : v \in V\}$ is unimodal (that is, every local minimum is a global minimum) for any weight function π from V to the nonnegative reals.*

In analogy with the fixed-cube property of median graphs, every automorphism φ of a Helly graph has a fixed simplex C (that is, $\varphi(C) = C$) [**155**]. For a comprehensive survey concerning fixed subgraphs, see [**151**].

The strong product is the l_∞ version of the Cartesian product. Thus, when we turn all k-cubes of the Cartesian product of k paths into simplices, then we have the corresponding strong product of k paths. More generally, a similar operator transforms median graphs into Helly graphs: let G^Δ be the graph having the same vertex set as G, where two vertices are adjacent if and only if they belong to a common cube of G. The *clique graph* $K(H)$ of a graph H is the intersection graph of the cliques (=maximal simplices) of H. The clique graph of a Helly graph is Helly again [**38**]. Applied to $H = G^\Delta$, we see that the cliques of G^Δ are exactly the maximal hypercubes in G, whence $K(G^\Delta)$ is the intersection graph of the maximal hypercubes in G.

PROPOSITION 3.2. [**45**] *If G is a median graph, then G^Δ and $K(G^\Delta)$ are Helly graphs.*

A *dually chordal graph* is a Helly graph in which the intersection graph of balls is chordal (i.e., without induced cycles of lengths > 3). Dually chordal graphs are exactly those graphs G which admit a spanning tree T such that any ball (or any clique) induces a subtree in T [**60**]. There is a close relationship between chordal graphs, dually chordal graphs, and graphs of acyclic cubical complexes which is set up by the following operators: taking the simplex graph, or the clique graph, or the intersection graph of the maximal hypercubes, respectively. Then dually chordal graphs are precisely the clique graphs of chordal graphs, and clique-Helly chordal graphs are precisely the clique graphs of dually chordal graphs [**60, 62**]. The intersection graph of the maximal cubes in any acyclic cubical complex is dually chordal [**16**]. Conversely, every dually chordal graph can be realized in such a way [**64**].

3.2. Irreflexive case: Helly n-chromatic graphs. The study of absolute retracts of bipartite graphs (alias Helly bipartite graphs) was initiated in [**120**] by establishing that they are precisely the retracts of direct ("relational") products of (irreflexive) paths. Another characterization, similar to Theorem 3.1(e), was given in [**149**]. Here we present the structural characterizations of absolute bipartite retracts provided in [**26**]. A bipartite graph G is of *breadth at most two* if, for any vertex u with a family of intervals $I(u, v_i)$, $i = 1, \ldots n$ having intersection $\{u\}$, two of the members of the family have intersection $\{u\}$. The graph B_n is the graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching; then let $\hat{B}_n$ be the extension of B_n obtained from $K_{n+1,n+1}$ by deleting a matching with n edges. The intersections of balls with one of the two color classes of the bipartite graph G are referred to as *half-balls*. G is called *modular* if any three pairwise intersecting half-balls have a nonempty intersection. It is easy to see that this definition is equivalent to the conventional one requiring that any three intervals $I(u, v), I(u, w)$, and $I(v, w)$ have a nonempty intersection.

THEOREM 3.3. [**26**] *For a finite bipartite graph $G = (V, E)$ the following statements are equivalent:*

(a) *G is a Helly bipartite graph, i.e., the collection of half-balls of G has the Helly property;*

(b) *G is modular and the collection of the neighborhoods of vertices of G has the Helly property;*
(c) *G is a modular graph of breadth at most two;*
(d) *G is a modular graph such that every induced subgraph B_n $(n \geq 4)$ extends to $\hat{B}_n$;*
(e) *for any vertices u and v with $d(u,v) \geq 3$, the neighbors of v in $I(u,v)$ have a second common neighbor in $I(u,v)$.*

Among finite bipartite graphs G, the Helly bipartite graphs can be characterized as those graphs that admit Condorcet solutions in the following voting location game. Assume that some voters are arbitrarily assigned to vertices of G (thus realizing nonnegative integer weights). A *Condorcet vertex* is a vertex x such that no absolute majority of voters would ever be closer to a rival vertex y than to x. In other words, x is not a Condorcet vertex exactly when there exists a vertex y such that the the sum of weights of the vertices closer to y than to x (representing the voters preferring y over x) exceeds half of the total weight. Now, Condorcet vertices exist for every distribution of voters in G exactly when G is Helly bipartite [**26**].

Many of the above (and further) equivalent descriptions of Helly bipartite graphs are perfect analogues of corresponding descriptions of Helly graphs. This correspondence has been made explicit in [**28**] by exhibiting four transformations between bipartite (irreflexive) graphs and reflexive graphs that preserve the property of being an absolute retract in the respective category.

The concept of absolute retract of bipartite graphs naturally extends to the n-chromatic (irreflexive) case. In order to develop criteria for absolute retracts in this scenario, the graphs under consideration should be endowed with n-colorings (which can be constructed canonically for absolute retracts). Then the characterization of n-chromatic absolute retracts either uses the Helly property of balls pairwise intersecting in color i plus certain local conditions prescribing the existence of vertices with color i $(i = 1, \dots, n)$ in the neighborhood of vertices (as a kind of color-variant of Menger-convexity) or uses recursive procedures or dismantling schemes; the results are though somewhat more involved and the proofs more complex than in the bipartite case [**40**].

4. Bridged graphs

A graph is called *bridged* if all isometric cycles of G have length three [**108**]. In particular, all chordal graphs are bridged. Every cycle (regarded as a set of edges) of a bridged graph is the modulo 2 sum of triangles, i.e., bridged graphs are *null-homotopic* [**128**].

THEOREM 4.1. [**108, 162**] *For a graph $G = (V,E)$, the following statements are equivalent:*
(a) *G is bridged;*
(b) *the balls $N_r(S)$, $r \geq 1$, centered at convex sets S are convex;*
(c) *$N_r(v)$ and $N_r(e)$ are convex for all $v \in V$, $e \in E$, and $r \geq 1$.*

A shortest path between two vertices x, y of a cycle C of G is called a *bridge* of C if its length is smaller than the distance between x and y measured along C. A cycle C is called *well-bridged* (in G) if for every vertex $x \in C$ there exists a bridge from x to some vertex of C or the two neighbors of x from C are adjacent (thus forming a chord). A cycle $C = C_n$ of length $n = 4$ or $n = 5$ is well-bridged exactly when it is not induced in G, that is, it has some chord, but a non-induced 6-cycle, for instance, is not necessarily well-bridged. All cycles of a bridged graph are well-bridged. In the Petersen graph all cycles except 5-cycles are well-bridged (although there are many induced 6-cycles). The property that all cycles of G except 5-cycles are well-bridged can be translated into a convexity property of balls.

THEOREM 4.2. [**108, 162**] *For a graph $G = (V, E)$, the following statements are equivalent:*

(a) *all balls of G are convex;*
(b) *all cycles C_n, $n \neq 5$, of G are well-bridged;*
(c) *G does not contain isometric cycles of length $n > 5$, and for any two vertices x, y the neighbors of x from the interval $I(x, y)$ are pairwise adjacent.*

THEOREM 4.3. [**1**] *The finite bridged graphs are exactly the cop-win graphs without induced 4-wheels and 5-wheels.*

A simple algorithmic proof of this result was given in [**82**], where it is shown that any ordering of the vertices of a finite bridged graph G produced by breadth-first search (BFS) starting from an arbitrary vertex b is a cop-win ordering, namely every vertex x is dominated by its father $f(x)$ with respect to BFS. The proof is based on the following property of BFS orderings of G: *if v and w are adjacent vertices of G, then their fathers $f(v)$ and $f(w)$ with respect to BFS either are adjacent or coincide.* This property turns out to be closely related with retraction questions (see Section 7) and with a combing property of graphs, which originated in the geometric theory of groups [**107**]. Let b be a distinguished vertex ("base point") of a graph G. A *geodesic* 1-*combing* of G with respect to the base point b comprises shortest paths P_x between b and all other vertices x such that for any edge uv of G the paths P_u and P_v stay close when traveling towards the base point, i.e., if u' and v' are vertices on P_u and P_v, respectively, with $d(u, u') = d(v, v')$, then $d(u', v') \leq 1$. One can select the combing paths so that their union constitutes a spanning tree T_b of G which is rooted at b and preserves the distances from b to all vertices. The neighbor $f(x)$ of a vertex $x \neq b$ in the unique path of T_b connecting x with b will be called the *father* of x; it is stipulated that the father map f fixes b. This kind of (nonexpansive) map f has also been referred to as a *mooring* in G onto $\{b\}$ (see [**75, 165**]). Conversely, in the case that G is bridged, a spanning tree T_b providing a 1-combing is obtained by taking the edges of G connecting a vertex x and its father $f(x)$ with respect to BFS. (For extensions of dismantling and combing properties to other classes of graphs and to infinite bridged graphs, see [**75, 83, 84**].)

The simplicial complex $\mathcal{X}(G)$ of a bridged graph G is called a *bridged complex.* Theorem 4.3 implies that finite bridged complexes are contractible. Bridged complexes can be characterized among flag complexes in the following way:

THEOREM 4.4. [**84**] *For a flag complex $\mathcal{X}$ the following conditions are equivalent:*

(a) *$\mathcal{X}$ is simply connected and the neighborhood of every vertex v in the underlying graph $G(\mathcal{X})$ does not contain induced 4–cycles and 5–cycles;*
(b) *$G(\mathcal{X})$ is weakly modular and does not contain induced 4–cycles and 5–cycles;*

(c) $G(\mathcal{X})$ *is bridged.*

Therefore it follows (from statement (a)) that the plane graphs in which all inner faces are triangles and all inner vertices have degrees ≥ 6 are bridged. Note that the 2-dimensional bridged complexes are precisely the CAT(0) 2-dimensional polyhedral complexes obtained from 2-dimensional simplicial complexes by replacing triangular faces by equilateral triangles of the same size.

5. Distance-hereditary and hyperbolic graphs

Trees (with weighted edges) constitute basic examples of metric spaces, and as such they play an important role in various fields, e.g., computer science [**147**], classification [**47**], and in particular, biology [**158**]. The characteristic metric feature of weighted trees is the "classical" 4-point condition, asserting that a finite metric space (X, d) can be embedded into a tree with edges weighted by positive reals if and only if for any four points u, v, w, x the two larger ones of the sums $d(u, v) + d(w, x)$, $d(u, w) + d(v, x)$, $d(u, x) + d(v, w)$ are equal. In the case that d is the shortest-path metric of a graph G, this 4-point condition is satisfied exactly when G is a *block graph,* that is, all 2-connected components of G are complete subgraphs [**31, 123**].

There are two options for relaxing the 4-point condition: one is to drop the positivity constraint for edge weights, which leads to *distance-hereditary graphs* [**122**]; their original definition states that every induced path is a shortest path. The second option is to allow the two larger distance sums to differ slightly, by some integer $\delta > 0$: thus, the graph G is δ-*hyperbolic* [**113, 114, 118**] (or tree-like with defect at most δ) if and only if for any four vertices u, v, w, x,

$$d(u, v) + d(w, x) \leq d(u, w) + d(v, x) \leq d(u, x) + d(v, w)$$

$$\text{implies } d(u, x) + d(v, w) - d(u, w) - d(v, x) \leq \delta.$$

Another notion of tree-likeness directly relaxes the distance requirement for a tree representation: for a near-isometric embedding of a graph G with shortest-path metric d into a (positively) weighted tree T with path-length metric d', it is required that the absolute error is bounded by some $\epsilon > 0$, that is, $||d - d'||_\infty < \epsilon$. An error bound $\epsilon = 2$, for instance, can be achieved for chordal graphs [**61**].

