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On the spectrum of the Laplacian on regular metric trees

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Abstract

A metric tree Γ is a tree whose edges are viewed as non-degenerate line segments. The Laplacian $\boldsymbol{\Delta}$ on such a tree is the operator of second order differentiation on each edge, complemented by the Kirchhoff matching conditions at the vertices. The spectrum of $\boldsymbol{\Delta}$ can be quite varied, reflecting the geometry of a tree.

We consider a special class of trees, namely the so-called regular metric trees. Any such tree Γ possesses a rich group of symmetries. As a result, the space $\mathsf{L}^2(\Gamma)$ decomposes into the orthogonal sum of subspaces reducing the operator $\boldsymbol{\Delta}$. This leads to detailed spectral analysis of $\boldsymbol{\Delta}$. We survey recent results on this subject.

1. Introduction

In classical graph theory a graph is considered as a combinatorial object. A function on such a graph is defined on the set of its vertices, and the Laplacian is a discrete operator.

In contrast, a metric graph $\mathcal{G}$ is a graph whose edges are regarded as non-degenerate line segments. A function on $\mathcal{G}$ is a family of functions defined on its edges, and the Laplacian on $\mathcal{G}$ acts as $\boldsymbol{\Delta} f = -f''$ on each edge; we include the sign '−' in the definition of the Laplacian. Functions in its domain satisfy certain matching conditions at the vertices. The spectral theory of the Laplacian on metric graphs is much less developed than its counterpart for the discrete Laplacian.

Regular metric trees form an important subclass of general metric graphs. We define regular trees in definition 2.1. The reader should keep in mind that the term 'regular' is universally overloaded. In particular, it may have different meanings in different branches of graph theory.

A regular tree Γ has a rich group of symmetries. This allows one to construct an orthogonal decomposition of the space $\mathsf{L}^2(\Gamma)$ (*the basic decomposition*) which reduces the Laplacian. With its help a detailed spectral analysis of the Laplacian is possible.

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Our goal is to survey the recent results on this subject. The paper can be considered as an expanded version of the article [15]. Many results are presented here in more detail, but we often give references and informal explanations rather than rigorous proofs. Among the new results we specially mention theorem 5.3 and example 6.2. On the other hand, we do not reproduce the results of [15] concerning the Schrödinger operator.

There are several papers devoted to differential operators on regular metric trees and related problems. In [9] the weighted spectral problems of the type $\boldsymbol{\Delta} f = \lambda V f$ were investigated, with a non-negative weight function $V \in \mathsf{L}^1(\Gamma)$. For this equation the eigenvalue estimates were obtained in terms of the geometry of Γ and properties of V. The tree Γ was not assumed to be regular. Some of the results of [9] for general trees were later refined in the paper [5].

In the same paper [9] the notion of a regular metric tree was singled out and the basic decomposition of the space $\mathsf{L}^2(\Gamma)$ on such trees was constructed. It is reproduced here as the first part of the formula (3.5). With its help much more precise results were obtained for regular trees than for arbitrary trees. Let us note that for the combinatorial trees a decomposition similar to (3.5) was known before, see e.g. [1] and [12].

In the paper [3] the basic decomposition was discovered independently and applied to the spectral analysis of the Laplacian and the Sturm–Liouville operator on the regular trees Γ of finite height. In particular, it was proved in [3] that for such trees each operator $\mathfrak{A}_k$ appearing in the decomposition (3.13) is compact and the counting functions for its eigenvalues satisfies the estimate $N(\lambda; \mathfrak{A}_k) = \mathrm{O}(\lambda^{1/2+\varepsilon})$ with any $\varepsilon > 0$. Our theorem 4.1 substantially refines this result.

In the same way as in the $\mathbb{R}^d$-setting, the spectral theory of the Laplacian on trees is closely related to the theory of Sobolev spaces. One has to distinguish between two types of such spaces on a tree. Let Γ be a rooted tree with the root o. We define the spaces

$$\mathsf{H}^{1,\bullet}(\Gamma) = \left\{ f \in \mathsf{C}(\Gamma) : f(o) = 0, \int_\Gamma (|f'|^2 + |f|^2)\,\mathrm{d}x < \infty \right\}$$

(cf section 3.1 for the detailed description), and

$$\mathcal{H}^1(\Gamma) = \left\{ f \in \mathsf{H}^{1,\bullet}_{\mathrm{loc}}(\Gamma) : \int_\Gamma |f'|^2\,\mathrm{d}x < \infty \right\},$$

see section 7.3. The space $\mathsf{H}^{1,\bullet}(\Gamma)$ is the quadratic domain of the Laplacian as an operator in $\mathsf{L}^2(\Gamma)$, and it is important that it is not always dense in $\mathcal{H}^1(\Gamma)$. The trees which meet this density condition are called recurrent; otherwise they are called transient.

