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Faber–Krahn type inequalities for trees ☆

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Abstract

The Faber–Krahn theorem states that the ball has lowest first Dirichlet eigenvalue amongst all bounded domains of the same volume in $\mathbb{R}^n$ (with the standard Euclidean metric). It has been shown that a similar result holds for (semi-) regular trees. In this article we show that such a theorem also holds for other classes of (not necessarily regular) trees, for example for trees with the same degree sequence. Then the resulting trees possess a spiral like ordering of their vertices, i.e., are ball approximations.

Keywords: Graph Laplacian; Dirichlet eigenvalue problem; Faber–Krahn type inequality; Tree; Degree sequence

1. Introduction

In recent years the eigenvectors of the graph Laplacian have received increasing attention. While its eigenvalues have been investigated for fifty years (see, e.g., [1,5,6]), there is little known about the eigenvectors. The graph Laplacian can be seen as the discrete analog of the continuous Laplace–Beltrami-operator on manifolds. When using an appropriate definition for the gradient on a graph, rules similar to the classical Laplace operator can be formulated, e.g., Green's formula. During the last years some results for eigenfunctions of the Laplace–Beltrami-operator have been shown to hold also for eigenvectors of the graph Laplacian; for example

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Cheeger-type inequalities [8] or nodal domain theorems [7] exist. However, it has turned out that there are small but subtle differences between the discrete and the continuous case.

The Faber–Krahn inequality is another well-known result. It states that among all bounded domains with the same volume in $\mathbb{R}^n$ (with the standard Euclidean metric), a ball has lowest first Dirichlet eigenvalue [4]. Friedman [9] introduced the idea of a "graph with boundary" (see below). With this concept he was able to formulate Dirichlet and Neumann eigenvalue problems for graphs. He also conjectured an analog to the Faber–Krahn inequality for regular trees. Amazingly Friedman's conjecture is false, i.e., in general these trees are similar but not equal to "balls," see [13,15] for counterexamples and [14] for a statement of the result. This example (as well as the nodal domain theorem where also some wrong conjectures exist, see [7]) shows that there is much more structure in graphs than in manifolds. Conclusions from this fact are twofold: First, some care is necessary since one's intuition, trained on manifolds, may lead to wrong conjectures. On the other hand, we can use the opportunity to go further and try to find these new structural properties where no analog exists in the world of elliptic operators on manifolds. It is this second conclusion that motivates this paper. We want to leave the world of regular graphs and look what happens when we drop this regularity assumption.

In this article we formulate Faber–Krahn type theorems for trees which need not be regular any more. We show that trees that have smallest first Dirichlet eigenvalue for a given number of vertices have an SLO (*spiral like ordering*) structure, i.e., are ball approximations. It is notable that the vertex degrees are as small as possible for vertices near the center of these trees. In particular if there are no other restrictions but the number of interior and exterior vertices then the resulting trees are paths with a star attached to one end, i.e., comets, see Fig. 2. Analogous results for the Laplace–Beltrami-operators on manifolds with non-constant curvature are rare (see, e.g., the work of Carron [2,3]). Additionally we also show in Theorem 5 the remarkable property that a Dirichlet eigenvalue is a weighted average of the number of boundary vertices to which an interior vertex is connected.

2. Discrete Dirichlet operator and Faber–Krahn property

Let $G(V, E)$ be a simple (finite) undirected graph with vertex set V and edge set E. The *Laplacian* of G is the matrix

$$\Delta(G) = D(G) - A(G), \tag{1}$$

where $A(G)$ denotes the adjacency matrix of the graph and $D(G)$ is the diagonal matrix whose entries are the vertex degrees, i.e., $D_{vv} = d_v$, where d_v denotes the degree of vertex v. We write Δ for short if there is no risk of confusion. To state a Faber–Krahn type inequality we need a Dirichlet operator which itself requires the notion of a boundary of a graph.

A *graph with boundary* $G(V_0 \cup \partial V, E_0 \cup \partial E)$ consists of a set of interior vertices V_0, boundary vertices ∂V, interior edges E_0 that connect interior vertices, and boundary edges ∂E that join interior vertices with boundary vertices [9]. There are no edges between two boundary vertices.

In the following we assume that every boundary vertex has degree 1 and every interior vertex has degree at least 2, i.e., a vertex is a boundary vertex if and only if it has degree 1. We also assume that both the set of interior vertices V_0 and the set of boundary vertices ∂V are not empty. Balls are of particular interest for our investigations. A *ball* $B(v_0, r)$ with center v_0 and radius $r \in \mathbb{N}$ is a connected graph where every boundary vertex w has geodesic distance $\mathrm{dist}(v_0, w) = r$.

A discrete Dirichlet operator is the graph Laplacian Δ which acts only on vectors that vanish in all boundary vertices. For a motivation of this definition see [9].

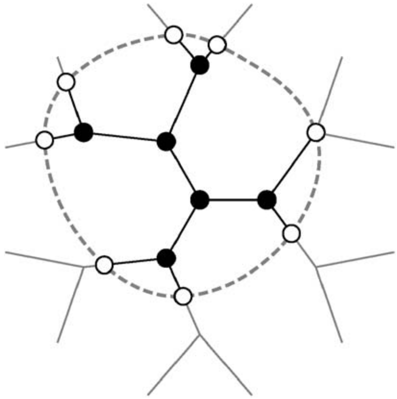

Fig. 1. The class of trees considered by Friedman [9] can be obtained by cutting connected subsets out of the geometric representation of an infinite d-regular tree (• … interior vertices, ∘ … boundary vertices).

Definition 1. A *discrete Dirichlet operator* Δ_0 is the graph Laplacian restricted to interior vertices, i.e.,

$$\Delta_0 = D_0 - A_0, \tag{2}$$

where A_0 is the adjacency matrix of the graph induced by the interior vertices, $G(V_0, E_0)$, and where D_0 is the degree matrix D restricted to the interior vertices V_0.

Notice that Δ_0 is obtained from the graph Laplacian Δ by deleting all rows and columns that correspond to boundary vertices. Thus any edges between two boundary vertices have no influence on the Dirichlet operator. Thus we have eliminated such edges by definition for the sake of simplicity.

Definition 2 *(Faber–Krahn property).* We say that a graph with boundary has the *Faber–Krahn property* if it has lowest first Dirichlet eigenvalue among all graphs with the same "volume" in a particular graph class.

This informal definition raises two questions: (1) What is the "volume" of a graph, and (2) what is an appropriate graph class (besides the trivial requirement that it must contain the graph G in question)?

Pruss [15] used the number of edges of an unweighted tree as volume and the class of semi-d-regular trees with boundary. In such a tree every interior vertex has the same degree d whereas every boundary vertex has degree 1. This idea can be extended to weighted trees [9], where edge weights are represented by the reciprocal lengths of arcs in a geometric representation of the tree. The volume is then defined as the sum of all the arc lengths of the geometric representation. Friedman [9] looked at the class of all trees, where the interior vertices have the same degree d, all interior edges have length (weight) 1 and all boundary edges have length at most 1. Such graphs can be obtained by cutting out a subset of the geometric representation of an infinite (unweighted) d-regular tree, see Fig. 1.