5.1. Distance-hereditary graphs.

Finite distance-hereditary graphs can be generated from the one-vertex graph K_1 by means of two operations, viz., adjoining a new pendant vertex or splitting a vertex into two copies that are either adjacent (referred to as "true twins") or nonadjacent ("false twins") . Alternatively, these graphs can be characterized in terms of forbidden isometric subgraphs or the generalized 4-point condition:

THEOREM 5.1. [**31**] *A graph G is distance-hereditary if and only if G does not contain the following graphs as isometric subgraphs: n-cycles C_n of length $n > 4$, the house (i.e., the 5-cycle with a unique chord), the fan (i.e., the 5-cycle with two incident chords), and the domino (i.e., the Cartesian product of K_2 and the 3-vertex path $K_{1,2}$).*

THEOREM 5.2. *For a finite graph G the following assertions are equivalent:*

(a) *G is distance-hereditary;*

(b) [**42**] *the vertex set of* G *is included in a tree* T *with edges weighted with positive or negative real numbers such that the shortest-path metric* d *of* G *is realized as the (additive) path-length function within the weighted tree* T;

(c) [**31**] G *can be generated from* K_1 *by successive applications of adjoining new pendant vertices and splitting vertices.*

The weighted tree T that represents the metric d of a distance-hereditary graph is unique provided that T contains no redundant vertices and edges; namely, all "unlabeled" vertices of T (i.e., those which do not represent vertices of G) should have degree > 2, and all edges should have nonzero weights. Then the edge weights are necessarily from the set $\{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$; see [**42**]. Further, every path in T connecting a labeled vertex (which represents a vertex from G) with an unlabeled vertex has non-negative length, and all paths connecting unlabeled vertices have lengths at least -1. This can easily be seen by induction. In fact, adjoining a new pendant vertex x to G by making it adjacent to some vertex u amounts to adjoining it with an edge of length 1 to the vertex labeled u in the tree representation. The splitting operation that turns some vertex v of G into a pair x_1, x_2 of twins is realized by the following tree expansion: remove the label v from the vertex that represented the vertex v of G in T and attach a new unlabeled vertex as a neighbor to this vertex; this new neighbor in turn gets two new neighbors labeled with x_1 and x_2, respectively; the new interior edge gets weight $-\lambda$ and the two new pendant edges have length λ; for true twins one sets $\lambda = \frac{1}{2}$ and for false twins $\lambda = 1$. After having performed this extension of the weighted tree T, the single unlabeled vertex of degree 2, which occurs only if the vertex v of G was a pendant vertex, is suppressed and the weights of those two incident edges are added up. On the other hand, the tree representation of G permits one to spot immediately either a pendant vertex or a pair of twins. If the former does not exist, then necessarily some end vertices of the representing weighted tree have a common (unlabeled) neighbor and thus constitute twins.

5.2. 1-Hyperbolic graphs. Gluing together any two shortest paths along a common terminal edge does not necessarily result in a shortest path: take a 4-cycle, for instance. The graphs in which this operation always returns shortest paths are exactly the distance-hereditary chordal graphs, also known as *ptolemaic graphs* [**122, 168**]. For arbitrary 1-hyperbolic graphs, however, this terminal gluing of shortest paths in general yields only near-shortest paths with defect 1, that is, the following property holds:

(α_1) if $v \in I(u,w)$ and $w \in I(v,x)$ are adjacent, then $d(u,x) \geq d(u,v) + d(w,x)$.

Under this condition all balls are convex, so that all cycles except C_5 must be well-bridged. Actually, this requirement on cycles plus one forbidden isometric subgraph characterize the graphs fulfilling (α_1) [**168**]. For a full characterization of 1-hyperbolic graphs, another five forbidden subgraphs come into play, so that altogether well-bridgedness of all C_n ($n \neq 5$) and six forbidden isometric subgraphs do the job [**20**].

All of these six forbidden (bridged) graphs include the fan as an induced subgraph. Hence a distance-hereditary graph is 1-hyperbolic exactly when it is chordal (since the 4-cycle yields the three distance sums $2 = 2 < 4$). Non-chordal distance-hereditary graphs G are 2-hyperbolic. To see this, take the subtree connecting a quartet of vertices from G in the (real-weighted) tree representation: the interior edge from the weighted quartet tree (corresponding to a path from the weighted tree) has length at least -1, and therefore the largest two distance sums for the quartet cannot differ by more than 2.

A characterization of all 2-hyperbolic graphs by forbidden isometric subgraphs is not in sight, inasmuch as isometric cycles of lengths up to 7 may occur, thus complicating the picture.

6. Meshed graphs and weak modularity

A graph $G = (V, E)$ is called *meshed* [**35**] if for any three vertices u, v, w with $d(v, w) = 2$, there exists a common neighbor x of v and w such that $2d(u, x) \leq d(u, v) + d(u, w)$. Meshed graphs are thus characterized by some (weak) convexity property of the radius functions $d(\cdot, u)$ for $u \in V$. This condition ensures that all balls centered at simplices in a meshed graph G induce isometric subgraphs and that every cycle can be written as a modulo 2 sum of cycles of lengths 3 and 4.

A graph G is *weakly modular* [**15, 33, 78**] if its distance function d satisfies the following conditions:

(*triangle condition*) for any three vertices u, v, w with $1 = d(v, w) < d(u, v) = d(u, w)$ there exists a common neighbor x of v and w such that $d(u, x) = d(u, v) - 1$;

(*quadrangle condition*) for any four vertices u, v, w, z with $d(v, z) = d(w, z) = 1$ and $2 = d(v, w) \leq d(u, v) = d(u, w) = d(u, z) - 1$, there exists a common neighbor x of v and w such that $d(u, x) = d(u, v) - 1$.

The quadrangle condition was first exhibited in a geometric context (as axiom (A6) in [**71**]). All weakly modular graphs are meshed. Basis graphs of matroids and even Δ-matroids (see below) as well as the graphs in which all median sets induce connected or isometric subgraphs [**19**] are meshed but in general not weakly modular. The icosahedron graph constitutes another example of a meshed graph that is not weakly modular. Meshed graphs, in which $K_{1,2}$ is a forbidden convex subgraph and $K_{2,3}$ plus no, one or two non-incident chords are forbidden induced subgraphs, have been characterized as the (weak) Cartesian products of icosahedra, 5-wheels, and joins of hyperoctahedra and complete graphs [**35**].

6.1. Weakly modular graphs.

Modular graphs [**46**], pseudo-modular graphs [**32**], pre-median, weakly median, and quasi-median graphs (Section 7), dual polar graphs [**71**], median, distance-hereditary, bridged, and Helly graphs (Sections 1-4) are all instances of weakly modular graphs.

The metric triangles of meshed graphs are equilateral [**19**]. Metric triangles of weakly modular graphs are somewhat more special: namely, a graph G is weakly modular if and only if for every metric triangle uvw all vertices of the interval $I(v, w)$ are at the same distance $k = d_G(u, v)$ from u [**78**]. In pseudo-modular graphs, every metric triangle has size at most 1 [**32**], and in modular graphs, every metric triangle is degenerate, that is, has size 0. A graph G is modular exactly when it is triangle-free and satisfies the quadrangle condition [**10, 142**].

A metric triangle uvw of G is a *quasi–median* of the triplet x, y, z if the following metric equalities are satisfied:

$$d(x, y) = d(x, u) + d(u, v) + d(v, y),$$

$$d(y, z) = d(y, v) + d(v, w) + d(w, z),$$

$$d(z,x) = d(z,w) + d(w,u) + d(u,x).$$

Every triplet x, y, z of a graph has at least one quasi-median, for instance, the metric triangle uvw constructed in the following way: first select any vertex u from $I(x,y) \cap I(x,z)$ at maximal distance to x, then select a vertex v from $I(y,u) \cap I(y,z)$ at maximal distance to y, and finally select any vertex w from $I(z,u) \cap I(z,v)$ at maximal distance to z. A *weakly median* graph is a weakly modular graph in which every triplet of vertices has a unique quasi-median. Weakly median graphs are exactly the weakly modular graphs not containing any two distinct vertices with an unconnected triplet of common neighbors; the four minimal obstructions are displayed in Fig.1 of [**18, 21**].

6.2. Basis graphs of matroids and even Δ-matroids. According to one of the many equivalent definitions, a *matroid* of rank k on a set I of n elements is a collection $\mathcal{B}$ of subsets of I of size k, called *bases,* which satisfy the following *exchange property:*

for $A, B \in \mathcal{B}$ *and* $i \in A - B$ *there exists* $j \in B - A$ *such that* $(A - \{i\}) \cup \{j\} \in \mathcal{B}$.

One says that the base $A - \{i\} \cup \{j\}$ is obtained from the base A by an *elementary exchange.* The *basis graph* $G = G(\mathcal{B})$ of a matroid $\mathcal{B}$ is the graph whose vertices are the bases of $\mathcal{B}$ and edges are the pairs A, B of bases differing by an elementary exchange. The exchange property implies that $G(\mathcal{B})$ is an isometric subgraph of the Johnson graph $J_{n,k}$. It is well known that basis graphs faithfully represent their matroids, thus studying the basis graph amounts to studying the matroid itself. An induced subgraph $G = (V, E)$ of $J_{n,k}$ can be compared to the 1-skeleton of the convex hull of the characteristic vectors of all $v \in V$: then G is the basis graph of a (rank k) matroid exactly when G coincides with the 1-skeleton of this polytope [**112**].

A Δ-*matroid* [**59, 76, 104**] (alias Lagrangian matroid in the terminology of [**58**]) is a collection $\mathcal{B}$ of subsets of a set I with $\#I = n$, called *bases*, not necessarily equicardinal, satisfying the following *symmetric exchange property:*

for $A, B \in \mathcal{B}$ *and* $i \in A\Delta B$, *there exists* $j \in B\Delta A$ *such that* $A\Delta\{i,j\} \in \mathcal{B}$.

A Δ-matroid whose bases all have the same cardinality modulo 2 is called an *even* Δ-*matroid.* The matroids are precisely the (even) Δ-matroids for which all members of $\mathcal{B}$ have the same cardinality. For a subset J of I denote $\mathcal{B}\Delta J := \{B\Delta J : B \in \mathcal{B}\}$ and say that the Δ-matroid $\mathcal{B}\Delta J$ is obtained by applying a *twisting* to $\mathcal{B}$.

If A, B are two bases of an even Δ-matroid $\mathcal{B}$ and $B = A\Delta\{i,j\}$ we say that B is obtained from A by an *elementary exchange.* Following the terminology for ordinary matroids, the *basis graph* $G = G(\mathcal{B})$ of an even Δ-matroid $\mathcal{B}$ is the graph whose vertices are the bases of $\mathcal{B}$ and edges are the pairs A, B of bases such that $\#A\Delta B = 2$. Some properties of these graphs have been used and investigated in [**172**]. It is clear that $G(\mathcal{B})$ is an isometric subgraph of the half-cube $\frac{1}{2}H_n$.

The following two conditions play a key role in the characterizations of basis graphs. A graph G satisfies the *interval condition* (ICm) ($m \geq 2$) if for any vertices u, v at distance 2, the interval $I(u,v)$ includes an induced square $C_4 = K_{2\times 2}$ and is contained in the m-octahedron $K_{m\times 2}$. A graph G satisfies the *positioning condition* (PC) if for each vertex b and each square $v_1v_2v_3v_4$ of G the equality $d(b,v_1) + d(b,v_3) = d(b,v_2) + d(b,v_4)$ holds. Obviously, a graph in which every interval between vertices at distance 2 includes a 4-cycle satisfies the positioning condition exactly when G is meshed.

THEOREM 6.1. *Let $G =$ be a finite graph.*

(a) [**140**] *G is a basis graph of a matroid if and only if it satisfies the interval condition* (IC3)*, the positioning condition* (PC)*, and the subgraph induced by the neighborhood $N(v)$ of any vertex v is the line graph of a bipartite graph;*
(b) [**85**] *G is a basis graph of an even Δ-matroid if and only if it satisfies the interval condition* (IC4)*, the positioning condition* (PC)*, and the subgraph induced by the neighborhood $N(v)$ of any vertex v is the line graph of some graph. If $G \neq K_4$, then all even Δ-matroids having G as a basis graph can be obtained each from other by a twisting.*

We outline the idea of the proof of this theorem given in [**85**]. To establish that a graph G satisfying the conditions of the theorem is a basis graph, pick a vertex b of G such that the subgraph induced by $N(b)$ is the line graph of a graph Γ and define the following mapping $\varphi : V \to 2^I$. Set $\varphi(b) = \emptyset$. Each vertex $x \in N(b)$ encodes some edge ij of Γ; put $\varphi(x) = \{i, j\}$. For any other vertex v, let $\varphi(v) = \cup\{\varphi(x) : x \in I(b, v) \cap N(b)\}$. One first shows that φ is injective and that all sets $\varphi(v)$ have even cardinality. It then turns out that φ is an edge-preserving map from G to the half-cube $\frac{1}{2}H_n$, which implies that $\mathcal{B}_\varphi := \{\varphi(v) : v \in V\}$ is an even Δ-matroid. If Γ is a bipartite graph with two color classes A and B, then $\mathcal{B}_\varphi \Delta A$ is a matroid of rank $\#A$. This encoding scheme is different from that used by S. Maurer. He encodes the vertex b by A, a vertex $x \in N(b)$ representing the edge ij of Γ with $i \in A$ and $j \in B$ is labeled by the set $(A - \{i\}) \cup \{j\}$. Then the encoding is inductively expanded to the whole graph using certain squares (among other things, in establishing that this labeling is well-defined, it is necessary to show that it does not depend of the choice of squares); see [**140**] for all details.