The difference between these two types of tree is important when studying Hardy's inequalities. A function $V \geqslant 0$ is called a Hardy weight on a tree Γ with the root o if the inequality

$$\int_\Gamma V|f|^2\,\mathrm{d}x \leqslant c \int_\Gamma |f'|^2\,\mathrm{d}x, \qquad c > 0 \tag{1.1}$$

holds for all $f \in \mathcal{H}^1(\Gamma)$. In the paper [6] a criterion on a function $V \geqslant 0$ to be a Hardy weight on an arbitrary tree was established. The similar problem in L^p, $1 < p < \infty$, was also solved there.

The positive definiteness of the Laplacian is equivalent to the inequality (1.1) with $V \equiv 1$ but for the functions from the narrower space $\mathsf{H}^{1,\bullet}(\Gamma)$. For this space the class of admissible weights is wider, and the result of [6] does not apply. The description of all *symmetric* (i.e. depending only on $\mathrm{dist}(x, o)$) admissible weights on $\mathsf{H}^{1,\bullet}(\Gamma)$ was given in [10]. The criterion of positive definiteness of the Laplacian on Γ (see theorem 5.2 of the present paper) is a particular case of the description obtained. Note that the conditions of [6] are never satisfied for $V \equiv 1$, provided that Γ is a tree of infinite height.

A tree is called *homogeneous* if all its edges have equal length and all its vertices are of the same degree. This is a special case of a regular tree. Homogeneity of a tree can be regarded as a sort of periodic structure. Hence, it is natural to expect some resemblance between the spectral properties of the Laplacian on such trees and those of the periodic differential operators on the real axis. Such resemblance was revealed in the paper [2]. In particular, the bandgap structure of the spectrum of the Laplacian on the homogeneous trees was established there.

A complete description of the spectrum of the Laplacian on homogeneous trees was given in [13] where also the Schrödinger operator with a symmetric potential was analysed. This description is reproduced below as example 6.3.

In the paper [14] a new approach to the eigenvalue estimates for the equation $\boldsymbol{\Delta} f = \lambda V f$ on metric graphs of finite length (not necessarily trees) was developed. As one application, the Weyl-type asymptotics for this equation was justified. Our theorem 4.1(ii) is a special case of this result. Another proof was given in [15]. The results of the paper [5] on asymptotics can also serve as a basis for obtaining this theorem.

The approach of [14] is based upon a special technique of approximation of functions from the Sobolev space $\mathsf{H}^1 = \mathsf{W}_2^1$ on graphs. In [16] this approach was extended to the spaces W_p^1 with $p \neq 2$.

In this short survey we listed only the results directly related to the material of the work presented. In the paper [7] the reader can find a detailed survey of the whole field, with the comprehensive list of literature.

2. Regular rooted metric trees

2.1. Geometry of a tree

Let Γ be a rooted tree with the root o, the set of vertices $\mathcal{V} = \mathcal{V}(\Gamma)$ and the set of edges $\mathcal{E} = \mathcal{E}(\Gamma)$. We suppose that $\#\mathcal{V} = \#\mathcal{E} = \infty$. Each edge e of a metric tree is viewed as a non-degenerate line segment of length $|e|$. The distance $\rho(x, y)$ between any two points $x, y \in \Gamma$, and thus the metric topology on Γ, is introduced in a natural way, and $|x|$ stands for $\rho(x, o)$. A subset $E \subset \Gamma$ is compact if and only if it is closed and has non-empty intersections with only a finite number of edges.

For any two points $x, y \in \Gamma$ there exists a simple polygonal path in Γ which starts at x and terminates at y. This path is unique and we denote it by $\langle x, y\rangle$. We write $x \prec y$ if $x \in \langle o, y\rangle$ and $x \neq y$. The relation $\prec$ defines a partial ordering on Γ.

For any vertex v its *generation* $\operatorname{gen}(v)$ is defined as

$$\operatorname{gen}(v) = \#\{x \in \mathcal{V}(\Gamma) : x \prec v\}.$$

In particular, $v = o$ is the only vertex such that $\operatorname{gen}(v) = 0$. For any edge emanating from vertex v (which means that $e = \langle v, w\rangle$ and $v \prec w$) we define the generation as $\operatorname{gen}(e) := \operatorname{gen}(v)$.

The *branching number* $b(v)$ of a vertex v is defined as the number of edges emanating from v. We assume that $\operatorname{gen}(v) < \infty$ for any v and $b(v) > 1$ for $v \neq o$. We denote by e_v^- the only edge which terminates at a vertex $v \neq o$, and by $e_v^1, \dots, e_v^{b(v)}$ the edges emanating from $v \in \mathcal{V}$.