In this article we want to formulate Faber–Krahn type theorems for (non-regular) trees. When we generalize the Faber–Krahn type theorems to arbitrary trees, we have to solve the following (roughly formulated) problem.

Problem 1. Give a characterization of all graphs in a given class $\mathcal{C}$ with the Faber–Krahn property, i.e., characterize those graphs in $\mathcal{C}$ which have minimal first Dirichlet eigenvalue for a given "volume."

Making the graph class $\mathcal{C}$ too large leads to quite simple (non-interesting) graphs. For example, if $\mathcal{C}$ is the set of all connected graphs with a given number of vertices as the "volume" of the graph, then graphs with the Faber–Krahn property are paths with one terminating triangle [12]. If we restrict this class to trees, then we arrive at simple paths [11,12].[2]

It seems natural to use the number of vertices as measure for the "volume" of a graph. (Notice that this is equivalent to using the number of edges for an unweighted tree.) Moreover, we will consider only graph classes where both the total numbers of interior vertices, $|V_0|$, and boundary vertices, $|\partial V|$, are fixed. (For semiregular trees this is always the case when we fix the total number of vertices.) We will drop this requirement at the end of this article and state some additional results in Section 4. Hence we will look at the following classes of graphs with boundaries:

$$\mathcal{T}^{(n,k)} = \{G \text{ is a tree, with } |V| = n \text{ and } |V_0| = k\}, \tag{3}$$

$$\mathcal{T}_d^{(n,k)} = \{G \in \mathcal{T}^{(n,k)}\colon d_v \geqslant d \text{ for all } v \in V_0\}. \tag{4}$$

As it is clear that we always look at a particular class $\mathcal{T}^{(n,k)}$ or $\mathcal{T}_d^{(n,k)}$ we will write $\mathcal{T}$ and $\mathcal{T}_d$ for short; n and k have then to be selected accordingly. We always assume that $1 \leqslant k \leqslant n-2$.

Another interesting class is based on so-called degree sequences. A sequence $\pi = (d_0, \dots, d_{n-1})$ of non-negative integers is called a *degree sequence* if there exists a graph G with n vertices for which $d_0, \dots, d_{n-1}$ are the degrees of its vertices. For trees the following characterization exists.

Lemma 1. [10] *A degree sequence $\pi = (d_0, \dots, d_{n-1})$ is a tree sequence* (*i.e., a degree sequence of some tree*) *if and only if every $d_i > 0$ and $\sum_{i=0}^{n-1} d_i = 2(n-1)$.*

Using this notion we can introduce another interesting graph class for which we want to formulate a Faber–Krahn like theorem,

$$\mathcal{T}_\pi = \{G \text{ is a tree with boundary and with degree sequence } \pi\}. \tag{5}$$

Notice that for a particular degree sequence π we have

$$\mathcal{T}_\pi \subseteq \mathcal{T}_{d_\pi} \subseteq \mathcal{T}_2 = \mathcal{T}, \tag{6}$$

where d_π is the minimal degree for interior vertices of the degree sequence π.

For the class $\mathcal{T}$ of all trees we find a simple structure for graphs with the Faber–Krahn property.

Theorem 1 *(Klobürštel theorem). A tree G has the Faber–Krahn property in the class $\mathcal{T}$ if and only if G is a star with a long tail, i.e., a comet, see Fig.* 2. *G is then uniquely determined up to isomorphism.*

Graphs with the Faber–Krahn property in $\mathcal{T}_d$ or $\mathcal{T}_\pi$ have a richer structure. For its description we need additional notions. For a tree G with root v_0 the *height* $h(v)$ of a vertex v is defined

[2] To be precise Katsuda and Urakawa [12] used the more general "non-separation property."

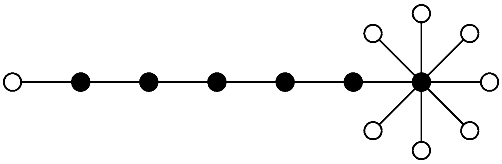

Fig. 2. A comet has the Faber–Krahn property in class $\mathcal{T}$. It consists of a star with diameter 2 and a path attached to it (• … interior vertices, ∘ … boundary vertices).

by $h(v) = \mathrm{dist}(v, v_0)$. For two adjacent vertices v and w with $h(w) = h(v) + 1$ we call v the *parent* of w, and w a *child* of v. Notice that every vertex $v \neq v_0$ has exactly one parent, and every interior vertex w has at least one child vertex. The main notion for describing trees with the Faber–Krahn property is *spiral-like ordering* of its vertices, first introduced by Pruss [15]. We give a slightly modified and extended definition.

Definition 3 *(SLO-ordering).* Let $G(V_0 \cup \partial V, E_0 \cup \partial E)$ be a tree with boundary with root v_0. Then a well-ordering $\prec$ of the vertices is called *spiral-like* (SLO-ordering for short) if the following holds for all vertices $v, v_1, v_2, w, w_1, w_2 \in V$:

(S1) $v \prec w$ implies $h(v) \leqslant h(w)$;
(S2) if $v_1 \prec v_2$ then for all children w_1 of v_1 and all children w_2 of v_2, $w_1 \prec w_2$;
(S3) if $v \prec w$ and $v \in \partial V$, then $w \in \partial V$.

It is called *spiral-like with increasing degrees* (SLO*-ordering for short) if additionally the following holds:

(S4) if $v \prec w$ for interior vertices $v, w \in V_0$, then $d_v \leqslant d_w$.

We call trees that have an SLO- or SLO*-ordering of its vertices *SLO-trees* and *SLO*-trees*, respectively.

Notice that SLO-trees are almost balls, that is, there exists a radius r such that $\mathrm{dist}(v, v_0) \in \{r, r+1\}$ for all boundary vertices $v \in \partial V$, see Fig. 3 for an example. With this concept we can formulate Faber–Krahn type theorems for the other graph classes, $\mathcal{T}_d$ and $\mathcal{T}_\pi$.

Theorem 2. *A graph G has the Faber–Krahn property in a class $\mathcal{T}_d$ if and only if it is an SLO*-tree where at most one interior vertex has degree d° exceeding d and all other interior vertices have degree d. G is then uniquely determined up to isomorphism.*

Theorem 3. *A graph G with degree sequence π has the Faber–Krahn property in the class $\mathcal{T}_\pi$ if and only if it is an SLO*-tree. G is then uniquely determined up to isomorphism.*

As an immediate corollary we get the result of Pruss [15].