6.3. Dual polar graphs. Polar spaces represent one of the fundamental types of incidence geometries [**69**]. Polar spaces of rank at least 3 with thick lines (i.e., all lines contain at least three points) have been classified in the seminal work by Tits [**167**]: they can be constructed from sesquilinear or pseudoquadratic forms on vector spaces; cf. [**90**]. To every abstract polar space of rank ≥ 2 one can associate a certain graph G of diameter 2 and radius 2, in which adjacency expresses collinearity. Thus the points of the space are the vertices of G and the lines are certain simplices (complete subgraphs) of G such that any two adjacent vertices are in some line, and every vertex outside a line L is adjacent to either exactly one vertex or all vertices of L. In the rank 2 case ("generalized quadrangle") lines are required to be cliques, that is, maximal simplices. For rank $n \geq 3$, additional requirements are imposed, where a hierarchy of simplices, distinguished as (singular) subspaces play a role. Therefore the graph alone captures the structure of the polar space only poorly in this case.

In contrast, interesting graphs of diameter n arise, when the order of the subspaces of a polar space of rank n is reversed (yielding a "dual polar space"), so that the vertices now represent the $(n-1)$-dimensional (projective) subspaces and the edges pairs of such subspaces intersecting in a $(n-2)$-dimensional subspace. According to [**71**] the graph G of a dual polar space of rank n is characterized by the following conditions, rephrased here:

(A1) *every clique of G is gated, that is, the triangle condition is fulfilled and the kite K_4^- (K_4 minus one edge) does not occur as an induced subgraph;*

(A2) *G has diameter n;*

(A3&4) *the gated hull $\langle\langle u, v\rangle\rangle$ of two vertices u, v at distance 2 has diameter 2, and hence the quadrangle condition is satisfied;*

(A5) *every pair of nonadjacent vertices u, v together with every neighbor x of u in $I(u, v)$ belong to the corners of some metric rectangle in G.*

The joint formulation of Cameron's two axioms (A3) and (A4) rests on the observation that the gated hull of a subset S in a weakly modular graph G is obtained as the smallest set including S and containing any common neighbor of any two of its vertices [**78**]. Axioms (A1),(A3), and (A4) can then be formulated as a single condition:

(A1&3&4) *the gated hull $\langle\langle u, v\rangle\rangle$ of any two vertices u, v at distance $1 \leq k \leq 2$ has diameter k.*

Axiom (A5) is a weak variant of the sphericity condition studied in [**52**]. In the case of rank $n = 2$, this axiom together with (A1) and specifying $n = 2$ in (A2) then describes the generalized quadrangles [**146**]. For example, the generalized quadrangle of order $(2, 2)$ is the familiar 15-vertex "doily" (for a diagram, see http://www.maths.monash.edu.au/∼bpolster/gq1.html, or http://home.wlu.edu/∼mcraea/Finite_Geometry/Applications/Prob33Walks/problem33.html, or [**65**]), whereas the one of order (4,2) is the 27-vertex Schläfli graph G_{27} (cf. [**68**]). On the other hand, the graphs of finite dual polar spaces of rank $r \geq 3$ in which two points at distance 2 have more than two neighbors are known explicitly (see Theorem 2 of [**71**]) due to Tits' classification of thick polar spaces of rank at least 3.

6.4. Modular graphs and orientability. A bipartite graph is the graph of a dual polar space if and only if it is modular, satisfies the above axiom (A5), and does not contain the graph $K_{3,3}^-$ ($K_{3,3}$ minus an edge) as an induced subgraph. Indeed, the gated hull $\langle\langle u, v\rangle\rangle$ of two vertices u, v at distance 2 in a modular graph has diameter 2 (and hence equals $K_{2,m}$, for some cardinal m) exactly when there is no induced $K_{3,3}^-$. The graphs of dual polar spaces that are bipartite have been classified completely [**71**]. Every interval $I(u, v)$ in such a graph G necessarily is the covering graph of a complemented modular lattice (with bounds u and v). Therefore G is *weakly spherical* [**52**] in the sense that for every vertex x between two vertices u and v there exists some vertex x' such that v, x, u, x' form a metric rectangle.

When dropping (A5) intervals still constitute modular lattices [**46**]. The whole graph cannot always be organized as the covering graph of a 4-crown-free ordered set (by a *4-crown* we mean the height 1 orientation of a 4-cycle; see [164, Fig. 5.12]). If a modular graph G admits a realization as the covering graph of an ordered set (with intervals being modular lattices) that has no 4-crown, then G is called *orientable*. The orientable hereditary modular graphs played a crucial role in the characterization of the so-called minimizable metrics in a version of the multifacility location problem (also known as the 0-extension problem); see [**25, 130**]. A *hereditary modular graph* is a bipartite graph for which all isometric cycles have length four and consequently is a Helly bipartite graph [**10**]. The hereditary modular graphs without induced $K_{3,3}^-$ characterize yet another property of the 0-extension problem [**131**].

The characterization of minimizable metrics took advantage of the following feature of modularity. A metric space (X, d) is said to be *modular* if the metric betweenness segments $[u, v], [u, w]$, and $[v, w]$ have a nonempty intersection for any three points u, v, and w. In the finite (or more generally, discrete) case, the metric space (X, d) can then be regarded as a modular graph for which graph intervals coincide with the corresponding segments and edges are weighted by some positive length function under which 4-cycles are turned into metric rectangles [**9**].

7. Fiber-complemented graphs

As we mentioned above, median graphs are generalized to quasi-median graphs and further to weakly median graphs. How do the results for median graphs extend to these two (or other intermediate) classes of graphs? Decomposition, retraction, and algebraic aspects of quasi-median and weakly median graphs have been settled in [**18, 21, 22, 36, 174**]. The gist of the methods and arguments employed there has been distilled in [**73, 74**] by presenting a general framework under which decomposition, retraction, and fixed-box theorems can be derived. This would only leave the question unanswered of describing the prime graphs in each specific case. A graph is called *prime* if it neither is a gated amalgam of any proper subgraphs nor can be represented as a nontrivial Cartesian product.

The main observation in [**73**] then is that those decomposition schemes hinge upon a property of the gated sets in the graphs under study. The term "prefiber" alludes to a property of the fibers in Cartesian products and had (unfortunately) been used as a mere synonym for gated set; cf. [**72, 165**]. An arbitrary gated set in a graph is however quite far from a genuine product fiber. We have therefore called a gated set A a *prefiber* [**21**] if it satisfies the following property investigated in [**73**]:

(Chastand condition) *each set of all vertices in the given graph G that share the same gate in the gated set A of G is itself gated in G.*

Graphs in which all gated sets are prefibers are called *fiber-complemented* [**73**].

THEOREM 7.1. [**73**] *A finite connected graph G is fiber-complemented if and only if G can be obtained by successive applications of gated amalgams from Cartesian products of prime graphs.*

In algebraic terms, this result asserts that G is a subdirect product of prime fiber-complemented graphs; see [**21**]. The main step in the proof of this theorem is to show that if G contains a proper gated set, then either G is a Cartesian product or a gated amalgam of smaller subgraphs. This is thus quite analogous to the strategy that was employed in the particular case of weakly median graphs [**18**]. The advantage of the general approach involving fiber-complementedness is that it provides a blueprint for proving the same sort of decomposition theorem for related or more general graph classes, such as the pre-median graphs [**73**]. Moreover, further analysis [**74**] shows that representations in terms of retracts from Cartesian products (e.g. Theorem 2.1 above) follow this scheme under some additional hypothesis:

THEOREM 7.2. [**74**] *Every finite fiber-complemented graph G which has a geodesic 1-combing is a retract of the Cartesian product of its prime constituents.*

We outline a shorter proof of this theorem, which is based on the following result from [**110**]: a graph G is a retract of Cartesian product $H = H_1\Box\ldots\Box H_n$ exactly when the following two projection criteria are met:

(1) G coincides with the largest induced subgraph of H that has the same images under the projections to all H_i $(1 \le i \le n)$ and $H_i\Box H_j$ $(1 \le i < j \le n)$ as G; and

(2) each of these images constitutes a retract of the corresponding factor H_i or product $H_i\Box H_j$.

Since the prime factors H_i do not have any nontrivial gated subgraphs in the case of a fiber-complemented graph G, the projections to the factors are either trivial or surjective. Then, as G can be interpreted as a subdirect product of algebras satisfying the absorption laws, it follows from [**51**] that criterion (1) is met. Moreover, one can easily show that the image of G under any projection to $H_i\Box H_j$ is either a singleton, a fiber, the whole product, or a point amalgam of two fibers. In general, it is difficult to decide whether a point amalgam (i.e., the gated amalgam of two graphs along a common vertex) is a retract of the Cartesian product of the two graphs - even when the second graph is fixed to be K_2: this decision problem is known to be NP-complete [**110**]. Now, a retraction from $H_i\Box K_2$ to H_i exists exactly when H_i admits a geodesic 1-combing [**21, 74**]. Therefore criterion (2) is satisfied by virtue of the hypothesis on G.

7.1. Weakly median graphs and algebras. One of the appealing features of weakly median graphs is that one can determine all prime constituents.

THEOREM 7.3. [**18**] *K_2, the 5-wheel, induced subgraphs of hyperoctahedra that include either K_4 or a 4-wheel, and two-connected K_4- and $K_{1,1,3}$-free bridged graphs are the prime finite weakly median graphs. The latter bridged graphs are exactly the graphs which can be realized as plane graphs such that all inner faces are triangles and all inner vertices have degrees larger than 5.*

The bridged prime constituents of weakly median graphs are therefore exactly the (3,6)-graphs (see the next section) that have only triangles as their inner faces. Although the construction of a weakly median graph from its prime constituents through gated amalgamation and Cartesian multiplication requires some finiteness hypothesis, the prime constituents are nevertheless manifest also in the infinite case as the blocks of a certain canonical "tolerance" (that is, a reflexive and symmetric binary relation compatible with the following "apex operation" [**21**]).

Universal algebra enters the study of weakly median graphs in a natural way, just as in the case of median structures. Notice that every graph G with vertex set V can be turned into a ternary algebra, called an *apex algebra* of G [**36**]: an *apex operation* $(\ldots) : V^3 \to V$ maps any triplets x, y, z and x, z, y to some vertex $u = (xyz) = (xzy) \in I(x, y) \cap I(x, z)$ such that $I(x, u)$ is maximal with respect to inclusion. For weakly median graphs the apex algebra is uniquely defined, because for any triplet x, y, z the vertices $(xyz), (yxz)$, and (zxy) form the unique quasi-median of the triplet. The equational description follows the lines of median algebras by skipping the left symmetry axiom (M3) and adding one (more complex) axiom for compensation:

THEOREM 7.4. [**22**] *Weakly median graphs with their apex operations are exactly the discrete ternary algebras satisfying the four axioms (M1), (M5), and*

(M8) $((uwx)(vwx)u) = (uwx)$,

(M9) $((wux)(uvw)v) = (w(uvw)(vu(wux)))$.

Again, we can freely exchange (M5) by the pair (M2), (M4). We outline the idea of the proof of Theorem 7.4. To show that the apex algebra $\mathcal{A}$ of a weakly median graph G satisfies the axioms (M8) and (M9) (as well as further identities), we first establish that $\mathcal{A}$ is a subdirect product of the apex algebras of its prime constituents, which constitute simple algebras. It suffices to consider only finite weakly median graphs because the subalgebras generated by finite sets in weakly median graphs are included in finite (weakly median) isometric subgraphs. This is quite straightforward to show for the 5-wheel and induced subgraphs of hyperoctahedra but quite laborious in the case of the prime bridged weakly median graphs since the required metric properties have to be established from scratch by employing their geometric structure (which has much in common with the hexagonal plane). To prove the converse, we first show that a discrete ternary algebra $\mathcal{A}$ satisfying the four axioms (M1),(M5),(M8), and (M9) defines a discrete "geometric interval space" (sensu [**170**]) obeying the interval version of the triangle condition. Then, using a result from [**15**], we show that this interval space leads to a weakly median graph G such that the apex algebra of G coincides with $\mathcal{A}$; for details, see [**22**].

Weakly median graphs are known to be so-called join spaces relative to the interval operation, which in turn can be expressed by two betweenness conditions, referred to as the Pasch and Peano axioms for interval spaces [**22, 79, 80, 161, 170**]. This underscores the naturalness of weakly median graphs and algebras with regard to some geometric features of classical Euclidean geometry.

Inasmuch as the class of weakly median graphs possesses a variety of prime members, there is quite some flexibility in defining special subclasses (tailored to specific problems) which are closed under gated amalgamation and weak Cartesian multiplication. For instance, one such subclass, briefly studied in [**22**], consists of the weakly median graphs that do not contain the 6-vertex sun, so that in this class there is only one isomorphism type for the convex hull of metric triangles of each size k, viz., the k-th Cartesian power of K_3.