Definition 2.1. *We call a tree Γ regular if all the vertices of the same generation have equal branching numbers, and all the edges of the same generation are of the same length.*

Any regular tree is fully determined by specifying two number sequences (*generating sequences*) $\{b_n\} = \{b_n(\Gamma)\}$ and $\{t_n\} = \{t_n(\Gamma)\}$, $n = 0, 1, \dots$ such that

$$b(v) = b_{\operatorname{gen}(v)}, \qquad |v| = t_{\operatorname{gen}(v)} \qquad \forall v \in \mathcal{V}(\Gamma).$$

We assume that $b_n \geqslant 2$ for any $n > 0$. This means that we ignore 'unessential' vertices $v \neq o$, such that $b(v) = 1$. It is clear that $t_0 = 0$ and the sequence $\{t_n\}$ is strictly increasing, and we denote

$$h_\Gamma = \lim_{n\to\infty} t_n = \sup_{x\in\Gamma} |x|.$$

We call h_Γ *the height of* Γ. Another useful characteristic of the regular tree is its *branching function*

$$g_\Gamma(t) = \#\{x \in \Gamma : |x| = t\}, \qquad 0 \leqslant t < h_\Gamma.$$

Clearly,

$$g_\Gamma(0) = 1; \qquad g_\Gamma(t) = b_0 \cdots b_n, \qquad t_n < t \leqslant t_{n+1}, n = 0, 1, \ldots.$$

We also introduce *the reduced height of* Γ

$$L_\Gamma = \int_0^{h_\Gamma} \frac{\mathrm{d}t}{g_\Gamma(t)}.$$

It follows from $g_\Gamma(t) \geqslant 1$ that $L_\Gamma \leqslant h_\Gamma$, so that $h_\Gamma < \infty$ implies $L_\Gamma < \infty$. For the trees of infinite height both $L_\Gamma < \infty$ and $L_\Gamma = \infty$ are possible.

The natural measure $\mathrm{d}x$ on Γ is induced by the Lebesgue measure on the edges. The spaces $\mathsf{L}^p(\Gamma)$ are understood as L^p-spaces with respect to this measure. We denote by $|E|$ the measure of a (measurable) subset $E \subset \Gamma$ and call the number $|\Gamma|$ *the total length* of Γ. It is clear that $\rho(x, y) = |\langle x, y\rangle|$ for any pair of points $x, y \in \Gamma$ and that $|\Gamma| = \int_\Gamma g_\Gamma(t)\,\mathrm{d}t$.

2.2. *Special subtrees of* Γ

Subtrees $T \subset \Gamma$ of the following two types play a special part in the further analysis. For any vertex v and for any edge $e = \langle v, w\rangle$, $v \prec w$ we set

$$T_v = \{x \in \Gamma : x \succeq v\}, \qquad T_e = e \cup T_w.$$

Evidently, $T_o = \Gamma$ and

$$T_v = \bigcup_{1\leqslant j\leqslant b(v)} T_{e_v^j}, \qquad \forall v \in \mathcal{V}(\Gamma).$$

Along with the function g_Γ, define the functions

$$g_k(t) = \#\{x \in T_e : |x| = t\}, \qquad \forall e \in \mathcal{E}(\Gamma) : \mathrm{gen}(e) = k. \tag{2.1}$$

It is clear that $g_0 = g_\Gamma$ and

$$g_k(t) = (b_0 \cdots b_k)^{-1} g_\Gamma(t), \qquad t \in [t_k, h_\Gamma), k \in \mathbb{N}. \tag{2.2}$$

3. The Laplacian on a regular tree

The notion of differential operator on any metric graph, in particular on a tree, is well known. Still, for the sake of completeness we present here the variational definition of the Laplacian on a tree.

3.1. *The Sobolev space on* Γ

We say that a scalar-valued function f on Γ belongs to the Sobolev space $\mathsf{H}^1 = \mathsf{H}^1(\Gamma)$ if f is continuous, $f \restriction e \in \mathsf{H}^1(e)$ for each edge e and

$$\|f\|^2_{\mathsf{H}^1} := \int_\Gamma (|f'(x)|^2 + |f(x)|^2)\,\mathrm{d}x < \infty.$$

The derivative of a function $f \restriction e$ at an interior point $x \in e$ is always taken in the direction compatible with the partial ordering on Γ. This agreement is irrelevant for the definition of H^1 but we shall use it later. By $\mathsf{H}^{1,\bullet} = \mathsf{H}^{1,\bullet}(\Gamma)$ we denote the subspace $\{f \in \mathsf{H}^1 : f(o) = 0\}$. We do not impose any conditions on the behaviour of functions $f \in \mathsf{H}^{1,\bullet}$ as $|x| \to h_\Gamma$, other than those implied by the requirement $f, f' \in \mathsf{L}^2$. For this reason, we prefer to avoid the notation $\mathsf{H}^{1,0}$.

Actually, this precaution is necessary only if $h_\Gamma < \infty$. This is implied by the following simple fact.

