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A note on Laplacian graph eigenvalues

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Abstract

Let $G = (V, E)$ be a graph on n vertices. Denote by $d(v)$ the degree of $v \in V$ and by $m(v)$ the average of the degrees of the vertices of G adjacent to v. Then $b(G) = \max\{m(v) + d(v)\colon v \in V\}$ is an upper bound for the Laplacian spectral radius of G; hence, $n - b(G^C)$ is a lower bound for the algebraic connectivity of G in terms of the vertex degrees of its complement.

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Let $G = (V, E)$ be a graph on vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The Laplacian matrix $L(G) = D(G) - A(G)$ is the difference of $D(G) = \mathrm{diag}\,(d(v_1), d(v_2), \ldots, d(v_n))$, the diagonal matrix of vertex degrees, and the adjacency matrix. Because its rows sum to 0, $L(G)$ is singular. Denoting its eigenvalues by $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$, we see from Geršgorin's Theorem that $\lambda_n = 0$ and $\lambda_1 \leqslant 2\max\{d(v)\colon v \in V\}$. Anderson and Morley [1] improved this upper bound on the spectral radius by showing that

$$\lambda_1 \leqslant \max\{d(u) + d(v)\colon uv \in E\}. \tag{1}$$

If $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$ are the degrees of the vertices of G (we are not necessarily assuming that $d_i = d(v_i)$), then the "main result" of [2] is

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$$\lambda_1 \leqslant 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)}. \tag{2}$$

Denote by $m(v)$ the average of the degrees of the vertices adjacent to v. Then $d(v)m(v)$ is the "2-degree" of vertex v [3].

Theorem. *If G is a graph then*

$$\lambda_1 \leqslant \max\{m(v) + d(v): \ v \in V\}. \tag{3}$$

Proof. If G has no edges, both sides of Eq. (3) are zero. Otherwise, it suffices to prove the result for connected graphs, and the inequality follows by applying Geršgorin's Theorem to the rows of $D(G)^{-1}L(G)D(G)$, the matrix whose (i, j)-entry is $d(v_i)$ if $i = j$, $-d(v_j)/d(v_i)$ if $v_iv_j \in E$, and 0 otherwise. □

Remark. That Eq. (3) improves Eq. (1) is clear. It is less obvious that Eq. (3) improves Eq. (2) provided $G \neq P_4$, the path on four vertices.

Let $a(G) = \lambda_{n-1}$, the "algebraic connectivity" of G and denote the right-hand side of Eq. (3) by $b(G)$. Because $\lambda_1(G^C) = n - a(G)$, where G^C is the complement of G, we see from Eq. (3) that

$$a(G) \geqslant n - b(G^C). \tag{4}$$

Denote the right-hand side of Eq. (1) by r and suppose $xy \in E$ satisfies $d(x) + d(y) = r$. Let $s = \max\{d(u) + d(v): \ uv \in E \setminus xy\}$. Having observed that the proof of Eq. (2) "is similar to the argument in" [1], the authors of [2] assert that the following can be obtained "in a similar way":

$$\lambda_1 \leqslant 2 + \sqrt{(r - 2)(s - 2)}. \tag{5}$$

Examples. Values of λ_1 and of the various bounds for the two graphs illustrated in Fig. 1 are given (to two decimal places) in Fig. 2. Note that Eq. (5) is better than Eq. (3) in the case of G_1. Extensive empirical evidence suggests, however, that Eq. (3) is usually the better bound.

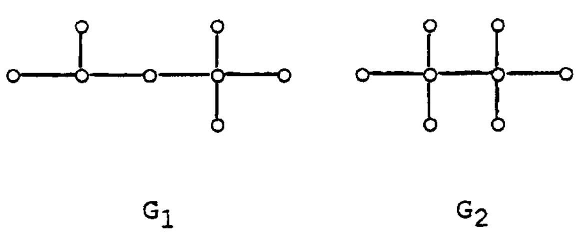

Fig. 1.

	λ_1	(1)	(2)	(3)	(5)
G_1	5.12	6	6.47	5.50	5.46
G_2	5.65	8	6.24	5.75	6.24

Fig. 2.

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