7.2. Quasi-median graphs. The theory developed for arbitrary weakly median graphs immediately specializes to quasi-median graphs (by forbidding the kite K_4^- as an induced subgraph), where the results often become much simpler. Finite quasi-median graphs were first constructed through a sequence of "quasi-median expansions" [**142**], which can be re-organized in a condensed sequence of "gated expansions" [**36**]. A gated expansion amounts to taking Cartesian products with a new complete graph and gated amalgamations along a common gated subgraph, which, algebraically speaking, is equivalent to the introduction of a new subdirect factor. Quasi-median graphs are precisely the retracts of (weak) Hamming graphs [**72, 88, 89, 174**]. This fact was used in a characterization of quasi-median graphs in terms of a dynamic location game [**88, 89**]. Further characteristic properties of quasi-median graphs can be found in [**36, 66, 125**]. Here is a list of some more recent descriptions, which are, of course, variations of the same theme:

THEOREM 7.5. *The following statements are equivalent for a graph* $G = (V, E)$*:*

(a) *G is quasi-median;*
(b) *G is meshed and without induced $K_{2,3}$ and K_4^-;*
(c) *G has no induced $K_{2,3}$, and the gated hull of any two vertices u and v at distance $1 \leq k \leq 2$ has diameter k;*
(d) *the interval $I(u,v)$ is convex and the gated hull $\langle\langle u,v\rangle\rangle$ has diameter k, for any vertices u and v with $d(u,v)=k$;*
(e) [**22**] *every clique is a prefiber.*

It needs only few arguments to show that the statements in the preceding theorem are equivalent to previous characterizations of quasi-median graphs. For instance, observe that a meshed graph G without induced K_4^- satisfies the quadrangle condition and hence is weakly modular, so that one can substitute "meshed" by "weakly modular" in statement (b). Indeed, proceed by induction and consider an instance of the quadrangle condition: if the required vertex is not obtained through meshedness right away, then an induced 4-cycle is obtained that violates the positioning condition, so that one is invited to use the triangle condition twice and then the induction hypothesis; two further applications of the triangle condition eventually lead to a forbidden K_4^-. To see that the gated hull operator on pairs of vertices does not increase distance/diameter in a quasi-median graph G, embed G isometrically in a Hamming graph H. Then two vertices u and v with $d(u,v)=k$ embedded in H differ in exactly k coordinates, so that the embedded gated hull $\langle\langle u,v\rangle\rangle$ stays within a k-dimensional Hamming subgraph of H.

Notice that the condition requiring that all intervals be gated, which is stronger than (d), in fact characterizes median graphs. On the other hand, if one relaxes the trivial consequence (c) of (d) by allowing induced $K_{2,3}$ subgraphs, then one arrives at the key axiom (A1&3&4) for dual polar graphs.

THEOREM 7.6. [**22**] *Quasi-median graphs with their apex operations are exactly the discrete ternary algebras satisfying (M1), (M5), and (M7).*

In view of this result every finite set of vertices in a quasi-median graph G generates a finite subalgebra of the apex algebra and hence yields a quasi-median graph in its own right. It is not yet clear though whether all possible algebraic descriptions of finite quasi-median graphs are also equivalent in the non-discrete case and, in particular, whether the free four-generated algebra is finite (and hence quasi-median) in each case [**36**]. The free quasi-median algebras with four and five generators have been described in [**150**]: they have 868 (cf. [**30**]) and 97,916,730,716,165 elements, respectively. The growth with the number of generators is thus enormous compared to the corresponding free median algebras, where the 5-generated one has only 81 elements [**45, 150**]. The total number of vertices of a subalgebra that is generated by some subset X of the vertex set of a Hamming graph is easy to determine by analyzing the projections of X to each pair of factors [**14**]. An Euler-type formula for the numbers of k-regular Hamming subgraphs has been established in [**66**].

A finite Hamming space, that is, a finite Hamming graph with positive coordinate weights is the natural host model for molecular sequences such as DNA or amino acid sequences. In this context, one considers a set X of k aligned sequences, which are forced to have the same length by inserting some gaps (encoded by "$-$") where necessary. In the case of DNA, the positions of the k sequences thus carry letters from the alphabet $\{A,G,C,T,-\}$, which is regarded as a complete graph. Then the Hamming graph K^X may be weighted coordinatewise, whence a mismatch at the i-th coordinate contributes $\lambda_i > 0$ to the weighted Hamming distance between two sequences. The structure of the data set X comprising aligned DNA sequences embedded in K^X can be elucidated by investigating the quasi-median closure of X in the corresponding Hamming space [**12, 14**]. For the task of phylogenetic analysis, one is typically interested in trees that connect the observed data and are optimal with regard to length (or likelihood). In particular, the maximum parsimony problem [**158**] corresponds to the (NP-hard [**119**]) Steiner problem in Hamming space, which has subsets of the vertex set of a Hamming space as its instances and requires a shortest connection ("minimum Steiner tree") in the Hamming space by using suitable additional vertices that serve as branching points ("Steiner points"). With respect to unit weights per coordinate, every quasi-median subgraph of the Hamming graph trivially contains a minimum Steiner tree by way of retraction. This can also be seen more directly by employing the apex operation in a two-phase labelling algorithm for finding suitable Steiner points, so that the result immediately carries over to the weighted situation.

PROPOSITION 7.7. [**41**] *The quasi-median closure of any set X in a finite Hamming space includes a minimal Steiner tree. More generally, every X-tree has a shortest-length realization within the quasi-median closure of X in the Hamming space.*

8. l_1-Graphs

Some structural properties of graphs, especially for Hamming graphs (and hypercubes, in particular), carry over to their isometric subgraphs - a theme that has been emphasized in the books [**94**] and [**125**]. In the former book the polyhedral view is predominant, whereas in the latter one the so-called canonical embedding of graphs (see below) receives particular attention.

A dissimilarity function d on $X = \{1, \ldots, n\}$ is said to be *hypermetric* if it satisfies the inequalities $\sum_{1 \le i < j \le n} b_i b_j d(i,j) \le 0$ for all integers $b_1, \ldots, b_n$ with $\sum_{i=1}^{n} b_i = 1$ $(n \ge 3)$ [**94**]. In particular, the triangle inequalities are captured by b coefficients from $\{-1, 0, 1\}$ such that exactly three coefficients are nonzero. The *pentagonal inequalities*, for example, arise with b coefficients from $\{-1, 0, 1\}$ such that exactly five coefficients are nonzero. Obviously, every split pseudometric (alias cut semimetric) $\delta_{A,B}$ associated with a split (alias bipartition or cut) $\{A, B\}$ of X is hypermetric (where $\delta_{A,B}$ can be thought of being lifted from the metric of the graph K_2 for which one vertex is labeled with the points from A and the other vertex with the points from B). Therefore hypermetricity is a necessary condition for l_1-embeddability.

An l_1-*(pseudo)metric* d on a finite set X is any positive linear combination of split pseudometrics. Thus, a finite metric space (X, d) is isometrically embeddable in $\mathbb{R}^n$ endowed with the l_1-norm exactly when d is an l_1-metric. For an l_1-metric, the positive linear combinations of split pseudometrics need not be unique as the example of the K_4 metric shows. In the case of uniqueness, one speaks of l_1-*rigid (pseudo)metrics* [**93, 94**].

A *totally decomposable (pseudo)metric* d on a finite set X [**27**] is any positive linear combination of split metrics with respect to triplewise weakly compatible splits: three splits $\{A_i, B_i\}$ of X $(i = 1, 2, 3)$ are said to be *weakly compatible* if $A_1 \cap A_2 \cap A_3 \neq \emptyset$ implies $B_1 \cap B_2 \cap B_3 = B_i \cap B_j$ for some i, j, that is, the traces of the three splits $\{A_i, B_i\}$ $(i = 1, 2, 3)$ on any 4-point subset $\{t, u, v, w\}$ of X do not constitute the three nontrivial 4-point splits $\{\{t,u\},\{v,w\}\},\{\{t,v\},\{u,w\}\}$, and $\{\{t,w\},\{u,v\}\}$. The weighted system of weakly compatible splits can then be retrieved from d [**27**].

We briefly say that a graph is a *hypermetric graph* or an l_1-*graph* or is l_1-*rigid* when its shortest-path metric d is a hypermetric or an l_1-metric or is l_1-rigid, respectively. Despite the embeddability results reported below, one still lacks a structural characterization of general l_1-graphs in terms of forbidden isometric subgraphs [94, p. 326].

8.1. Canonical isometric embedding. Even though a graph may be isometrically embedded into different Cartesian products, there is a finest choice, viz., the *canonical isometric embedding* [**117**]. Two edges xy and uv of a graph $G = (V,E)$ are in the Djokovic-Winkler relation Θ if $d(x,u)+d(y,v) \neq d(x,v)+d(y,u)$. This relation on the edge set E is trivially reflexive and symmetric but not necessarily transitive. Let Θ^* denote the transitive closure of Θ, and let $E_1,\dots,E_k$ be the blocks of Θ^*. Let G_i $(i = 1,\dots,k)$ be the graph having the connected components of the graph $(V, E-E_i)$ as its vertices, with two different components being adjacent when connected by an edge from E_i; alternatively, one can view G_i as the graph resulting from the contraction of all edges in $E-E_i$. This contraction induces a natural projection α_i from G onto G_i.

THEOREM 8.1. [**117**] *The map* $\alpha : G \to G_1\square\dots\square G_k$ *defined by* $\alpha(v) = (\alpha_1(v),\dots,\alpha_k(v))$ *constitutes an isometric embedding, which is the finest isometric embedding of* G *into a Cartesian product (whence the name "canonical").*

The previous construction can be turned into an efficient algorithm for canonical isometric embedding of a graph into a Cartesian product of indecomposable factors, because the transitive closure of Θ can be computed in polynomial time [**109, 110, 117**].

Every edge ab of a graph $G = (V,E)$ induces a partition of V into $V = a/b \cup a\backslash b \cup a|b$, where $a|b = \{v \in V : d(v,a) = d(v,b)\}$. Clearly, if the graph G is bipartite, then $a|b = \emptyset$ for each $ab \in E$.

THEOREM 8.2. *Let* G *be a graph.*

(a) [**77**] *G can be isometrically embedded into a Hamming graph if and only if the sets* $a/b, a\backslash b, a/b \cup a|b$ *and* $a\backslash b \cup a|b$ *are convex for all edges* ab *of* G.

(b) [**175**] *G can be isometrically embedded into a Hamming graph* $H_{3,3,\dots,3}$ *(Cartesian power of* K_3*) if and only if the relation* Θ *is transitive.*

THEOREM 8.3. *For a bipartite graph* G *the following conditions are equivalent:*

(a) *G can be isometrically embedded into a hypercube;*

(b) [**56**] *G is an* l_1*-graph;*

(c) [**175**] *G is* l_1*-rigid;*

(d) [**154**] *G is hypermetric;*

(e) [**7, 154**] *the pentagonal inequalities are satisfied in* G;

(f) [**97**] $a/b, a\backslash b$ *are convex for each edge* ab *of* G;

(g) [**97**] *the relation* Θ *is transitive on the edge set of* G.

The dimension k of the smallest hypercube into which a finite graph $G = (V,E)$ may be isometrically embedded equals the number of blocks $E_1,\dots,E_k$ of the equivalence relation Θ. While it is NP-complete to decide if G isometrically embeds into the Cartesian product of three trees, it is possible to determine in polynomial time the minimum dimension m of the integer lattice $\mathbb{Z}^m$ into which G isometrically embeds [**106**]: m equals k minus the maximum matching in the graph Γ whose vertices are the connected components of the graphs $(V, E - E_i), i = 1 \ldots, k$, where two such components are adjacent in Γ if and only if they are not disjoint but together cover V.

Further characterizations of isometric subgraphs of Hamming graphs are presented in [**173**] and [**63**]. For other characterizations of isometric subgraphs of hypercubes see [**94, 115, 125, 176**]. Recognition algorithms for graphs that are isometrically embeddable into (hypercubes and) Hamming graphs are surveyed in [**125**].

In relation with the canonical decomposition of graphs, the following task arises: given a class of graphs determine the factors in the canonical isometric embedding of its members. For weakly median graphs, these factors are the prime weakly median graphs described in Section 5. More generally, the prime fiber-complemented graphs are exactly the factors occurring in the canonical decomposition of fiber-complemented graphs [**73**].