Lemma 3.1. *Let $h_\Gamma = \infty$, then any function $f \in H^1(\Gamma)$ vanishes as $|x| \to \infty$. More exactly, for any $\varepsilon > 0$ there exists a number $t_\varepsilon = t_\varepsilon(f) > 0$, such that*

$$|f(x)| < \varepsilon \qquad \forall x \in \Gamma : |x| > t_\varepsilon. \tag{3.1}$$

Proof. Take any infinite path $\Lambda \subset \Gamma$ starting at the root o. Any such path can be identified with $\mathbb{R}_+$, and $f \in \mathsf{H}^1(\Gamma)$ implies $f \restriction \Lambda \in \mathsf{H}^1(\mathbb{R}_+)$. Hence, $f(x) \to 0$ along Λ, and therefore for any $x \in \Lambda$

$$|f(x)|^2 = -\int_{\Lambda_x} 2\,\mathrm{Re}(f'(y)\overline{f(y)})\,\mathrm{d}y, \qquad \Lambda_x = \{y \in \Lambda : x \prec y\}.$$

Given an $\varepsilon > 0$, choose t_ε such that

$$\int_{x\in\Gamma:|x|>t_\varepsilon} (|f'(x)|^2 + |f(x)|^2)\,\mathrm{d}x < \varepsilon^2.$$

Then $\int_{\Lambda_x} (|f'(x)|^2 + |f(x)|^2)\,\mathrm{d}x < \varepsilon^2$ for any $x \in \Gamma$ with $x > t_\varepsilon$ and any path $\Lambda \ni x$. This yields (3.1) by Cauchy's inequality. □

3.2. *The operator* $\boldsymbol{\Delta}$

We define the (positive) Laplacian $\boldsymbol{\Delta}$ as the self-adjoint operator in the space $\mathsf{L}^2(\Gamma)$, associated with the quadratic form $\int_\Gamma |f'|^2\,\mathrm{d}x$ considered on the form domain $\mathrm{Quad}(\boldsymbol{\Delta}) = \mathsf{H}^{1,\bullet}(\Gamma)$.

Let us describe the operator domain $\mathrm{Dom}(\boldsymbol{\Delta})$, though we do not use this description in our further reasonings.

Evidently $f \in \mathrm{Dom}(\boldsymbol{\Delta}) \Rightarrow f \restriction e \in \mathsf{H}^2(e)$ for each edge e and the Euler–Lagrange equation reduces on e to $\boldsymbol{\Delta} f = -f''$. Hence,

$$\sum_{e\in\mathcal{E}(\Gamma)} \int_e (|f''|^2 + |f'|^2 + |f|^2)\,\mathrm{d}x < \infty, \qquad \forall f \in \mathrm{Dom}(\boldsymbol{\Delta}).$$

At the root we have the boundary condition $f(o) = 0$. At each vertex $v \neq o$ the functions $f \in \mathrm{Dom}(\boldsymbol{\Delta})$ satisfy certain matching conditions. In order to describe them, denote by f_- the restriction $f \restriction e_v^-$ and by f_j, $j = 1, \dots, b(v)$ the restrictions $f \restriction e_v^j$. The matching conditions at $v \neq o$ are

$$f_-(v) = f_1(v) = \cdots = f_{b(v)}(v); \qquad f_1'(v) + \cdots + f_{b(v)}'(v) = f_-'(v). \tag{3.2}$$

These are nothing but the Kirchhoff laws well known in the theory of electrical networks. The first condition in (3.2) comes from the requirement $f \in \mathsf{H}^1(\Gamma)$ which includes continuity of f, and the second arises as the natural condition in the sense of calculus of variations. If $h_\Gamma = \infty$, the functions $f \in \mathrm{Dom}(\boldsymbol{\Delta})$ vanish as $|x| \to \infty$ by lemma 3.1, and the conditions listed give the complete description of $\mathrm{Dom}(\boldsymbol{\Delta})$.

If $h_\Gamma < \infty$, the functions $f \in \mathrm{Dom}(\boldsymbol{\Delta})$ satisfy the natural condition (again, in the sense of calculus of variations) as $|x| \to h_\Gamma$. Roughly speaking, it consists in the requirement $g(|x|)f'(x) \to 0$. We give the precise formulation in section 3.4. In this paper we do not discuss the conditions at $|x| = h_\Gamma$, other than the natural condition; see [3] in this connection.

Due to the boundary condition $f(o) = 0$, the Laplacian on Γ splits into the orthogonal sum of the Laplacians on the subtrees whose initial edges are $e_o^1, \ldots, e_o^{b(o)}$. For this reason, in what follows we assume $b(o) = 1$. Only in the last section 7, where we discuss the Neumann Laplacian and the Laplacian on trees without boundary, will this assumption be discarded.

3.3. Reduction of the Laplacian

Our further analysis is based upon an orthogonal decomposition of the space $\mathsf{L}^2(\Gamma)$ which, for the case of regular trees, reduces the Laplacian.