Corollary 4. [15, Theorem 6.2] *In the class of semi-d-regular trees a graph G has the Faber–Krahn property if and only if it is an SLO*-tree. G is then uniquely determined up to isomorphism.*

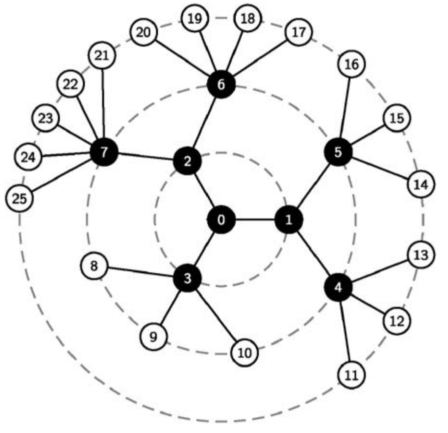

Fig. 3. An SLO*-tree with 8 interior and 18 boundary vertices. The SLO*-ordering $\prec$ is indicated by numbers. Degree sequence $\pi = (3, 3, 3, 4, 4, 4, 5, 6, 1, 1, \ldots, 1)$.

Before we prove these theorems we first want to show that each of these two classes indeed contains an SLO*-tree.

Lemma 2. *Each class $\mathcal{T}_\pi$ contains an SLO*-tree that is uniquely determined up to isomorphism.*

Proof. First we prove the existence of an SLO*-tree. This is trivial for a star with one interior vertices and $n-1$ boundary vertices: the central vertex is chosen as root for the SLO-ordering. Such stars have degree sequence $(n-1, 1, \ldots, 1)$. For all other trees (which have at least two interior vertices) we show this statement by induction on $|\pi|$ (the number of vertices of π).

Now we assume by induction that each $\mathcal{T}_{\pi'}$ with $|\pi'| \leqslant n-1$ has an SLO*-tree. Let $\pi = (d_0, d_1, \ldots, d_{k-1}, d_k, \ldots, d_{n-1})$, $k \geqslant 2$, be the degree sequence of $\mathcal{T}_\pi$, where $2 \leqslant d_0 \leqslant d_1 \leqslant \cdots \leqslant d_{k-1}$ and $d_k = \cdots = d_{n-1} = 1$ (i.e., correspond to boundary vertices); $|\pi| = n$. Notice that d_{k-1} is the last degree for interior vertices and thus the corresponding vertex v_{k-1} is adjacent to $d_{k-1} - 1$ boundary vertices, which correspond to the last entries in π. Therefore, we can construct a new degree sequence π' by deleting the last $d_{k-1} - 1$ elements from π and by replacing d_{k-1} by $d'_{k-1} = 1$. Obviously π' has $n - (d_{k-1} - 1) < n$ elements.

By Lemma 1, π' is a tree sequence. By induction $\mathcal{T}_{\pi'}$ has an SLO*-tree T'. Let v be the first vertex of T' w.r.t. the SLO-ordering that is adjacent to some boundary vertex w. We replace w by an interior vertex u and add $d_{k-1} - 1$ boundary vertices and get a tree T. Obviously u has degree d_{k-1} and thus T has degree sequence π. Moreover, T has an SLO*-ordering which can be derived from the ordering in T' by inserting the new vertex u as the last interior vertex and the new boundary vertices as the last $d_{k-1} - 1$ vertices in the ordering. It is then easy to see that the properties (S1)–(S4) are satisfied.

To show that two SLO*-trees G and G' in a class $\mathcal{T}_\pi$ are isomorphic we use a function ϕ that maps the vertex v_i in the ith position in the SLO*-ordering of G to the vertex w_i in the ith position in the SLO*-ordering of G'. By the properties of the SLO*-ordering, ϕ is an isomorphism, as v_i and w_i have the same degree and the images of all children of v_i are exactly the children of w_i. The latter can be seen by looking on all interior vertices of G in the reverse SLO*-ordering. Thus the proposition follows. □

3. Proof of the theorems

We first recall some basic results. By definition the Laplace operator Δ is symmetric. Its associate Rayleigh quotient on real-valued functions f on V is the fraction

$$\mathcal{R}_G(f) = \frac{\langle \Delta f, f\rangle}{\langle f, f\rangle} = \frac{\sum_{(u,v)\in E}(f(u)-f(v))^2}{\sum_{v\in V} f(v)^2}. \tag{7}$$

For the Dirichlet operator Δ_0 we get a similar Rayleigh quotient. However, it is much simpler to consider $\mathcal{R}_G(f)$ again but restrict the set of functions f such that $f(v)=0$ for all boundary vertices $v \in \partial V$. We denote the first Dirichlet eigenvalue of $\Delta_0(G)$ by $\lambda(G)$. The following proposition states a well-known fact about Rayleigh quotients.

Proposition 3. *For a graph with boundary $G(V_0 \cup \partial V, E_0 \cup \partial E)$ we have*

$$\lambda(G) = \min_{f\in\mathcal{S}} \mathcal{R}_G(f) = \min_{f\in\mathcal{S}} \frac{\langle \Delta f, f\rangle}{\langle f, f\rangle}, \tag{8}$$

where $\mathcal{S}$ is the set of all real-valued functions on V with the constraint $f|_{\partial V}=0$. Moreover, if $\mathcal{R}_G(f)=\lambda(G)$ for a function $f\in\mathcal{S}$, then f is an eigenfunction of the first Dirichlet eigenvalue of Δ_0.

For eigenfunctions of the Dirichlet operator the following remarkable property holds.

Theorem 5. *Let $G(V_0 \cup \partial V, E_0 \cup \partial E)$ be a connected graph with boundary and f an eigenfunction corresponding to some eigenvalue λ of the Dirichlet operator. Let b_v denote the number of boundary vertices adjacent to v, i.e., $b_v = |\{w \in \partial V\colon (v,w)\in E\}|$. Then either $\sum_{v\in V} f(v) = \sum_{v\in V} b_v f(v) = 0$, or*

$$\lambda = \frac{\sum_{v\in V} b_v f(v)}{\sum_{v\in V} f(v)}.$$

Proof. Let $\mathbf{1} = (1,\dots,1)'$ and $i_v = |\{w \in V_0\colon (v,w)\in E\}|$ be the number of interior vertices adjacent to v. Thus $b_v + i_v = d_v$. A straightforward computation gives

$$\begin{aligned}
\langle \mathbf{1}, \Delta_0 f\rangle &= \sum_{v\in V_0} d_v f(v) - \sum_{v\in V_0} \sum_{\substack{(v,w)\in E\\ w\in V_0}} f(w)\\
&= \sum_{v\in V_0} d_v f(v) - \sum_{w\in V_0} f(w) \sum_{\substack{(w,v)\in E\\ v\in V_0}} 1\\
&= \sum_{v\in V_0} d_v f(v) - \sum_{w\in V_0} i_w f(w) = \sum_{v\in V_0} b_v f(v).
\end{aligned}$$

Since f is an eigenfunction we find $\langle \mathbf{1}, \Delta_0 f\rangle = \lambda \sum_{v\in V_0} f(v)$. As $f(v)=0$ for all boundary vertices $v\in\partial V$ the result follows. □

Proposition 4. (Friedman [9]) *Let $G(V_0 \cup \partial V, E_0 \cup \partial E)$ be a connected graph with boundary.*

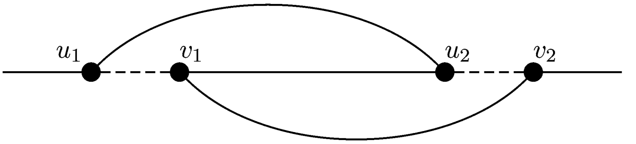

Fig. 4. Switching: edges (v_1, u_1) and (v_2, u_2) are replaced by edges (v_1, v_2) and (u_1, u_2).