8.2. Scale embeddings. A triangle K_3 with its metric d cannot be embedded into a hypercube but the doubled distance $2d$ can. This sort of property turns out to be a general feature of l_1-graphs. A graph $G = (V, E)$ is *scale λ embeddable* into a graph $H = (W, F)$ for some positive integer λ if there exists a mapping $\varphi : V \to W$ such that $d_H(\varphi(u), \varphi(v)) = \lambda d_G(u, v)$ for all vertices $u, v \in V$ [**94**]. In the particular case $\lambda = 1$ we obtain the notion of isometric embedding. For instance, Hamming graphs as well as half-cubes and the Johnson graphs are scale 2 embedded in hypercubes. On the other hand, the m-octahedron $K_{m\times 2}$ is scale embeddable into a hypercube, but its scale grows with m.

PROPOSITION 8.4. [**5**] *G is an l_1-graph if and only if it admits a scale embedding into a hypercube.*

This result paves the way to a description of l_1-graphs and, more generally, hypermetric graphs via the canonical embedding. The extra indecomposable factors that come into play when passing from l_1-graphs to the more general hypermetric graphs involve the Gosset graph G_{56}, which is the skeleton of the 7-dimensional Gosset polytope: its 56 vertices are obtained by putting signs in the eight possible ways to the characteristic vectors of the Fano plane; see [**94, 166**] for details.

THEOREM 8.5. *Let G be a graph.*

(a) [**92, 166**] *G is hypermetric if and only if G is an isometric subgraph of a Cartesian product of half-cubes, hyperoctahedra and copies of the Gosset graph.*
(b) [**92, 159**] *G is an l_1-graph if and only if G is an isometric subgraph of a Cartesian product of half-cubes and hyperoctahedra.*
(c) [**86**] *Every K_5-free l_1-graph is isometrically embeddable into a half-cube, and every K_4-free l_1-graph is l_1-rigid. In particular, every planar l_1-graph is isometrically embeddable into a half-cube.*

The proof of (b) given in [**159**] is graph-theoretical and can be adapted to test in time $O(\#V\#E)$ whether a given graph $G = (V, E)$ is an l_1-graph and whether it can be isometrically embedded into a half-cube; see [**95**]. This contrast with the NP-completeness of recognizing l_1-embeddable finite metrics. The proof of (a) given in [**166**] and the alternative proof of (b) given in [**92**] are geometrical and exploit some beautiful connections between hypermetric graphs and Delaunay polytopes of root systems, which can be sketched as follows (notice that [**86**] presents two proofs of (c), one graph-theoretical and the other geometrical). According to a result of Assouad [**4**], a finite metric space (V, d) is hypermetric if and only if V can be mapped to the vertices of the Delaunay polytope P_d of some lattice L_d of $\mathbb{R}^k$ so that $(V, \sqrt{d})$ is isometrically l_2-embedded into $\mathbb{R}^k$ (for basic definitions related to lattices and Delaunay polytopes, see [**94**]). By a result of [**166**], if the hypermetric space (V, d) comes from a graph $G = (V, E)$, then the Delaunay polytope P_d associated with (V, d) is generated by a root lattice L_d and G is isometrically embedded in the underlying graph (1-skeleton) of P_d. Every root lattice L_d is the direct sum of irreducible root lattices, and, by a well-known result of Witt, the irreducible root lattices have one of the forms A_n $(n \geq 0)$, D_n $(n \geq 4)$, E_8, E_7, and E_6. The underlying graphs of the Delaunay polytopes of A_n are the Johnson graphs $J_{n+1,k}$ with $1 \leq k \leq \lfloor \frac{n+1}{k} \rfloor$, those of D_n are the hyperoctahedra $K_{n\times 2}$ and the half-cubes $\frac{1}{2}H_n$, those of E_8 are K_9 and $K_{8\times 2}$, those of E_7 are K_8 and the Gosset graph G_{56}, and that of E_6 is the Schläfli graph G_{27}. To conclude the proof, it remains to notice that G_{27} is an isometric subgraph of G_{56} and that G_{27} and G_{56} are not l_1-graphs.

Some particular classes of plane graphs which can be isometrically embedded into a half-cube are worth mentioning explicitly:

(4,4)-graphs [**152**] *are plane graphs in which all inner faces have lengths* ≥ 4 *and all inner vertices have degrees* ≥ 4;

(6,3)-graphs [**87**] *are plane graphs in which all inner faces have length* ≥ 6 *and all inner vertices have degrees* ≥ 3;

(3,6)-graphs [**87**] *are plane graphs in which (all inner faces have lengths* ≥ 3 *and) all inner vertices have degrees* ≥ 6.

Proposition 8.6. *Let G be a plane graph in which the degrees of all inner vertices are at least p and all inner faces have lengths at least q.*

(a) [**118, 137**] *If each inner face of G with k sides is replaced by a regular k-gon of the Euclidean plane with side length 1, then the resulting cell complex $|G|$ is CAT(0) exactly when G is either a (4,4)-, or a (6,3)- , or a (3,6)-graph.*
(b) [**87, 152**] *The (4,4)-, (6,3),-, and (3,6)-graphs G are scale 2 embeddable into hypercubes.*

The proof of (b) rests on the analysis of the so-called alternating splits in those plane graphs [**86, 152**], which are constructed in the following way. Give every odd face (cycle) of the plane graph G a clockwise orientation. Start with an edge u_0v_0 on the boundary of the outer face of G and keep selecting the edges opposite in the next inner face F_i that contains the edge u_iv_i selected last but no other selected edge (- if there is no such face, u_iv_i is on the boundary, and the edge selection is completed). If F_i is odd, there are two candidate edges (incident with the unique vertex opposite to u_iv_i): select the one that comes either first or second in the chosen orientation, depending on the last choice for an odd face in the series of faces processed so far. Namely, let first and second choices alternate in the series

$F_0, F_1, \ldots, F_i$ (ignoring even faces, which are unoriented). If at least one odd face belongs to this series, then there are thus exactly two feasible selections starting from u_0v_0. In either case, the corresponding *zone* Z is defined as the union of all faces $F_0, F_1, \ldots, F_k$ that contain a pair of selected cut edges. Removing these edges from G results in two connected components, defining the *alternating split* $\{A, B\}$ associated with Z. Under the hypothesis that G is (4,4) or (6,3) or (3,6), the strip-like zone $Z = Z(A, B)$ as well as the split parts A and B are convex. It then turns out that every edge of G is separated by exactly two alternating splits (if the associated zone includes at least one odd face) or just one alternating split (separating only even faces), thus yielding the desired scale 2 embedding.

The (4,4)-, (3,6)-, and (6,3)-graphs have been introduced in the context of combinatorial group theory [**137**]. The following maximum principle for such graphs G was established in [**136**]: for any vertex v of G, the vertices x such that the interval $I(v, x)$ is not properly contained in any other interval $I(v, x')$ all belong to the outer face of G; in other words, every shortest path of G can be extended to a shortest path having both end vertices on the outer face of G. This result actually holds for all non-positively curved plane graphs [**49, 50**].

8.3. Tope graphs, cellular graphs, and benzenoids. Isometric subgraphs of hypercubes naturally arise in several contexts. We now briefly mention three classes of such graphs with specific features, which are derived from oriented matroids or 2-dimensional cubical complexes, respectively.

An (affine) *arrangement of hyperplanes* in $\mathbb{R}^k$ is a finite family $\mathcal{A} = (H_i)_{i \in I}$ of (affine) hyperplanes in $\mathbb{R}^k$. One can choose a "positive side" for every hyperplane in $\mathcal{A}$, which is realized by a sign vector $s(x) \in \{+, -, 0\}^I$ for every point $x \in \mathbb{R}^n$, where $x_i(e)$ denotes whether x is on the positive side of H_i, on the negative side, or lies on H_i. The set of all points $x \in \mathbb{R}^k$ having the same sign vector s forms a *cell* in the decomposition of $\mathbb{R}^k$ induced by $\mathcal{A}$. The maximal (i.e., the n-dimensional) cells are called *regions* (or *topes*). The *tope graph* (or *graph of regions*) $\mathcal{T}(\mathcal{A})$ of an arrangement $\mathcal{A}$ has as vertices the set of regions of $\mathcal{A}$ and as edges the pairs of regions which share a common facet. Tope graphs of central arrangements, i.e., arrangements of hyperplanes through the origin, are the 1-skeletons of zonotopes (n-polytopes which can be expressed as the Minkowski sum of n line segments); for details see the book [**55**] on oriented matroids. Tope graphs of any arrangements of n hyperplanes embed isometrically into the hypercube H_n, namely, the graph distance between two regions equals the number of hyperplanes of $\mathcal{A}$ which separate these regions [**54**] (for similar results in the theory of Coxeter matroids, see [**58**]). Using this property and the uniqueness of the isometric embedding into hypercubes, it can be shown that an arrangement of hyperplanes is uniquely determined (up to reorientation) by its unlabeled tope graph [**54**]. As a corollary, two zonotopes are combinatorially equivalent if and only if their 1-skeletons are isomorphic graphs. These results can be extended to the graphs of topes (i.e., maximal covectors) of oriented matroids (for definitions, results, and references, see Section 4.1 and Exercises to Chapter 4 of [**55**]). The converse question of characterizing the isometric subgraphs of hypercubes which are tope graphs of oriented matroids is still open; the rank 3 case has been settled in [**111**]: these graphs are the planar antipodal isometric subgraphs of hypercubes.

Graphs for which the shortest-path metrics are totally decomposable constitute certain low-dimensional polyhedral complexes. In the finite bipartite case, they are known to be built up from cycles (their "cells"), whence the name "cellular". Since graphs are discrete objects, we may extend the notion of total decomposability to the infinite case: then a graph G is *cellular* if there exists a positively weighted system of triplewise weakly compatible splits such that for any two vertices u and v the distance $d(u,v)$ can be expressed as the sum of all split weights for those (finitely many) splits that separate u and v (i.e., for which $\{u,v\}$ is not contained in a split part). Because the split weights can be reconstructed from the metric d by comparing distance sums for quartets of vertices, the split weights are equal to $\frac{1}{2}$ or 1. Therefore all cellular graphs are scale 2 embedded in hypercubes.

THEOREM 8.7. [**17**] *For a bipartite graph $G=(V,E)$ with at least two vertices, the following conditions are equivalent:*

(a) *G is cellular;*
(b) *every isometric cycle of G is gated, and G does not contain any three isometric cycles C_1, C_2, C_3 and three distinct edges e_1, e_2, e_3 sharing a common vertex such that e_i belongs to C_j exactly when $i \neq j$;*
(c) *G can be obtained from a collection of single edges and even cycles by successive gated amalgamations;*
(d) *the system of splits $\{u/v, u\backslash v\}$ is triplewise weakly compatible.*

Every finite cellular bipartite graph different from a single vertex, edge, or even cycle contains a gated cutset that is a tree.

To establish this result it is shown, among other things, that a finite bipartite graph in which every isometric cycle is gated is isometrically embeddable into a hypercube. Note that if one replaces every gated (i.e., isometric) cycle of a cellular graph by a regular polygon with side length 1, then the resulting 2-dimensional cell complex is CAT(0). The precise structure of all (not necessarily bipartite or finite) cellular graphs has not yet been determined. Among cellular graphs are, for example, the 3-octahedron, any Cartesian product of two block graphs (being scale 2 embeddable into the Cartesian product of two trees), and certain planar graphs:

PROPOSITION 8.8. *(4,4)-graphs are cellular.*

The collection of alternating splits providing the l_1-embedding of a (4,4)-graph G actually consists of triplewise weakly compatible splits. To show this, consider four arbitrary vertices t,u,v,w and an alternating split $\{A,B\}$ such that $t,u \in A$ and $v,w \in B$. As we noticed above, the associated zone $Z = Z(A,B)$ is a strip (a path of faces) consisting of two convex paths $\partial A = A \cap Z$ and $\partial B = B \cap Z$. These three sets are pseudo-gated in the following sense. A set Y is *pseudo-gated* in a graph if for every vertex x outside Y there exists a simplex π_x (the *pseudo-gate*) in Y such that every shortest path from x to a vertex of Y passes through π_x; cf. [**34**]. The pseudo-gate π_x of any vertex x of G in Z comprises a single vertex or an edge [**87**]. For each vertex x of A and each vertex y of B there exists a shortest (x,y)-path traversing both pseudo-gates π_x and π_y. Therefore, for any mutual location of the pseudo-gates π_t, π_u on ∂A and the pseudo-gates π_v, π_w on ∂B, we can select a vertex from $\{t,u\}$, say u, and a vertex from $\{v,w\}$, say v, such that the intervals $I(u,v)$ and $I(t,w)$ intersect. Since the alternating splits of G are convex, there is no such split separating $\{u,v\}$ from $\{t,w\}$.

In the context of chemical graph theory, a *benzenoid* is a particular (6,3)-graph, viz., a plane graph in which all inner faces are regular hexagons and all inner vertices have degree 3. These graphs are not necessarily cellular but they admit isometric embeddings into hypercubes [**132**] and, in particular, into the Cartesian product of three trees [**81**]. With the latter coordinatization in hand, one can compute the Wiener index (the sum of pairwise distances) of benzenoids in optimal time [**125**].