Given a subtree $T \subset \Gamma$, we say that a function $f \in \mathsf{L}^2(\Gamma)$ belongs to the set (a closed subspace) $\mathcal{F}_T$ if and only if

$$f(x) = 0 \qquad \text{for } x \notin T; \qquad f(x) = f(y) \qquad \text{if } x, y \in T \text{ and } |x| = |y|.$$

In particular, $\mathcal{F}_\Gamma$ consists of all *symmetric* (i.e. depending only on $|x|$) functions from $\mathsf{L}^2(\Gamma)$.

We need the subspaces $\mathcal{F}_T$ associated with the subtrees T_e and T_v, introduced in section 2.2. To simplify our notations, we shall write $\mathcal{F}_e, \mathcal{F}_v$ instead of $\mathcal{F}_{T_e}, \mathcal{F}_{T_v}$.

Let $e \in \mathcal{E}(\Gamma)$ and $\mathrm{gen}(e) = k$. Each function $f \in \mathcal{F}_e$ can be identified with a function $\hat{\varphi}_f$ on $[t_k, h_\Gamma)$ such that

$$f(x) = \hat{\varphi}_f(t), \qquad \forall x \in T_e : |x| = t.$$

It is clear that

$$\int_\Gamma |f(x)|^2\,\mathrm{d}x = \int_{t_k}^{h_\Gamma} |\hat{\varphi}_f(t)|^2 g_k(t)\,\mathrm{d}t$$

where g_k is the function introduced in (2.1). In order to deal with a single weight function which does not depend on k, define

$$\varphi_f(t) = (b_0 \cdots b_k)^{-1/2}\hat{\varphi}_f(t).$$

Then

$$\int_\Gamma |f(x)|^2\,\mathrm{d}x = \int_{t_k}^{h_\Gamma} |\varphi_f(t)|^2 g_\Gamma(t)\,\mathrm{d}t, \qquad \forall f \in \mathcal{F}_e, \mathrm{gen}(e) = k$$

and the operator $J_e : f \mapsto \varphi_f$ acts as the natural isometry of $\mathcal{F}_e$ onto the weighted space $\mathsf{L}^2((t_k, h_\Gamma); g_\Gamma)$.

Let now $v \in \mathcal{V}(\Gamma)$ and $\mathrm{gen}(v) = k$. Let e_v^j, $j = 1, \ldots, b_k$ be the edges emanating from v. The corresponding subspaces $\mathcal{F}_{e_v^j}$ are mutually orthogonal and their orthogonal sum $\tilde{\mathcal{F}}_v$ contains $\mathcal{F}_v$:

$$\tilde{\mathcal{F}}_v := \mathcal{F}_{e_v^1} \oplus \cdots \oplus \mathcal{F}_{e_v^{b_k}} \supset \mathcal{F}_v.$$

Given a function $f \in \tilde{\mathcal{F}}_v$, denote by f^j its component in the subspace $\mathcal{F}_{e_v^j}$, $j = 1, \ldots, b_k$. Consider the operator

$$J_v : f \mapsto \boldsymbol{\varphi}_f := \{\varphi_{f^1}, \ldots, \varphi_{f^{b_k}}\}.$$

The operator J_v defines an isometry of the subspace $\tilde{\mathcal{F}}_v$ onto the Hilbert space $\mathbb{C}^{b_k} \otimes \mathsf{L}^2((t_k, h_\Gamma); g_\Gamma)$.

We need a special orthogonal decomposition of $\tilde{\mathcal{F}}_v$. Denote $\omega_k = \mathrm{e}^{2\pi\mathrm{i}/b_k}$. The vectors

$$\boldsymbol{h}^{\langle s\rangle} = b_k^{-1/2}\{\omega_k^s, \ldots, \omega_k^{s(b_k-1)}, 1\} \qquad s = 1, \ldots, b_k$$

form an orthogonal basis in $\mathbb{C}^{b_k}$. With each s and each scalar-valued function $\varphi \in \mathsf{L}^2((t_k, h_\Gamma); g_\Gamma)$, let us associate the vector-valued function

$$\boldsymbol{\varphi}^{\langle s\rangle} = \boldsymbol{h}^{\langle s\rangle} \otimes \varphi.$$

The set of all such functions $\boldsymbol{\varphi}^{\langle s\rangle}$ is a subspace in $\mathbb{C}^{b_k} \otimes \mathsf{L}^2((t_k, h_\Gamma); g_\Gamma)$; we denote it by $\mathcal{G}_k^{\langle s\rangle}$. Note that

$$\|\boldsymbol{\varphi}^{\langle s\rangle}(t)\|_{\mathbb{C}^{b_k}} = |\varphi(t)| \qquad \text{a.e. on } (t_k, h_\Gamma). \tag{3.3}$$

Define the subspaces $\mathcal{F}_v^{\langle s\rangle} = J_v^{-1}\mathcal{G}_k^{\langle s\rangle} \subset \tilde{\mathcal{F}}_v$. According to this definition, $f \in \mathcal{F}_v^{\langle s\rangle}$ if and only if there exists a function $\varphi \in \mathsf{L}^2((t_k, h_\Gamma); g_\Gamma)$ such that $J_v f = \boldsymbol{h}^{\langle s\rangle} \otimes \varphi$. Taking (3.3) into account, we see that the operator

$$J_v^{\langle s\rangle} = J_v \restriction \mathcal{F}_v^{\langle s\rangle} \tag{3.4}$$

acts as an isometry of $\mathcal{F}_v^{\langle s\rangle}$ onto $\mathsf{L}^2((t_k, h_\Gamma); g_\Gamma)$.