(1) *$\Delta_0(G)$ is a positive operator, i.e., $\lambda(G) > 0$.*
(2) *An eigenfunction f of the eigenvalue $\lambda(G)$ is either positive or negative on all interior vertices of G.*
(3) *$\lambda(G)$ is monotone in G, i.e., if $G \subset G'$ then $\lambda(G) > \lambda(G')$.*
(4) *$\lambda(G)$ is a simple eigenvalue.*

The main techniques for proving our theorems is *rearranging* of edges. We need two different types of rearrangement steps that we call *switching* and *shifting*, respectively, in the following.

Lemma 5 *(Switching).* (See also [13, Lemma 5].) *Let $G(V, E)$ be a tree with boundary in some class $\mathcal{T}_\pi$. Let $(v_1, u_1), (v_2, u_2) \in E$ be edges such that u_2 is in the geodesic path from v_1 to v_2, but u_1 is not, see Fig.* 4. *Then by replacing edges (v_1, u_1) and (v_2, u_2) by the edges (v_1, v_2) and (u_1, u_2) we get a new tree $G'(V, E')$ which is also contained in $\mathcal{T}_\pi$ with the same set of boundary vertices. Moreover, we find for a function $f \in \mathcal{S}$*

$$\mathcal{R}_{G'}(f) \leqslant \mathcal{R}_G(f) \tag{9}$$

whenever $f(v_1) \geqslant f(u_2)$ and $f(v_2) \geqslant f(u_1)$. Inequality (9) *is strict if both inequalities are strict.*

Proof. Since by assumption u_2 is in the geodesic path from v_1 to v_2 and u_1 is not, $G'(V, E')$ is again a tree. The set of vertices does not change by construction. Moreover, since this switching does not change the degrees of the vertices, the degree sequence remains unchanged. To verify inequality (9) we have to compute the effects of removing and inserting edges and get

$$\begin{aligned}\langle \Delta(G')f, f\rangle - \langle \Delta(G)f, f\rangle &= \left[\left(f(v_1) - f(v_2)\right)^2 + \left(f(u_1) - f(u_2)\right)^2\right] \\ &\quad - \left[\left(f(v_1) - f(u_1)\right)^2 + \left(f(v_2) - f(u_2)\right)^2\right] \\ &= 2\left(f(u_1) - f(v_2)\right) \cdot \left(f(v_1) - f(u_2)\right) \\ &\leqslant 0,\end{aligned}$$

where the last inequality is strict if both inequalities $f(v_1) \geqslant f(u_2)$ and $f(v_2) \geqslant f(u_1)$ are strict. Thus the proposition follows. □

Lemma 6. *Let $G(V, E)$ be a tree with boundary in some $\mathcal{T}_\pi$ and let $G'(V, E')$ be a tree obtained from G by applying switching as defined in Lemma* 5. *If f is a non-negative eigenfunction of the first Dirichlet eigenvalue of G then $\lambda(G') \leqslant \lambda(G)$ whenever $f(v_1) \geqslant f(u_2)$ and $f(v_2) \geqslant f(u_1)$. Moreover, $\lambda(G') < \lambda(G)$ if one of these two inequalities is strict.*

Proof. The first inequality is an immediate consequence of Lemma 5 and Proposition 3

$$\lambda(G') \leqslant \mathcal{R}_{G'}(f) \leqslant \mathcal{R}_G(f) = \lambda(G).$$

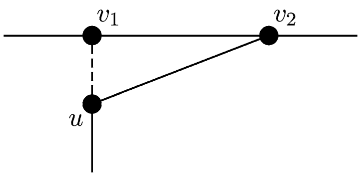

Fig. 5. Shifting: edge (u, v_1) is replaced by edge (u, v_2).

For the second statement notice that $\lambda(G') = \lambda(G)$ if and only if $\mathcal{R}_{G'}(f) = \mathcal{R}_G(f)$ and f is an eigenfunction corresponding to $\lambda(G')$ on G', since $\lambda(G')$ is simple (Propositions 3 and 4). Therefore, if $\lambda(G') = \lambda(G)$ we find

$$\lambda(G)f(v_1) = \Delta(G)f(v_1) = d_{v_1}f(v_1) - f(u_1) - \sum_{\substack{(v_1,w)\in E\\ w\neq u_1}} f(w)$$

$$= \lambda(G')f(v_1) = \Delta(G')f(v_1) = d_{v_1}f(v_1) - f(v_2) - \sum_{\substack{(v_1,w)\in E'\\ w\neq v_2}} f(w).$$

Since the summation is done over the same neighbors of v_1 in this equation we find $f(u_1) = f(v_2)$. Analogously we derive from $\Delta(G)f(u_1) = \Delta(G')f(u_1)$, $f(v_1) = f(u_2)$. Thus the proposition follows. □

Lemma 7 *(Shifting). Let $G(V, E)$ be a tree with boundary in graph class $\mathcal{T}$. Let $(u, v_1) \in E$ be an edge and $v_2 \in V$ some vertex such that u is not in the geodesic path from v_1 to v_2, see Fig.* 5. *Then by replacing edge (u, v_1) by the edge (u, v_2) we get a new tree $G'(V, E')$ which is also contained in $\mathcal{T}$. If $v_2 \in V_0$ is an interior vertex and $d_{v_1} \geqslant 3$ then the number of boundary vertices remains unchanged. Moreover, we find for a non-negative function $f \in \mathcal{S}$*

$$\mathcal{R}_{G'}(f) \leqslant \mathcal{R}_G(f) \tag{10}$$

if $f(v_1) \geqslant f(v_2) \geqslant f(u)$. The inequality is strict if $f(v_1) > f(v_2)$.

Notice that if G is in some class $\mathcal{T}_d$ (or $\mathcal{T}_\pi$) then in general G' need not be a member of this graph class any more.

Proof. Analogously to the proof of Lemma 5. □

Remark 6. Lemmata 5, 6, and 7 hold analogously for arbitrary graphs.

We now can use a sequence of switchings and shiftings to transform any tree G with boundary in some class $\mathcal{T}_\pi$ into an SLO*-tree $G^* \in \mathcal{T}_\pi$.

Lemma 8. *Let $G(V, E)$ be a tree with boundary in some class $\mathcal{T}_\pi$. Then there exists an SLO-tree $G'(V, E')$ in $\mathcal{T}_\pi$ with $\lambda(G') \leqslant \lambda(G)$.*

Furthermore, if G has the Faber–Krahn property then there exists already an SLO-ordering $\prec$ of the vertices (i.e., G is an SLO-tree). If, moreover, f is a non-negative eigenfunction of $\lambda(G)$, then $v \prec w$ implies $f(v) \geqslant f(w)$.