8.4. Lopsided sets. An important, though elementary, feature of median subgraphs G of n-cubes (the n-th Cartesian powers of K_2) is that every k-cube image under any k-fold projection has, a forteriori, a k-cube within G as a pre-image [**169**]. This (weak) projectivity property of hypercubes is in fact behind the basic counting formula that equates the number of vertices of a median graph G with the number of those k-fold projections ($k = 0, 1, \dots, n$) to the K_2 powers which are surjective [**103**], alluded to in Section 2.1. Conversely, when this projectivity property of all k-cubes is stipulated, then one arrives at a class of isometric subgraphs of hypercubes, closed under taking induced subgraphs of complementary vertex sets, which is thus more general than finite median graphs but still has canonical properties with regard to geometric l_1 realizations. These constitute the lopsided sets introduced in [**134**] in order to investigate the subgraphs $\mathcal{S}$ of the n-cube with vertex set $\{0,1\}^n$ which encode the intersection pattern of a given convex set K in n-dimensional Euclidean space with the (closed) orthants of $\mathbb{R}^n$. Specifically, a vertex x of H_n belongs to $\mathcal{S} = \mathcal{S}(K)$ exactly when K meets the orthant of $\mathbb{R}^n$ in which the ±1 ("sign") vector $2x-\mathbf{1}$ lies. Every subgraph $\mathcal{S}$ of H_n which arises in this way is hereditarily asymmetric in the following sense (whence the name "lopsided") [**134**]:

(Lawrence condition) *each face F of H_n which intersects $\mathcal{S}$ properly includes a vertex pair u, v diametrical in F such that $\mathcal{S}$ contains exactly one of u and v.*

One can then define $\mathcal{S}$ to be *lopsided* if it satisfies the preceding characteristic property. It is obvious that the Lawrence condition rejects any minimal obstruction to isometry of $\mathcal{S}$ (realized in some face F of dimension > 1 as a diametrical vertex pair), whence lopsided sets induce isometric subgraphs of H_n. Moreover, a hereditary version of isometry ("superisometry" [**23**]) then characterizes lopsidedness. To express this, factor the n-cube 2^X hosting $\mathcal{S}$ into 2^Y and 2^{X-Y} with respect to any subset Y of $X = \{1, \dots, n\}$ and let the subset $\mathcal{S}^Y$ of 2^{X-Y} encode the location of all 2^Y fibers within $\mathcal{S}$:

$$\mathcal{S}^Y := \{t \in 2^{X-Y} : \text{ every extension } s \in 2^X \text{ of } t \text{ belongs to } \mathcal{S}\}.$$

The coordinate subsets Y of X for which some 2^Y fiber exists in $\mathcal{S}$ form a simplicial complex:

$$\underline{\mathcal{X}}(\mathcal{S}) := \{Y \subseteq X : \mathcal{S}^Y \text{ is nonempty }\}.$$

The sets $\mathcal{S}^Y$ and $\underline{\mathcal{X}}(\mathcal{S})$ can be compared to the sets

$$\mathcal{S}_Y := \mathcal{S}|_{X-Y} \text{ and } \overline{\mathcal{X}}(\mathcal{S}) := \{Y \subseteq X : \mathcal{S}_Y = 2^{X-Y}\},$$

which determines the image of $\mathcal{S}$ under the projection onto 2^{X-Y} and collects the coordinate subsets Y such that the projections onto 2^{X-Y} are surjective, respectively. To give an example, let $X = \{1, 2, 3\}$ and let $\mathcal{S}$ be an isometric 6-cycle in 2^X, then $\mathcal{S}_{\{e\}}$ yields a 4-cycle but $\mathcal{S}^{\{e\}}$ comprises only a diametrical pair of vertices in this 4-cycle for any $e \in X$.

The original definition of a lopsided set in [**134**] can be expressed in terms of the simplicial complexes $\overline{\mathcal{X}}(\mathcal{S})$ and $\underline{\mathcal{X}}(\mathcal{S})$ as the requirement that for each $Y \subseteq X$ the alternative $Y \in \underline{\mathcal{X}}(\mathcal{S})$ or $Y \notin \overline{\mathcal{X}}(\mathcal{S})$ holds. One of the basic observations is that the inequality $\#\underline{\mathcal{X}}(\mathcal{S}) \leq \#\mathcal{S} \leq \#\overline{\mathcal{X}}(\mathcal{S})$ holds for all subsets $\mathcal{S}$ of 2^X [**23**]. Any equality here then characterizes lopsidedness. In fact, the condition $\#\mathcal{S} = \#\overline{\mathcal{X}}(\mathcal{S})$ ("$\mathcal{S}$ is ample") led to a rediscovery of lopsided sets in the context of overlapping clustering [**100**]. A systematic investigation of the interplay between the subsets $\mathcal{S}$ of the n-cube and the associated simplicial complexes $\underline{\mathcal{X}}(\mathcal{S})$ and $\overline{\mathcal{X}}(\mathcal{S})$ then leads to an amazing number of equivalent conditions for lopsidedness. The following theorem then presents only a small sample of characteristic properties established in [**23**].

THEOREM 8.9. [**23**] *For a subset $\mathcal{S}$ of 2^X the following conditions are equivalent:*

(a) *$\mathcal{S}$ is lopsided;*
(b) *$2^X - \mathcal{S}$ is lopsided;*
(c) *$\mathcal{S}^Y$ is isometric for all $Y \subseteq X$;*
(d) *$\#\mathcal{S} = \#\overline{\mathcal{X}}(\mathcal{S})$;*
(e) *$\#\mathcal{S} = \#\underline{\mathcal{X}}(\mathcal{S})$;*
(f) *$\underline{\mathcal{X}}(\mathcal{S}) = \overline{\mathcal{X}}(\mathcal{S})$;*
(g) *$\mathcal{S}$ is connected, and $\mathcal{S}^{\{e\}}$ is lopsided for every $e \in X$;*
(h) *$\mathcal{S}$ is isometric, and both $\mathcal{S}_{\{e\}}$ and $\mathcal{S}^{\{e\}}$ are lopsided for some $e \in X$.*

As to the geometric interpretation of lopsided sets, it was already noted in [**134**] that not every lopsided set encodes the orthant intersection pattern for a convex set in Euclidean space. It comes close, though. In order to have a full geometric representation, one has to resort to a weaker concept ("ortho-convexity") of convexity. For a subset $\mathcal{S}$ of 2^X, let $|\mathcal{S}|$ be the polyhedral cubical complex obtained by replacing all faces of $\mathcal{S}$ by solid cubes. If $\mathcal{S}$ is connected, then $|\mathcal{S}|$ is connected as well, and therefore can be endowed with an intrinsic l_1-metric $d_{|\mathcal{S}|}$. The resulting metric space $(|\mathcal{S}|, d_{|\mathcal{S}|})$ is geodesic but not necessarily a metric subspace of $(\mathbb{R}^X, ||\cdot||_1)$. For example, if $\mathcal{S}$ comprises the six vertices of an isometric 6-cycle in the 3-cube, then $|\mathcal{S}|$ is a solid 6-cycle of $\mathbb{R}^3$. The l_1-distance between the midpoints of two opposite sides of this cycle is 2, while the intrinsic l_1-distance between the same points is 3. In fact, l_1-isometry of $|\mathcal{S}|$ is yet another characteristic feature of lopsidedness of $\mathcal{S}$:

THEOREM 8.10. [**24**] *For a subset $\mathcal{S}$ of 2^X the following conditions are equivalent:*

(a) *$\mathcal{S}$ is lopsided;*
(b) *$|\mathcal{S}|$ endowed with the intrinsic l_1-metric $d_{|\mathcal{S}|}$ is a metric subspace of the metric space $(\mathbb{R}^X, ||\cdot||_1)$;*
(c) *$\mathcal{S}$ is isometric in 2^X and every face of $|\mathcal{S}|$ is a gated subset of $(|\mathcal{S}|, d_{|\mathcal{S}|})$;*
(d) *$\mathcal{S}$ encodes the orthant intersection pattern for some geodesic metric subspace K of $(\mathbb{R}^n, ||\cdot||_1)$, that is, $x \in \mathcal{S}$ exactly when the orthant determined by the corresponding sign vector $2x - \mathbf{1}$ also includes a point from K.*

In the preceding condition (d), $K = |\mathcal{S}|$ can actually be chosen. By Theorem 8.9 or 8.10 it is clear that every median subgraph $\mathcal{S}$ of the cube 2^X is a lopsided set. Specifically, a subset $\mathcal{S}$ of 2^X forms a median subgraph of this $\#X$-cube exactly when $\underline{\mathcal{X}}(\mathcal{S}) = \overline{\mathcal{X}}(\mathcal{S})$ is a flag complex. There are many other interesting classes of lopsided sets for which the associated simplicial complexes are not necessarily flag complexes. For example, tope graphs of arrangements of pseudolines such that no more than two pseudolines meet at any intersection point constitute lopsided sets [**134**]. Antimatroids (i.e., closure systems satisfying the anti-exchange property) on a finite set X, also known under the name convex geometries [**105**], are instances of lopsided sets (when passing to the characteristic maps). The requirement that the set X itself belongs to the antimatroid can be dropped without losing lopsidedness: these set systems over X are then referred to as conditional antimatroids [**23**]. The set of all partial orders on a finite set M, for example, constitutes a conditional antimatroid $\mathcal{S}$ on $X = M^2$. Its geometric realization $|\mathcal{S}|$ could then in principle provide a framework for fuzzy partial orders.

Acknowledgments. We are grateful to the editors (E. Goodman, J. Pach, and R. Pollack) of this special volume of AMS Contemporary Mathematics for inviting us to contribute our survey. We are grateful to J. Matoušek for suggesting us to write this survey of metric graph theory. Finally, we would like to thank the two referees of a version of this survey initially prepared for a special issue of DCG (which never came into being) for careful reading and numerous helpful suggestions.

References

[1] R.P. Anstee, M. Farber, *On bridged graphs and cop-win graphs,* J. Combin. Th. Ser. B **44** (1988), 22–28.
[2] F. Ardila, C.J. Klivans, *The Bergman complex of a matroid and phylogenetic trees,* J. Combin. Th. Ser. B **96** (2006), 38–49.
[3] N. Aronszajn, P. Panitchpakdi, *Extensions of uniformly continuous transformations and hyperconvex metric spaces,* Pacific J. Math. **6** (1956), 405–439.
[4] P. Assouad, *Sous espaces de L^1 et inégalités hypermétriques,* Comptes Rendus de l'Académie des Sciences de Paris, **294**(A), 1982, 439–442.
[5] P. Assouad, M. Deza, *Espaces métriques plongeables dans un hypercube: aspects combinatoires,* Ann. Discr. Math. **8** (1980), 197–210.
[6] S.P. Avann, *Metric ternary distributive semi-lattices,* Proc. Amer. Math. Soc. **12** (1961), 407–414.
[7] D. Avis, *Hypermetric spaces and the Hamming cone,* Canad. J. Math. **33** (1981), 795–802.
[8] H.-J. Bandelt, *Retracts of hypercubes,* J. Graph Th. **8** (1984), 501–510.
[9] H.-J. Bandelt, *Networks with Condorcet solutions,* Europ. J. Oper. Res. **20** (1985), 314–326.
[10] H.-J. Bandelt, *Hereditary modular graphs,* Combinatorica **8** (1988), 149–157.
[11] H.-J. Bandelt, *Recognition of tree metrics,* SIAM J. Discr. Math. **3** (1990), 1–6.
[12] H.-J. Bandelt, *Phylogenetic networks,* Verh. naturwiss. Ver. Hamburg **34** (1994), 51–71.
[13] H.-J. Bandelt, *Combination of data in phylogenetic analysis,* Plant Syst. Evol. [Suppl.] **9** (1995), 355–361.
[14] H.-J. Bandelt, *Exploring reticulate patterns in DNA sequence data,* in Plant Species-Level Systematics: New Perspectives on Pattern & Process, F.T. Bakker, L.W. Chatrou, B. Gravendeel, P.B. Pelser (eds.), Regnum Vegetabile, vol. 142, Koeltz, Königstein, 2005, pp. 245–270.
[15] H.-J. Bandelt, V. Chepoi, *A Helly theorem in weakly modular space,* Discr. Math. **126** (1996), 25–39.
[16] H.-J. Bandelt, V. Chepoi, *Graphs of acyclic cubical complexes,* Europ. J. Combin. **17** (1996), 113–120.
[17] H.-J. Bandelt, V. Chepoi, *Cellular bipartite graphs,* Europ. J. Combin. **17** (1996), 121–134.
[18] H.-J. Bandelt, V. Chepoi, *Decomposition and l_1-embedding of weakly median graphs,* Europ. J. Combin. **21** (2000), 701–714.
[19] H.-J. Bandelt, V. Chepoi, *Graphs with connected medians,* SIAM J. Discr. Math. **15** (2002), 268–282.
[20] H.-J. Bandelt, V. Chepoi, *1-Hyperbolic graphs,* SIAM J. Discr. Math. **16** (2003), 323–334.