Given a vertex v, the subspaces $\mathcal{F}_v^{\langle s\rangle}$, $s = 1, \ldots, b_k$ are mutually orthogonal and $\mathcal{F}_v^{\langle b_k\rangle} = \mathcal{F}_v$, therefore

$$\mathcal{F}_v' := \tilde{\mathcal{F}}_v \ominus \mathcal{F}_v = \mathcal{F}_v^{\langle 1\rangle} \oplus \cdots \oplus \mathcal{F}_v^{\langle b_k-1\rangle}.$$

The following result was proved in [9, 10] and, in a slightly different form, in [3].

Theorem 3.2. *Let Γ be a regular metric tree and $b(o) = 1$. Then the subspaces $\mathcal{F}_v'$, $v \in \mathcal{V}(\Gamma)$ are mutually orthogonal and orthogonal to $\mathcal{F}_\Gamma$. Moreover,*

$$\mathsf{L}^2(\Gamma) = \mathcal{F}_\Gamma \oplus \sum_{v\in\mathcal{V}(\Gamma)} \oplus \mathcal{F}_v' = \mathcal{F}_\Gamma \oplus \sum_{k=1}^{\infty} \sum_{\mathrm{gen}(v)=k} \sum_{s=1}^{b_k-1} \oplus \mathcal{F}_v^{\langle s\rangle} \tag{3.5}$$

and both decompositions reduce the Laplacian on Γ.

3.4. Parts of the Laplacian in the reducing subspaces

Suppose that $v \in \mathcal{V}(\Gamma)$ and $1 \leqslant s \leqslant b(v)$. We call the pair (v, s) *admissible* if either $v = o$ and $s = 1$, or $v \neq o$ and $1 \leqslant s < b(v)$. Note that $\mathcal{F}_o^{\langle 1\rangle} = \mathcal{F}_\Gamma$. Our next goal is to understand the nature of each operator $\Delta \restriction \mathcal{F}_v^{\langle s\rangle}$ where (v, s) is an admissible pair.

To this end, let us consider the Hilbert space $\mathfrak{H}_k := \mathsf{L}^2((t_k, h_\Gamma); g_\Gamma)$ and the quadratic form in it,

$$\mathfrak{a}_k[\varphi] = \int_{t_k}^{h_\Gamma} |\varphi'(t)|^2 g_\Gamma(t)\,\mathrm{d}t, \qquad \varphi \in \mathsf{H}^{1,\bullet}((t_k, h_\Gamma); g_\Gamma). \tag{3.6}$$

Here $\mathsf{H}^{1,\bullet}((t_k, h_\Gamma); g_\Gamma)$ stands for the weighted Sobolev space which is defined by the following conditions: φ and its distributional derivative φ' belong to $\mathsf{L}^2((t_k, h_\Gamma); g_\Gamma)$, and $\varphi(t_k) = 0$. The quadratic form $\mathfrak{a}_k$ is non-negative and closed in $\mathfrak{H}_k$. Let $\mathfrak{A}_k$ be the corresponding self-adjoint operator.

Theorem 3.3. *Let Γ be a regular tree, $b_0(\Gamma) = 1$, (v, s) be an admissible pair and* $\mathrm{gen}(v) = k$. *Then the part of the operator $\boldsymbol{\Delta}$ in the reducing subspace $\mathcal{F}_v^{\langle s\rangle}$ is unitarily equivalent to the operator $\mathfrak{A}_k$.*

Proof. It is sufficient to show that the operator $J_v^{\langle s\rangle}$ (see (3.4)) sends the set $\mathcal{F}_v^{\langle s\rangle} \cap \mathsf{H}^{1,\bullet}(\Gamma)$ onto $\mathsf{H}^{1,\bullet}((t_k, h_\Gamma); g_\Gamma)$, and that $f \in \mathcal{F}_v^{\langle s\rangle} \cap \mathsf{H}^{1,\bullet}(\Gamma)$ yields $\int_\Gamma |f'|^2\, \mathrm{d}x = \mathfrak{a}_k[J_v^{\langle s\rangle} f]$. All these properties can be easily checked by direct inspection. □

The operator domain $\mathrm{Dom}(\mathfrak{A}_k)$ can be described with the help of the Euler–Lagrange procedure. When giving this description, we use the notation $I_j = (t_{j-1}, t_j)$, $j \in \mathbb{N}$.