Proof. Let $n = |V|$ and $k = |V_0|$ denote the number of vertices and of interior vertices of G, respectively, and let f be a non-negative eigenfunction of the first Dirichlet eigenvalue of G. We assume that the vertices of G, $V = \{v_0, v_1, \dots, v_{k-1}, v_k, \dots, v_{n-1}\}$, are numbered such that $f(v_i) \geqslant f(v_j)$ if $i \leqslant j$, i.e., they are sorted with respect to $f(v)$ in non-increasing order. We define a well-ordering $\prec$ on V by $v_i \prec v_j$ if and only if $i < j$.

Now we use a series of switchings to construct the desired new tree G'. This is done recursively such that we have a ball that already has the desired SLO-ordering in the central part of each intermediate graph. This ball grows in every recursion step until all vertices of the initial graph G are used.

We start with the first vertex v_0 of this ordered set of vertices. If v_0 is adjacent to v_1 there is nothing to do. Else, we check whether v_0 is adjacent to some vertex w with $f(w) = f(v_1)$ and $v_1 \prec w$. If there exists such a vertex we just exchange the positions of these two vertices in the ordering of V (and update the indices of the vertices). (In particular this is the case when v_1 is a boundary vertex then by our assumptions $0 \leqslant f(w) \leqslant f(v_1) = 0$ and thus $f(w) = f(v_1) = 0$ and this condition is satisfied.) Otherwise, there exists a child vertex u_0 of v_0 with $v_1 \prec u_0$ and a path $P_{0,1}$ from v_0 to v_1, since G is connected. There also exists a parent of v_1 (which is in this path $P_{0,1}$ and which cannot be v_0) and some child vertices (which are not in this path). The latter exists as v_1 cannot be a boundary vertex, since then one of the above two cases would apply. Now if $u_0 \in P_{0,1}$ then let u_1 be one these child vertices; else let u_1 be the parent of v_1. As, by the construction, $v_0 \prec v_1 \prec u_0, u_1$ we have $f(v_0) \geqslant f(v_1) \geqslant f(u_0), f(u_1)$ and hence we can apply Lemma 5, exchange edges (v_0, u_0) and (v_1, u_1) by (v_0, v_1) and (u_0, u_1), and get a new graph G_1 with $\mathcal{R}_{G_1}(f) \leqslant \mathcal{R}_G(f)$ which also belongs to $\mathcal{T}_\pi$.

By this switching step we have exchanged a child of v_0 by v_1 (if necessary) which then becomes a child of v_0. By the same procedure we can exchange all other vertices adjacent to v_0 with the respective vertices $v_2, v_3, \dots, v_{s_0}$, where $s_0 = d_{v_0}$, and get graphs $G_2, G_3, \dots, G_{s_0}$ in $\mathcal{T}_\pi$ with $\mathcal{R}_{G_i}(f) \leqslant \mathcal{R}_{G_{i-1}}(f)$.

Next we proceed in an analogous manner with all children u of v_1 with $v_1 \prec u$ and make all vertices $v_{s_0+1}, v_{s_0+2}, \dots, v_{s_1}$ adjacent to v_1, where $s_1 = s_0 + d_{v_1} - 1$, and get graphs G_{s_0+1}, $G_{s_0+2}, \dots, G_{s_1}$. By processing all interior vertices in this way we get a sequence of graphs

$$G = G_0 \to G_1 \to G_2 \to \dots \to G_k = G' \tag{11}$$

in $\mathcal{T}_\pi$ with

$$\lambda(G) = \mathcal{R}_{G_0}(f) \geqslant \mathcal{R}_{G_1}(f) \geqslant \dots \geqslant \mathcal{R}_{G_k}(f) \geqslant \lambda(G'). \tag{12}$$

In step $G_{r-1} \to G_r$ there is either nothing to do (when we assume that the vertices are already in the proper ordering), or switching is used to joint the vertex $v_{r-1} \prec v_r$ with vertex v_r: Let $P_{r-1,r}$ be the geodesic path from v_{r-1} to v_r. By construction of our sequence of graphs we have $h(v_{r-1}) \leqslant h(v_r)$ in graph G_{r-1} and thus the parent w_r of v_r must be in $P_{r-1,r}$. Moreover, v_r cannot be a boundary vertex (since otherwise we can use the argument from above and we only had to change the ordering of the vertices) and thus has some child u_r. Furthermore this path either contains some child u_{r-1} of v_{r-1}, or it contains the parent of v_{r-1}. In the latter case there exists at least one child u_{r-1}. Now we can use switching and replace either edges (v_{r-1}, u_{r-1}) and (v_r, u_r) by the edges (v_{r-1}, v_r) and (u_{r-1}, u_r) (if u_{r-1} is contained in $P_{r-1,r}$) or (otherwise) edges (v_{r-1}, u_{r-1}) and (w_r, v_r) by the edges (v_{r-1}, v_r) and (u_{r-1}, w_r). In both cases we can apply Lemma 5 as $f(v_{r-1}) \geqslant f(v_r) \geqslant f(u_{r-1}), f(w_r), f(u_r)$. (It cannot happen that v_r is adjacent to some vertex w with $w \prec v_{r-1}$.) In the consecutive steps edges between vertices u and w with $u \prec w \prec v_{r+1}$ are neither deleted nor inserted any more. Hence $\lambda(G') \leqslant \mathcal{R}_{G'}(f) \leqslant \mathcal{R}_G(f) = \lambda(G)$.

It remains to show that $\prec$ is an SLO-ordering of the vertices V in G'. Property (S3) holds by definition of the ordering $\prec$. By construction (S2) holds. Moreover, G' is built by stepwise adding layers to a ball. Thus property (S1) holds and the first statement follows.

Now assume that G has the Faber–Krahn property. Then equality holds in (12) everywhere. Furthermore, f must be an eigenfunction of the first Dirichlet eigenvalue for every graph G_i in this sequence. Otherwise, if f is not an eigenfunction of a graph G_i then $\lambda(G_i) < \mathcal{R}_{G_i}(f) = \lambda(G)$, a contradiction by Proposition 3.

For switching step $G_{r-1} \to G_r$ we have $f(v_r) \geqslant f(u_{r-1})$. If $f(v_r) = f(u_{r-1})$ there would be nothing to do (we only change the positions of v_r and u_{r-1} in the ordering $\prec$). Hence we have $f(v_r) > f(u_{r-1})$ and by Lemma 6, $\lambda(G_r) < \lambda(G_{r-1})$, a contradiction to the Faber–Krahn property of G.

The monotonicity property of f follows for the same reasons. □

Lemma 9. *Let $G(V,E)$ be a tree with boundary in some $\mathcal{T}_\pi$. Then there exists an SLO*-tree $G^*(V,E^*)$ in $\mathcal{T}_\pi$ with $\lambda(G^*) \leqslant \lambda(G)$.*

Proof. Let again $n = |V|$ and $k = |V_0|$ denote the number of vertices and of interior vertices of G, respectively, and let f be a non-negative eigenfunction of the first Dirichlet eigenvalue of G. Then by Lemma 8 there exists an SLO-tree $G'_0 = G'(V,E')$ in $\mathcal{T}_\pi$ with the SLO-ordering $\prec$. The vertices of G (and G') $V = \{v_0, v_1, \dots, v_{k-1}, v_k, \dots, v_{n-1}\}$ are numbered such that $v_i \prec v_j$ if and only if $i < j$. Moreover, by the construction in the proof of Lemma 8 we find $f(v) \geqslant f(w)$ if $v \prec w$. The degree sequence of G is given by $\pi = (d_0, d_1, \dots, d_{k-1}, d_k, \dots, d_{n-1})$ such that the degrees d_i are non-decreasing for $0 \leqslant i < k$, and $d_j = 1$ for $j \geqslant k$ (i.e., correspond to boundary vertices).