[21] H.-J. Bandelt, V. Chepoi, *The algebra of metric betweenness I: subdirect representation and retraction,* Europ. J. Combin. (to appear).
[22] H.-J. Bandelt, V. Chepoi, *The algebra of metric betweenness II: equational characterization of weakly median graphs* (submitted).
[23] H.-J. Bandelt, V. Chepoi, A. Dress, J. Koolen, *Combinatorics of lopsided sets,* Europ. J. Combin. **27** (2006), 669–689.
[24] H.-J. Bandelt, V. Chepoi, A. Dress, J. Koolen, *Geometry of lopsided sets* (in preparation).
[25] H.-J. Bandelt, V. Chepoi, A. Karzanov, *A characterization of minimizable metrics in the multifacility location problem,* Europ. J. Combin. **21** (2000), 715–725.
[26] H.-J. Bandelt, A. Dählmann, H. Schütte, *Absolute retracts of bipartite graphs,* Discr. Appl. Math. **16** (1987), 191–215.
[27] H.-J. Bandelt, A.W.M. Dress, *A canonical decomposition theory for metrics on a finite set,* Adv. Math. **92** (1992), 47–105.
[28] H.-J. Bandelt, M. Farber, P. Hell, *Absolute reflexive retracts and absolute bipartite retracts,* Discr. Appl. Math. **44** (1993), 9–20.
[29] H.-J. Bandelt, J. Hedlíková, *Median algebras,* Discr. Math. **45** (1983), 1–30.
[30] H.-J. Bandelt, K.T. Huber, V. Moulton, *Quasi-median graphs from sets of partitions,* Discr. Appl. Math. **122** (2002), 23–35.
[31] H.-J. Bandelt, H.M. Mulder, *Distance-hereditary graphs,* J. Combin. Th. Ser. B **41** (1986), 182–208.
[32] H.-J. Bandelt, H.M. Mulder, *Pseudo-modular graphs,* Discr. Math. **62** (1986), 245–260.
[33] H.-J. Bandelt, H.M. Mulder, *Cartesian factorization of interval-regular graphs having no long isometric odd cycles,* in Graph Theory, Combinatorics, and Applications, vol. 1, Y. Alavi, G. Chartrand, O.R. Oellermann, A.J. Schwenk (eds.), Wiley, New York, 1991, pp. 55–75.
[34] H.-J. Bandelt, H.M. Mulder, *Helly theorems for dismantlable graphs and pseudo-modular graphs,* in Topics in Combinatorics and Graph Theory, R. Bodendiek, R. Henn, (eds.), Physica-Verlag, Heidelberg, 1990, pp. 65–71.
[35] H.-J. Bandelt, H.M. Mulder, V. Soltan, *Weak Cartesian factorization with icosahedra, 5-wheels, and subhyperoctahedra as factors* (submitted).
[36] H.-J. Bandelt, H.M. Mulder, E. Wilkeit, *Quasi-median graphs and algebras,* J. Graph Th. **18** (1994), 681–703.
[37] H.-J. Bandelt, L. Quintana-Murci, A. Salas, V. Macaulay, *The fingerprint of phantom mutations in mitochondrial DNA data,* Am. J. Hum. Genet. **71** (2002), 1150–1160.
[38] H.-J. Bandelt, E. Prisner, *Clique graphs and Helly graphs,* J. Combin. Th. Ser. B **51** (1991), 34–45.
[39] H.-J. Bandelt, E. Pesch, *Dismantling absolute retracts of reflexive graphs,* Europ. J. Combin. **10** (1989), 211–220.
[40] H.-J. Bandelt, E. Pesch, *Efficient characterizations of n-chromatic absolute retracts,* J. Combin. Th. Ser. B **53** (1991), 5–31.
[41] H.-J. Bandelt, A. Röhl, *Quasi-median hulls in Hamming space are Steiner hulls* (submitted).
[42] H.-J. Bandelt, M. A. Steel, *Symmetric matrices representable by weighted trees over a cancellative abelian monoid,* SIAM J. Discr. Math. **8** (1995), 517–525.
[43] H.-J. Bandelt, M. van de Vel, *A fixed cube theorem for median graphs,* Discr. Math. **62** (1987), 129–137.
[44] H.-J. Bandelt, M. van de Vel, *Embedding topological median algebras in products of dendrons,* Proc. London Math. Soc. **58** (1989), 439–453.
[45] H.-J. Bandelt, M. van de Vel, *Superextensions and the depth of median graphs,* J. Combin. Th. Ser. A **57** (1991), 187–202.
[46] H.-J. Bandelt, M. van de Vel, E. Verheul, *Modular interval spaces,* Math. Nachr. **163** (1993), 177–201.
[47] J.-P. Barthélemy, A. Guénoche, *Trees and Proximity Representations,* Wiley, London, 1991.
[48] J.-P. Barthélemy, B. Leclerc, B. Monjardet, *On the use of ordered sets in problems of comparison and consensus of classifications,* J. Classif. **3** (1986), 187–224.
[49] O. Baues, N. Peyerimhoff, *Curvature and geometry of tessellating plane graphs,* Discr. Comput. Geom. **25** (2001), 141–159.
[50] O. Baues, N. Peyerimhoff, *Geodesics in non-positively curved plane tesselations,* Adv. Geom. **6** (2006), 243–263.

[51] G. Bergman, *On the existence of subalgebras of direct products with prescribed d-fold projections,* Algebra Universalis **7** (1977), 341–356.
[52] A. Berrachedi, I. Havel, H.M. Mulder, *Spherical and clockwise spherical graphs,* Czech. Math. J. **53** (2003), 295–309.
[53] L.J. Billera, S.P. Holmes, K. Vogtmann, *Geometry of the space of phylogenetic trees,* Adv. Appl. Math. **27** (2001), 733–767.
[54] A. Björner, P.H. Edelman, G.M. Ziegler, *Hyperplane arrangements with a lattice of regions,* Discr. Comput. Geom. **5** (1990), 263–288.
[55] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler, *Oriented Matroids,* Encyclopedia of Mathematics and its Applications, vol. 46, Cambridge University Press, Cambridge, 1993.
[56] I.F. Blake, J.H. Gilchrist, *Addresses for graphs,* IEEE Trans. Inform. Th. IT-**19** (1973), 683–688.
[57] V.G. Boltyanskii, P.S. Soltan, *Combinatorial Geometry of Some Classes of Convex Sets,* Ştiinţa, Chişinău, 1976 (Russian).
[58] A.V. Borovik, I.M. Gelfand, N. White, *Coxeter Matroids,* Progress in Mathematics, vol. 216, Birkhäuser, Boston, 2003.
[59] A. Bouchet, *Greedy algorithm and symmetric matroids,* Math. Program. **38** (1987), 147–159.
[60] A. Brandstädt, F. Dragan, V. Chepoi, V. Voloshin, *Dually chordal graphs,* SIAM J. Discr. Math. **11** (1998), 437–455.
[61] A. Brandstädt, V. Chepoi, F. Dragan, *Distance approximating trees for chordal and dually chordal graphs,* J. Algorithms **30** (1999), 166–184.
[62] A. Brandstädt, V.B. Le, J.P. Spinrad, *Graph Classes,* SIAM Monographs Discr. Math. and Appl. SIAM, Philadelphia, 1999.
[63] B. Brešar, *Partial Hamming graphs and expansion procedures,* Discr. Math. **237** (2001), 13–27.
[64] B. Brešar, *Intersection graphs of maximal hypercubes,* Europ. J. Combin. **24** (2003), 195–209.
[65] B. Brešar, *On clique-gated graphs,* Discr. Math. **199** (1999), 201–205.
[66] B. Brešar, S. Klavžar, R. Škrekovski, *Quasi-median graphs, their generalizations, and tree-like equalities,* Europ. J. Combin. **23** (2003), 557–572.
[67] M. Bridson, A. Haefliger, *Metric Spaces of Non-Positive Curvature,* Springer, Berlin, 1999.
[68] A.E. Brouwer, A.M. Cohen, A. Neumaier, *Distance-Regular Graphs, Springer,* Berlin, 1989.
[69] F. Buekenhout, P. Cameron, *Projective and affine geometry over division rings,* in Handbook of Incidence Geometry, F. Buekenhout (ed.), Elsevier, Amsterdam, 1995, pp. 27–62.
[70] P. Buneman, *A note on metric properties of trees,* J. Combin. Th. Ser. B **7** (1974), 48–50.
[71] P. Cameron, *Dual polar spaces,* Geom. Dedicata **12** (1982), 75–85.
[72] M. Chastand, *Retracts of infinite Hamming graphs,* J. Combin. Th. Ser. B **71** (1997), 54–66.
[73] M. Chastand, *Fiber-complemented graphs I: structure and invariant subgraphs,* Discr. Math. **226** (2001), 107–141.
[74] M. Chastand, *Fiber-complemented graphs II: retractions and endomorphisms,* Discr. Math. **268** (2003), 81–101.
[75] M. Chastand, F. Laviolette, N. Polat, *On constructible graphs, infinite bridged graphs and weakly cop-win graphs,* Discr. Math. **224** (2000), 61–78.
[76] R. Chandrasekaran, S.N. Kabadi, *Pseudomatroids,* Discr. Math. **71** (1988), 205–217.
[77] V. Chepoi, *Isometric subgraphs of Hamming graphs and d-convexity,* Cybernetics **24** (1988), 6–10 (Russian, English transl.).
[78] V. Chepoi, *Classifying graphs by metric triangles,* Metody Diskretnogo Analiza **49** (1989), 75–93 (Russian).
[79] V. Chepoi, *Convexity and local conditions on graphs,* in Studies in Applied Mathematics and Information Science, Ştiinţa, Chişinău, 1990, pp. 184–191 (Russian).
[80] V. Chepoi, *Separation of two convex sets in convexity structures,* J. Geometry **50** (1994), 30–51.
[81] V. Chepoi, *On distances in benzenoid systems,* J. Chem. Inf. Comput. Sci. **36** (1996), 1169–1172.
[82] V. Chepoi, *Bridged graphs are cop-win graphs: an algorithmic proof,* J. Combin. Th. Ser. B **69** (1997), 97–100.
[83] V. Chepoi, *On distance-preserving and domination elimination orderings,* SIAM J. Discr. Math. **11** (1998), 414–436.