Theorem 3.4. *A function φ lies in $\mathrm{Dom}(\mathfrak{A}_k)$ if and only if it satisfies the following conditions.*

(i) $\varphi \restriction I_j \in H^2(I_j)$ for all $j > k$ and

$$\sum_{j>k} \int_{I_j} (|\varphi''|^2 + |\varphi'|^2 + |\varphi|^2) g_\Gamma(t)\, \mathrm{d}t < \infty.$$

(ii) φ and $\varphi' g_\Gamma$ are continuous on $[t_k, h_\Gamma)$ and $\varphi(t_k) = 0$.

(iii) If $|\Gamma| = \infty$ (in particular, if $h_\Gamma = \infty$), then

$$\lim_{t\to h_\Gamma} \varphi(t) = 0. \tag{3.7}$$

If $|\Gamma| < \infty$ (and hence, $h_\Gamma < \infty$), then

$$\lim_{t\to h_\Gamma} \varphi'(t) g_\Gamma(t) = 0. \tag{3.8}$$

On this domain the operator acts as

$$(\mathfrak{A}_k\varphi)(t) = -\varphi''(t), \qquad t \neq t_k, t_{k+1}, \ldots. \tag{3.9}$$

The proof is standard and we confine ourselves to a few remarks, mostly concerning the conditions (3.7) and (3.8).

The continuity of $\varphi' g_\Gamma$ at the points t_j appears as the natural condition in the sense of calculus of variations. It can be re-written as

$$\varphi'(t_j+) = b_j^{-1}\varphi'(t_j-), \qquad \forall j > k;$$

this is an analogue of the second matching condition in (3.2).

If $h_\Gamma = \infty$, the condition (3.7) follows from lemma 3.1. Now let $h_\Gamma < \infty$ but $|\Gamma| = \infty$. Then we change the variables, taking

$$s = s(t) = \int_{t_k}^{t} \frac{\mathrm{d}\tau}{g_\Gamma(\tau)}.$$

For definiteness, in the further analysis we take $k = 0$ which corresponds to the operator $\mathfrak{A}_0$. Let $t(s)$ stand for the function inverse to $s(t)$ and $\psi(s) = \varphi(t(s))$. We have

$$\int_0^{h_\Gamma} |\varphi(t)|^2 g_\Gamma(t)\, \mathrm{d}t = \int_0^{L(\Gamma)} |\psi(s)|^2 W(s)\, \mathrm{d}s, \qquad W(s) = g_\Gamma^2(t(s)), \tag{3.10}$$

$$\int_0^{h_\Gamma} |\varphi'(t)|^2 g_\Gamma(t)\, \mathrm{d}t = \int_0^{L(\Gamma)} |\psi'(s)|^2\, \mathrm{d}s. \tag{3.11}$$

Since the latter integral is finite, the function ψ is continuous at $s = L_\Gamma$, and hence φ is continuous at $t = h_\Gamma$. Further,

$$\int_0^{L(\Gamma)} W(s)\, \mathrm{d}s = \int_0^{h_\Gamma} g_\Gamma(t)\, \mathrm{d}t = |\Gamma|.$$

If $|\Gamma| = \infty$ and the integral in the right-hand side of (3.10) is finite, then necessarily $\varphi(h_\Gamma) = \psi(L(\Gamma)) = 0$. This is exactly the condition (3.7).

The equality (3.7) for the case considered can also be derived from the analysis of deficiency indices, carried out in [3].

Now we turn to the case $|\Gamma| < \infty$. Since for $\varphi \in \mathrm{Dom}(\mathfrak{A}_k)$ the function $\varphi' g_\Gamma$ is continuous and $g'_\Gamma(t) = 0$ a.e., we have

$$\varphi'(t)g_\Gamma(t) - \varphi'(s)g_\Gamma(s) = \int_s^t \varphi''(\tau)g_\Gamma(\tau)\,\mathrm{d}\tau, \qquad t_k < s < t < h_\Gamma.$$

Hence,

$$\begin{aligned}|\varphi'(t)g_\Gamma(t) - \varphi'(s)g_\Gamma(s)|^2 &\leqslant \int_s^t g_\Gamma(\tau)\,\mathrm{d}\tau \int_s^t |\varphi''(\tau)|^2 g_\Gamma(\tau)\,\mathrm{d}\tau \\ &\leqslant |\Gamma| \int_s^t |\varphi''(\tau)|^2 g_\Gamma(\tau)\,\mathrm{d}\tau.\end{aligned}$$

It follows that the function $\varphi'(t)g_\Gamma(t)$ has a limit as $t \to h_\Gamma$. The equality (3.8) says that this limit must be zero; this is the natural boundary condition at the point $t = h_\Gamma$.

Theorem 3.4 also gives an implicit description of the natural boundary conditions at $t = h_\Gamma$ for the whole operator $\boldsymbol{\Delta}$.