Now we start with root v_0. If $d_{v_0} = d_0$ ($= \min_{0 \leqslant i \leqslant k} d_i$) then there is nothing to do. Otherwise, we can use shifting to replace all edges (v_0, v_{d_0+1}), (v_0, v_{d_0+2}), $\dots$, by the respective edges (v_1, v_{d_0+1}), (v_1, v_{d_0+2}), $\dots$. As $v_0 \prec v_1 \prec v_{d_0+1} \prec \cdots$ we have $f(v_0) \geqslant f(v_1) \geqslant f(v_{d_0+1}) \geqslant \cdots$ and thus we can apply Lemma 7 and get a new graph G'_1 with $\mathcal{R}_{G'_1}(f) \leqslant \mathcal{R}_{G'}(f)$. Notice that G'_1 is again an SLO-tree. Moreover, the number of boundary vertices remains unchanged, since either $d_0 = 2$ and there is nothing to do, or $d_0 \geqslant 3$ and the statement follows from Lemma 7. However, it might happen that the degree sequence has changed and $G'_1 \notin \mathcal{T}_\pi$.

Next we proceed in the same way with vertex v_1. We denote the degree of a vertex v_j in a graph G'_i with index i by $d^{(i)}_{v_j}$. Notice that $d^{(1)}_{v_1} \geqslant \min_{1 \leqslant i \leqslant k} d_i = d_1$. If $d^{(1)}_{v_1} = d_1$ there is nothing to do. Otherwise, we can use shifting to replace all edges (v_1, v_{s_1+1}), (v_1, v_{s_1+2}), $\dots$, by the respective edges (v_2, v_{s_1+1}), (v_2, v_{s_1+2}), $\dots$, where $s_1 = d_0 + d_1$. Again we can apply Lemma 7 and get a new graph G'_2 with $\mathcal{R}_{G'_2}(f) \leqslant \mathcal{R}_{G'_1}(f)$. We can continue in this way and get a sequence of SLO-trees

$$G \to G' = G'_0 \to G'_1 \to G'_2 \to \cdots \to G'_k = G^* \tag{13}$$

with

$$\lambda(G) = \mathcal{R}_G(f) \geqslant \mathcal{R}_{G'_0}(f) \geqslant \mathcal{R}_{G'_1}(f) \geqslant \cdots \geqslant \mathcal{R}_{G'_k}(f) \geqslant \lambda(G^*). \tag{14}$$

Notice that we always have $d^{(r)}_{v_r} \geqslant d_r$. This follows from the fact that $\sum_{j \leqslant r} d^{(0)}_{v_j} \geqslant \sum_{j \leqslant r} d_j$ as the right-hand side of this inequality is the minimum of any sum of degrees of j interior vertices of G'. Moreover, by our construction, $\sum_{j \leqslant r} d^{(r)}_{v_j} = \sum_{j \leqslant r} d^{(0)}_{v_j}$ and $\sum_{j<r} d^{(r)}_{v_j} = \sum_{j<r} d_j$. Hence $d^{(r)}_{v_r} = \sum_{j \leqslant r} d^{(r)}_{v_j} - \sum_{j<r} d^{(r)}_{v_j} = \sum_{j \leqslant r} d^{(0)}_{v_j} - \sum_{j<r} d_j \geqslant \sum_{j \leqslant r} d_j - \sum_{j<r} d_j = d_r$. In step

$G'_r \to G'_{r+1}$ there is either nothing to do, or edges are exchanged such that vertex v_r has the desired degree. In the consecutive steps edges that are incident to a vertex $u \prec v_{r+1}$ are neither deleted nor inserted.

The resulting SLO-tree G^* has the same degree sequence π as G and thus belongs to class $\mathcal{T}_\pi$. It also satisfies property $(S4)$, i.e., $\prec$ is an SLO*-ordering of the vertices. □

For our theorem on the class $\mathcal{T}_d$ we need a modified version of this lemma. To state this new proposition we need a partial ordering of degree sequences. Let $\pi = (d_0, d_1, \ldots, d_{k-1}, d_k, \ldots, d_{n-1})$ and $\pi' = (d'_0, d'_1, \ldots, d'_{k'-1}, d_{k'}, \ldots, d_{n-1})$ be two degree sequence of some trees with the same number of vertices n and respective numbers k and k' of interior vertices (not necessarily equal). Again we assume that the first k (and k', respectively) degrees correspond to the interior vertices and are ordered non-decreasingly. Then we write $\pi \trianglelefteq \pi'$ if the above condition holds and $\sum_{j \leqslant r} d_j \leqslant \sum_{j \leqslant r} d'_j$ for all $0 \leqslant r < n$.

Lemma 10. *Let $G(V, E)$ be a tree with boundary with degree sequence π and let π' another degree sequence with $\pi' \trianglelefteq \pi$. Then there exists an SLO*-tree $G^*(V, E^*)$ in $\mathcal{T}_{\pi'}$ with $\lambda(G^*) \leqslant \lambda(G)$.*

Proof. Completely analogous to the proof of Lemma 9. □

Notice that Lemma 9 is a special case of this lemma as $\pi \trianglelefteq \pi$. It can also be applied to prove Theorem 2 for class $\mathcal{T}_d$ as we immediately have $\pi^\circ \trianglelefteq \pi$ with $\pi^\circ = (d, d, \ldots, d, d^\circ, 1, \ldots, 1)$ where $d^\circ = d + \sum_{v \in V_0}(d_v - d)$.

Next we show that every tree with the Faber–Krahn property has an SLO*-ordering.

Lemma 11. *Let G be an SLO-tree with a non-negative eigenfunction f of $\lambda(G)$. Then every interior vertex v has a child w with $f(w) < f(v)$.*

Proof. First assume v is not the root of G. Let u be the parent of v. Then by Lemma 8 $f(v) \leqslant f(u)$ and $f(v) \geqslant f(w)$ for all children w of v. Now suppose that $f(v) = f(w)$ for all children of v. Then $\lambda(G) f(v) = \Delta f(v) = \sum_{(v,x) \in E}(f(v) - f(x)) = f(v) - f(u) \leqslant 0$, a contradiction as both $f(v) > 0$ and $\lambda(G) > 0$ by Proposition 4. If v is the root of G then all vertices adjacent to v are children of v. If we again suppose $f(w) = f(v)$ for all these children then we find analogously $\lambda(G) f(v) = 0$, again a contradiction. □

Lemma 12. *Let $G(V, E)$ be an SLO*-tree and f a non-negative eigenfunction to $\lambda(G)$. Let v and w be two vertices with $f(v) = f(w)$. Then the subtrees T_v and T_w rooted at v and w, respectively, are isomorphic.*

Proof. We prove this lemma by induction from boundary vertices to the root v_0. It is obviously trivial for boundary vertices. Without loss of generality we assume $v \prec w$.