[84] V. Chepoi, *Graphs of some CAT(0) complexes,* Adv. Appl. Math. **24** (2000), 125–179.
[85] V. Chepoi, *Basis graphs of even delta-matroids,* J. Combin. Th. Ser. B. (to appear).
[86] V. Chepoi, M. Deza, V. Grishukhin, *Clin d'oeil on L_1-embeddable planar graphs,* Discr. Appl. Math. **80** (1997), 3–19.
[87] V. Chepoi, F. Dragan, Y. Vaxès, *Distance and routing labeling schemes for non-positively curved plane graphs,* J. Algorithms **61** (2006), 1–29.
[88] F.R.K. Chung, R.L. Graham, M.E. Saks, *Dynamic search in graphs,* in Discrete Algorithms and Complexity, D.S. Johnson, T. Nishizeki, N. Takao, H.S. Wilf (eds.), Academic Press, San Diego, 1987, pp. 351–388.
[89] F.R.K. Chung, R.L. Graham, M.E. Saks, *A dynamic location problem for graphs,* Combinatorica **9** (1989), 111–132.
[90] A.M. Cohen, *Point-line spaces related to buildings,* in Handbook of Incidence Geometry, F. Buekenhout (ed.), Elsevier, Amsterdam, 1995, pp. 647–737.
[91] M. Develin, B. Sturmfels, *Tropical convexity,* Documenta Mathematica **9** (2004), 1–27 (Erratum in: Documenta Mathematica **9** (2004), 204–205).
[92] M. Deza, V.P. Grishukhin, *Hypermetric graphs,* Quart. J. Math. Oxford **2** (1993), 399–433.
[93] M. Deza, M. Laurent, *l_1-Rigid graphs,* J. Algebraic Combin. **3** (1994), 153–175.
[94] M. Deza, M. Laurent, *Geometry of Cuts and Metrics,* Springer-Verlag, Berlin, 1997.
[95] M. Deza, S. Shpectorov, *Recognition of l_1-graphs with complexity $O(nm)$, or football in a hypercube,* Europ. J. Combin. **17** (1996), 279–289.
[96] R. Diestel, *Graph Theory,* Springer, New York, 2000.
[97] D. Ž. Djoković, *Distance-preserving subgraphs of hypercubes,* J. Combin. Th. Ser. B **14** (1973), 263–267.
[98] F.F. Dragan, *Conditions for coincidence of local and global minima for eccentricity function on graphs and the Helly property,* in Studies in Applied Mathematics and Information Science, Ştiinţa, Chişinău, 1990, pp. 49–56 (Russian).
[99] A.W.M. Dress, *Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups,* Adv. Math. **53** (1984), 321–402.
[100] A.W.M. Dress, *Towards a theory of holistic clustering,* in Mathematical Hierarchies and Biology (Piscataway, NJ, 1996), DIMACS Ser. Discr. Math. Theoret. Comput. Sci., vol. 37, Amer. Math. Soc., Providence, RI, 1997, pp. 271–289.
[101] A. Dress, V. Moulton, W. Terhalle, *T-theory: an overview,* Europ. J. Combin. **17** (1996), 161–175.
[102] A.W.M. Dress, R. Scharlau, *Gated sets in metric spaces,* Aequationes Math. **34** (1987), 112–120.
[103] A. Dress, M. Hendy, K. Huber, V. Moulton, *On the number of vertices and edges in the Buneman graph,* Ann. Combin. **1** (1997), 329–337.
[104] A.W.M. Dress, T. Havel, *Some combinatorial properties of discriminants in metric vector spaces,* Adv. Math. **62** (1986), 285–312.
[105] P.H. Edelman, R.E. Jamison, *The theory of convex geometries,* Geom. Dedicata **19** (1985), 247–270.
[106] D. Eppstein, *The lattice dimension of a graph,* Europ. J. Combin. **26** (2005), 585–592.
[107] D.B.A. Epstein, J.W. Cannon, D.F. Holt, S.V.F. Levy, M.S. Paterson, W.P. Thurston, *Word Processing in Groups,* Jones and Bartlett, Boston, 1992.
[108] M. Farber, R.E. Jamison, *On local convexity in graphs,* Discr. Math. **66** (1987), 231–247.
[109] T. Feder, *Product graph representations,* J. Graph Th. **16** (1992), 467–488.
[110] T. Feder, *Stable Networks and Product Graphs,* Mem. Amer. Math. Soc. **555** (1995).
[111] K. Fukuda, K. Handa, *Antipodal graphs and oriented matroids,* Discr. Math. **111** (1993), 245–256.
[112] I.M. Gelfand, R.M. Goresky, R.D. MacPherson, V.V. Serganova, *Combinatorial geometries, convex polyhedrs, and Schubert cells,* Adv. Math. **63** (1987), 301–316.
[113] E. Ghys, A.H. Haefliger, A. Verjovsky (eds.), *Group Theory from a Geometric Viewpoint, World Scientific,* Singapore, 1991.
[114] E. Ghys, P. de la Harpe (eds.), *Les Groupes Hyperboliques d'après M. Gromov,* Progress in Mathematics, vol. 83, Birkhäuser, 1990.
[115] R.L. Graham, *Isometric embeddings of graphs,* in Selected Topics in Graph Theory, vol. 3, L.W. Beineke, R.L. Wilson (eds.), Academic Press, London, 1988, pp. 133–150.

[116] A.J. Goldman, C.J. Witzgall, *A localization theorem for optimal facility placement,* Transport. Sci. **4** (1970), 406–409.
[117] R.L. Graham, P.M. Winkler, *On isometric embeddings of graphs,* Trans. Amer. Math. Soc. **288** (1985), 527–536.
[118] M. Gromov, *Hyperbolic groups,* in Essays in Group theory, S.M. Gersten (ed.), MSRI Publ., vol. 8, Springer, Berlin, 1987, pp. 75–263.
[119] D. Gusfield, *Algorithms on Strings, Trees, and Sequences: Computer Science and Computational Biology,* Cambridge University Press, Cambridge, 1997.
[120] P. Hell, *Absolute retracts in graphs,* Lecture Notes in Math. **406** (1974), 291–301.
[121] P. Hell, I. Rival, *Absolute retracts and varieties of reflexive graphs,* Canad. J. Math. **39** (1987), 544–567.
[122] E. Howorka, *A characterization of distance-hereditary graphs,* Quart. J. Math. Oxford Ser. 2, **28** (1977), 417–420.
[123] E. Howorka, *A characterization of ptolemaic graphs,* J. Graph Th. **5** (1981), 323–331.
[124] W. Imrich, S. Klavžar, *On the complexity of recognizing Hamming graphs and related classes of graphs,* Europ. J. Combin. **17** (1996), 209–221.
[125] W. Imrich, S. Klavžar, *Product Graphs: Structure and Recognition,* Wiley-Interscience Publication, New York, 2000.
[126] J. Isbell, *Six theorems about metric spaces,* Comment. Math. Helv. **39** (1964), 65–74.
[127] J.R. Isbell, *Median algebra,* Trans. Amer. Math. Soc. **260** (1980), 319–362.
[128] R.E. Jamison, *On the null-homotopy of bridged graphs,* Europ. J. Combin. **8** (1987), 421–428.
[129] E. Jawhari, D. Misane, M. Pouzet, *Retracts: graphs and ordered sets from the metric point of view,* Contemp. Math., vol. 56, Amer. Math. Soc., Providence, RI, 1986, pp. 175–225.
[130] A.V. Karzanov, *Minimum 0-extensions of graph metrics,* Europ. J. Combin. **19** (1998), 71–101.
[131] A.V. Karzanov, *Metrics with finite sets of primitive extensions,* Ann. Combin. **2** (1998), 213–243.
[132] S. Klavžar, I. Gutman, B. Mohar, *Labeling of benzenoid systems which reflects the vertex-distance relations,* J. Chem. Inf. Comput. Sci. **35** (1995), 590–593.
[133] M. Kolibiar, T. Marcisová, *On a question of J. Hashimoto,* Mat. Časopis **24** (1974), 179–185.
[134] J. Lawrence, *Lopsided sets and orthant-intersection of convex sets,* Pacific J. Math. **104** (1983), 155–173.
[135] N. Linial, E. London, Y. Rabinovich, *The geometry of graphs and some of its algorithmic applications,* Combinatorica **15** (1995), 215–245.
[136] R.C. Lyndon, *A maximum principle for graphs,* J. Combin. Th. **3** (1967), 34–37.
[137] R. Lyndon, P. Schupp, *Combinatorial Group Theory,* Springer-Verlag, New York, 1977.
[138] J.-H. Mai, Y. Tang, *An injective metrization for collapsible polyhedra,* Proc. Amer. Math. Soc. **88** (1983), 333–337.
[139] J. Matoušek, *Lectures on Discrete Geometry,* Springer, New York, 2002.
[140] S.B. Maurer, *Matroid basis graphs* I, J. Combin. Th. Ser. B **14** (1973), 216–240; II, J. Combin. Th. Ser. B **15** (1973), 121–145.
[141] H.M. Mulder, *The structure of median graphs,* Discr. Math. **24** (1978), 197–204.
[142] H.M. Mulder, *The Interval Function of a Graph,* Math. Centre Tracts **132** (Amsterdam), 1980.
[143] H.M. Mulder, A. Schrijver, *Median graphs and Helly hypergraphs,* Discr. Math. **25** (1979), 41–50.
[144] R. Nowakowski, I. Rival, *The smallest graph variety containing all paths,* Discr. Math. **43** (1983), 223–234.
[145] R. Nowakowski, P. Winkler, *Vertex-to-vertex pursuit in a graph,* Discr. Math. **43** (1983), 235–239.
[146] S.E. Payne, J.A. Thas, *Finite Generalized Quadrangles,* Research Notes in Mathematics vol. 110, Pitman, Boston, 1984.
[147] D. Peleg, *Distributed Computing: A Locality-Sensitive Approach,* SIAM Monographs on Discr. Math. and Appl. SIAM, Philadelphia, 2000.
[148] E. Pesch, *Retracts of Graphs,* Athenaeum Verlag, Frankfurt, 1988.
[149] E. Pesch, W. Poguntke, *A characterization of absolute retracts of n-chromatic graphs,* Discr. Math. **43** (1983), 223–234.

[150] M. Ploščica, *A duality for isotropic median algebras,* Comment. Math. Univ. Carolin. **33** (1992), 541–550.
[151] N. Polat, *Invariant graphs for a family of endomorphisms–a survey,* in Combinatorics and Graph Theory'95, vol. 1 (Hefei), World Science, River Edge, NJ, 1995, pp. 313–331.
[152] Ch. Prisăcaru, P. Soltan, V. Chepoi, *On embeddings of planar graphs into hypercubes,* Proc. Moldavian Academy of Sci., Math. **1** (1990), 43–50 (Russian).
[153] M. Roller, *Poc sets, median algebras and group actions,* Preprint, University of Southampton, 1998.
[154] R.L. Roth, P.M. Winkler, *Collapse of the metric hierarchy for bipartite graphs,* Europ. J. Combin. **7** (1986), 371–375.
[155] A. Quilliot, *On the Helly property working as a compactness criterion on graphs,* J. Combin. Th. Ser. A **40** (1985), 186–193.
[156] T.J. Schaefer, *The complexity of satisfability problems,* Proc. 10th ACM Symp. on Theory of Computing (1978), 216–226.
[157] R. Scharlau, *Buildings,* in Handbook of Incidence Geometry, F. Buekenhout (ed.), Elsevier, Amsterdam, 1995, pp. 477–645.
[158] C. Semple, M. Steel, *Phylogenetics,* Oxford University Press, Oxford, 2003.
[159] S.V. Shpectorov, *On scale embeddings of graphs into hypercubes,* Europ. J. Combin. **14** (1993), 117–130.
[160] D. Speyer, B. Sturmfels, *The tropical Grassmannian,* Adv. Geom. **4** (2004), 389–411.
[161] V.P. Soltan, *Introduction to the Axiomatic Theory of Convexity,* Ştiinţa, Chişinău, 1984 (Russian).
[162] V.P. Soltan, V.D. Chepoi, *Conditions for invariance of set diameters under d-convexification in a graph,* Cybernetics **19** (1983), 750–756 (Russian, English transl.).
[163] P. Soltan, D. Zambitskii, C. Prisăcaru, *Extremal Problems on Graphs and Algorithms of their Solution,* Ştiinţa, Chişinău, 1973 (Russian).
[164] M. Stern, *Semimodular Lattices: Theory and Applications,* Encyclopedia of mathematics and its applications, vol. 73, Cambridge University Press, Cambridge, 1999.
[165] C. Tardif, *On compact median graphs,* J. Graph Th. **23** (1996), 325–336.
[166] P. Terwiliger, M. Deza, *Classification of finite connected hypermetric spaces,* Graphs Combin. **3** (1987), 293–298.
[167] J. Tits, *Buildings of Spherical Type and Finite BN-Pairs,* Lecture Notes in Math. **386,** Springer-Verlag, Berlin, 1974.
[168] S.V. Yushmanov, V. Chepoi, *A general method of investigation of metric graph properties related to the eccentricity,* in Mathematical Problems in Cybernetics, vol. 3, Nauka, Moscow, 1991, pp. 217–232 (Russian).
[169] M. van de Vel, *Matching binary convexities,* Topology Appl. **16** (1983), 207–235.
[170] M. van de Vel, *Theory of Convex Structures,* Elsevier Science Publishers, Amsterdam, 1993.
[171] M. van de Vel, *Collapsible polyhedra and median spaces,* Proc. Amer. Math. Soc. **126** (1998), 2811–2818.
[172] W. Wenzel, *Maurer's homotopy theory for even Δ-matroids and related combinatorial geometries,* J. Combin. Th. Ser. A **71** (1995), 19–59.
[173] E. Wilkeit, *Isometric embedding in Hamming graphs,* J. Combin. Th. Ser. B **50** (1990), 179–197.
[174] E. Wilkeit, *The retracts of Hamming graphs,* Discr. Math. **102** (1992), 197–218.
[175] P.M. Winkler, *Isometric embedding in the product of complete graphs,* Discr. Appl. Math. **7** (1984), 221–225.
[176] P.M. Winkler, *The metric structure of graphs: theory and applications.* in Survey in Combinatorics 1987, C. Whitehead (ed.), London Math. Soc. Lect. Notes Ser., vol. 123, Cambridge University Press, Cambridge, 1987, pp. 197–221.

Fachbereich Mathematik, Universität Hamburg, Bundesstr. 55, 20146 Hamburg, Germany,

E-mail address: bandelt@math.uni-hamburg.de

Laboratoire d'Informatique Fondamentale de Marseille, Université de la Méditerranée, Faculté des Sciences de Luminy, F-13288 Marseille Cedex 9, France

E-mail address: chepoi@lif.univ-mrs.fr