We start with the case where v is not the root v_0 of SLO*-ordering. Let u_v and u_w be the parents of v and w, respectively. Then from $\Delta(G) f(v)$ and $\Delta(G) f(w)$ we get $f(u_v) = (d_v - \lambda(G)) f(v) - \sum_{(v,x) \in E, x \neq u_v} f(x)$ and $f(u_w) = (d_w - \lambda(G)) f(w) - \sum_{(w,y) \in E, y \neq u_w} f(y)$.

By property (S2) and Lemma 8 we have $f(u_v) \geqslant f(u_w)$ and therefore it follows from $f(v) = f(w)$,

$$(d_w - d_v) f(v) \leqslant \sum_{\substack{(w,y)\in E\\ y\neq u_w}} f(y) - \sum_{\substack{(v,x)\in E\\ x\neq u_v}} f(x), \tag{15}$$

where the sums on the right-hand side are over all children of w and v, respectively. Let m be a child of v such that $f(m) \leqslant f(x)$ for all children x of v. Notice that by (S2) $x \prec y$ and thus by Lemma 8 $f(x) \geqslant f(y)$ for all children y of w; in particular $f(m) \geqslant f(y)$. Thus $\sum_{(v,x)\in E, x\neq u_v} f(x) \geqslant (d_v - 1) f(m)$ and $\sum_{(w,y)\in E, y\neq u_w} f(y) \leqslant (d_w - 1) f(m)$. Consequently

$$\sum_{\substack{(w,y)\in E\\ y\neq u_w}} f(y) - \sum_{\substack{(v,x)\in E\\ x\neq u_v}} f(x) \leqslant (d_w - d_v) f(m) \tag{16}$$

and by (15) $(d_w - d_v) f(v) \leqslant (d_w - d_v) f(m)$.

By Proposition 4 and Lemma 11, $0 < f(m) < f(v)$. By property (S4), $d_v \leqslant d_w$. Hence $d_v = d_w$. Then the right-hand side of (15) (and left-hand side of (16)) vanishes and f must have the same value for all children of v and w (in particular $f(x) = f(y)$). It then follows by induction that T_v and T_w are isomorphic.

The case where v is the root v_0 of SLO*-ordering, remains. Then we set $u_v = v_1$ and all estimations are still valid. Thus the proposition follows. □

Lemma 13. *If a tree $G(V,E)$ with boundary has the Faber–Krahn property in some class $\mathcal{T}_\pi$, then G is an SLO*-tree.*

Proof. By Lemma 8, G is an SLO-tree. In the proof of Lemma 9 we have produced sequence (13) of trees where inequalities (14) hold. Since G has the Faber–Krahn property, equality holds in each of these inequalities. Notice that G' and G^* are in class $\mathcal{T}_\pi$ while all other graphs G'_i need not. However, for every graph G'_i in this sequence that belongs to $\mathcal{T}_\pi$ we have by the Faber–Krahn property $\lambda(G'_i) = \lambda(G)$ and f is also an eigenfunction of the first Dirichlet eigenvalue of G'_i. Otherwise we had $\lambda(G'_i) < \mathcal{R}_{G'_i}(f) = \lambda(G)$, a contradiction.

Now suppose there is a graph $G'_r \in \mathcal{T}_\pi$ while $G'_{r+1} \notin \mathcal{T}_\pi$. We denote the children of vertex v_r in G'_r by $w_1, \ldots, w_s$ and its parent by u_r. In step $G'_r \to G'_{r+1}$ we replace the edges $(v_r, w_{d_r}), \ldots, (v_r, w_s)$ by the respective edges $(v_{r+1}, w_{d_r}), \ldots, (v_{r+1}, w_s)$. Hence $s > d_r - 1$, since otherwise there would be nothing to do and $G'_{r+1} = G'_r$, a contradiction to $G'_{r+1} \notin \mathcal{T}_\pi$. Notice that the neighbors of v_r in G'_{r+1} do not change any more in the subsequent steps. As f is an eigenfunction of both G'_r and G^* to the same eigenvalue $\lambda(G)$ it follows that $\Delta(G'_r) f(v_r) = \Delta(G^*) f(v_r)$, i.e.,

$$(s+1) f(v_r) - f(u_r) - \sum_{j=1}^{s} f(w_j) = d_r f(v_r) - f(u_r) - \sum_{j=1}^{d_r - 1} f(w_j)$$

and thus $(s - d_r + 1) f(v_r) = \sum_{j=d_r}^{s} f(w_j)$. Since $f(v_r) \geqslant f(w_1) \geqslant f(w_j) \geqslant f(w_s) \geqslant 0$ for all $j = 1, \ldots, s$ by Lemma 8, we find $f(v_r) = f(w_j)$ for all children w_j, a contradiction to Lemma 11. If $r = 0$, i.e., v_r is the root and there is no parent of v_r, the same argument holds analogously.

Hence there cannot be a graph $G'_r \in \mathcal{T}_\pi$ while $G'_{r+1} \notin \mathcal{T}_\pi$. Therefore each graph G'_i in sequence (13) belongs to class $\mathcal{T}_\pi$ and f is an eigenfunction for each of these. We show for each

r that G'_r is isomorphic to G'_{r+1} and consequently isomorphic to G^*. Thus all these graphs, in particular G'_0, are SLO*-trees. Notice that for step $G'_r \to G'_{r+1}$ we either find $G'_r = G'_{r+1}$, or $f(v_r) = f(v_{r+1})$, since otherwise we had $\mathcal{R}_{G'_r}(f) > \mathcal{R}_{G'_{r+1}}(f)$ by Lemma 7. In the first case there remains nothing to show. In the latter case the subtrees (of both G'_r and G'_{r+1}) rooted at the respective vertices v_r and v_{r+1} are isomorphic by Lemma 12. As only edges incident to v_r are shifted to v_{r+1} the isomorphism between G'_r and G'_{r+1} follows. □

Now we are ready to prove our theorems.

Proof of Theorem 3. The necessity of the condition has been shown in Lemma 13. The sufficiency follows from the fact that SLO*-trees are uniquely determined up to isomorphism (Lemma 2). □

Proof of Theorem 2. Let $\pi = (d_0, d_1, \ldots, d_{k-1}, 1, \ldots, 1)$ be the degree sequence of G, where $d \leqslant d_0 \leqslant d_1 \leqslant \cdots \leqslant d_{k-1}$ are the degrees of the interior vertices. Define a new degree sequence by $\pi^\circ = (d, d, \ldots, d, d^\circ, 1, \ldots, 1)$ where $d^\circ = d + \sum_{v \in V_0}(d_v - d)$. Then $\pi' \trianglelefteq \pi$ and we can apply Lemma 10. The necessity of the condition follows analogously to the proof Lemma 13. The sufficiency follows from the fact that SLO*-trees are uniquely determined up to isomorphism (Lemma 2). □

Proof of Theorem 1. This is an immediate corollary of Theorem 2 as $\mathcal{T} = \mathcal{T}_2$. □

Remark 7. The procedure that was used for the proof of Theorem 3 can also be stated by the algorithm below.

Algorithm Rearrange.

Input: Tree $G(V, E) \in \mathcal{T}_\pi$.
Output: Tree $G^*(V, E^*) \in \mathcal{T}_\pi$ with SLO*-ordering $\prec$ and $\lambda(G^*) \leqslant \lambda(G)$.

Compute non-negative eigenfunction f of lowest Dirichlet eigenvalue.
Enumerate vertices $v_0, v_1, \ldots, v_{n-1}$ such that $f(v_i) \geqslant f(v_j)$ if $i \leqslant j$.
Define a well-ordering $\prec$: $v_i \prec v_j$ if and only if $i < j$.
Set $s \leftarrow 0$.
for $r = 0, \ldots, k-1$ **do**
    for $i = 1, \ldots, d_0$ if $r = 0$ **do** $[i = 1, \ldots, d_r - 1]$
        Set $s \leftarrow s + 1$ (increment s).
        if v_s is not adjacent to v_r **then**
            Select an edge (v_r, w_r) such that $v_s \prec w_r$.
            Select an edge (v_s, w_s) such that $v_s \prec w_s$ and w_s is in the geodesic path from v_r to v_s if and only if w_r is not.
            Apply switching such that the new graph G_s has edges (v_r, v_s) and (w_r, w_s).
        end if
    end for
    for all $(v, v_r) \in E$ with $v_s \prec v$ **do**
        Apply shifting such that edge (v, v_r) is replaced by edge (v, v_{r+1}).
    end for
end for
Return $G^* = G_s$.

4. Further results

One might ask what happens when we relax the conditions in the classes $\mathcal{T}^{(n,k)}$ and $\mathcal{T}_d^{(n,k)}$. We then get the following classes:

$$\mathcal{T}^{(n,\cdot)} = \{G \text{ is a tree, with } |V| = n\}, \tag{17}$$

$$\mathcal{T}_d^{(n,\cdot)} = \{G \in \mathcal{T}^{(n,\cdot)}\colon\ d_v \geqslant d \text{ for all } v \in V_0\}, \tag{18}$$

where we keep the total number of vertices fixed, and

$$\mathcal{T}^{(\cdot,k)} = \{G \text{ is a tree, with } |V_0| = k\}, \tag{19}$$

$$\mathcal{T}_d^{(\cdot,k)} = \{G \in \mathcal{T}^{(\cdot,k)}\colon\ d_v \geqslant d \text{ for all } v \in V_0\}, \tag{20}$$

where we keep the number of interior vertices fixed. Using the arguments from the proofs of our theorems we find the following characterizations for graphs with the Faber–Krahn property.

Theorem 8. *A tree G with boundary has the Faber–Krahn property*:

(i) *In* $\mathcal{T}^{(n,\cdot)}$ *if and only if it is a path with n vertices.* (*This is the result of* [12].)
(ii) *In* $\mathcal{T}_d^{(n,\cdot)}$ *if and only if it is an SLO*-tree where exactly one interior vertex has degree* d° *with* $d \leqslant d^\circ < 2d$ *and all other interior vertices have degree* d. (*This is the SLO*-tree in* $\mathcal{T}_d^{(n,\cdot)}$ *with the greatest number of interior vertices.*)
(iii) *In* $\mathcal{T}^{(\cdot,k)}$ *if and only if it is a path with* $k+2$ *vertices.*
(iv) *In* $\mathcal{T}_d^{(\cdot,k)}$ *if and only if it is an SLO*-tree where all interior vertices have degree* d.

G is then uniquely determined up to isomorphism.

For the classes $\mathcal{T}_\pi$ we cannot give a similar theorem. However, we can ask whether we can compare the least first Dirichlet eigenvalue in classes with the same number of vertices. From Lemma 10 we can derive the following result.

Theorem 9. *Let* π *and* π' *be two tree sequences with* $|\pi| = |\pi'|$ *and let* G *and* G' *be trees with the Faber–Krahn property in* $\mathcal{T}_\pi$ *and* $\mathcal{T}_{\pi'}$, *respectively. If* $\pi' \trianglelefteq \pi$ *then* $\lambda(G) \leqslant \lambda(G')$ *where equality holds if and only if* $\pi = \pi'$.

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References

[1] N. Biggs, Algebraic Graph Theory, second ed., Cambridge Univ. Press, Cambridge, UK, 1994.
[2] G. Carron, Inégalités isopérimétriques de Faber–Krahn et conséquences, in: A.L. Besse (Ed.), Actes de la table ronde de géométrie différentielle en l'honneur de Marcel Berger, Luminy, France, 12–18 juillet, 1992, in: Sémin. Congr., vol. 1, Soc. Math. France, Paris, 1996, pp. 205–232.
[3] G. Carron, Inégalités de Faber–Krahn et inclusion de Sobolev–Orlicz, Potential Anal. 7 (2) (1997) 555–575.
[4] I. Chavel, Eigenvalues in Riemannian Geometry, Academic Press, Orlando, FL, 1984.
[5] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs – Theory and Applications, Academic Press, New York, 1980.

[6] D.M. Cvetković, M. Doob, I. Gutman, A. Torgašev, Recent Results in the Theory of Graph Spectra, Ann. Discrete Math., vol. 36, North Holland, Amsterdam, 1988.
[7] E.B. Davies, G.M.L. Gladwell, J. Leydold, P.F. Stadler, Discrete nodal domain theorems, Linear Algebra Appl. 336 (1–3) (2001) 51–60.
[8] Y.C. de Verdière, Le trou spectral des graphes et leurs propriétés d'expansion, Séminaire de théorie spectral et géométrie, 1993–1994, pp. 51–68.
[9] J. Friedman, Some geometric aspects of graphs and their eigenfunctions, Duke Math. J. 69 (3) (1993) 487–525.
[10] F. Harary, Graph Theory, Addison–Wesley, Reading, MA, 1969.
[11] A. Katsuda, H. Urakawa, The first eigenvalue of the discrete Dirichlet problem for a graph, J. Combin. Math. Combin. Comput. 27 (1998) 217–225.
[12] A. Katsuda, H. Urakawa, The Faber–Krahn type isoperimetric inequalities for a graph, Tohoku Math. J. (2) 51 (2) (1999) 267–281.
[13] J. Leydold, A Faber–Krahn-type inequality for regular trees, Geom. Funct. Anal. 7 (2) (1997) 364–378.
[14] J. Leydold, The geometry of regular trees with the Faber–Krahn property, Discrete Math. 245 (1–3) (2002) 155–172.
[15] A.R. Pruss, Discrete convolution-rearrangement inequalities and the Faber–Krahn inequality on regular trees, Duke Math. J. 91 (3) (1998) 463–514.