3.5. Second realization of $\boldsymbol{\Delta} \restriction \mathcal{F}_v^{\langle s\rangle}$

The operators $\mathfrak{A}_k$ act in the weighted spaces L^2. Often it is more convenient to pass on to operators acting in L^2 without weight. To this end, we make the substitution $u(t) = \varphi(t)\sqrt{g_\Gamma(t)}$. It defines an isometry of the space $\mathsf{L}^2((t_k, h_\Gamma); g_\Gamma)$ onto $\mathsf{L}^2(t_k, h_\Gamma)$. Under this substitution $\mathfrak{A}_k$ turns into the operator $\boldsymbol{A}_k$ whose description is as follows. Its domain $\mathrm{Dom}(\boldsymbol{A}_k)$ consists of all functions u on $[t_j, h_\Gamma)$, such that $u \restriction I_j \in \mathsf{H}^2(I_j)$ for any $j > k$,

$$\sum_{j>k} \int_{I_j} (|u''|^2 + |u'|^2 + |u|^2)\,\mathrm{d}t < \infty,$$

and the following boundary condition at t_k and the matching conditions at the points t_j, $j > k$ are satisfied:

$$\begin{aligned}&u(t_k) = 0; \qquad u(t_j+) = b_j^{1/2}u(t_j-), \qquad j > k;\\ &u'(t_j+) = b_j^{-1/2}u'(t_j-), \qquad j > k.\end{aligned} \tag{3.12}$$

If $|\Gamma| = \infty$, then by (3.7) $u(t) \to 0$ as $t \to h_\Gamma$; if $|\Gamma| < \infty$, then by (3.8) $\lim_{t\to h_\Gamma} u'(t)\sqrt{g_\Gamma(t)} = 0$. On this domain the operator $\boldsymbol{A}_k$ acts according to the same rule (3.9).

The quadratic form of $\boldsymbol{A}_k$ is

$$\boldsymbol{a}_k[u] = \sum_{j>k} \int_{I_j} |u'|^2\,\mathrm{d}t.$$

Its domain is

$$\begin{aligned}\mathrm{Quad}(\boldsymbol{A}_k) = \Bigg\{&u \in \mathsf{L}^2(t_k, h_\Gamma) : u \restriction I_j \in \mathsf{H}^1(I_j)\forall j > k,\\ &\sum_{j>k} \int_{I_j} |u'|^2\,\mathrm{d}t < \infty, \text{ and the conditions (3.12) are satisfied}\Bigg\}.\end{aligned}$$

This realization was used in the papers [9, 10, 13] and [14]. In the present paper we make use of the first realization, that is of the operators $\mathfrak{A}_k$.

3.6. Spectrum of $\mathfrak{A}_0$ and spectrum of $\boldsymbol{\Delta}$

The outcome of our analysis is the following result. It was proved in [10] for the general case of Schrödinger operators. Below $\mathfrak{A}^{[r]}$ stands for the orthogonal sum of r copies of a self-adjoint operator $\mathfrak{A}$, and '$\sim$' means unitary equivalence.

Theorem 3.5. *Let Γ be the regular tree with the generating sequences $\{b_n\}$, $\{t_n\}$ where $b_0 = 1$. Then*

$$\boldsymbol{\Delta} \sim \mathfrak{A}_0 \oplus \sum_{k=1}^{\infty} \oplus \mathfrak{A}_k{}^{[b_0 \cdots b_{k-1}(b_k - 1)]}. \tag{3.13}$$

The operators $\mathfrak{A}_k$ in (3.13) can be replaced by $\boldsymbol{A}_k$.

We conclude from the description of $\mathrm{Quad}(\mathfrak{A}_k)$ and (3.6) that

$$\mathrm{Quad}(\mathfrak{A}_0) \supset \mathrm{Quad}(\mathfrak{A}_1) \supset \mathrm{Quad}(\mathfrak{A}_2) \supset \cdots$$

and

$$\mathfrak{a}_k = \mathfrak{a}_0 \restriction \mathrm{Quad}(\mathfrak{A}_k), \qquad \forall k \in \mathbb{N}.$$

By the variational principle, this implies that the spectral properties of all the operators $\mathfrak{A}_k$ and of the whole operator $\boldsymbol{\Delta}$ are basically determined by the properties of the single operator $\mathfrak{A}_0$. In particular, the following statement holds. As usual, we denote by $\sigma(\mathfrak{A})$ and $\sigma_p(\mathfrak{A})$ the spectrum and the point spectrum of a self-adjoint operator $\mathfrak{A}$.

Theorem 3.6. *Let $\mathfrak{A}_k$, $k = 0, 1, \ldots$ be the above defined operators in $\mathsf{L}^2((t_k, h_\Gamma); g_\Gamma)$. Then we have the following.*

(i) If $\mathfrak{A}_0$ is positive definite, then the same is true for any operator $\mathfrak{A}_k$, $k \in \mathbb{N}$, and

$$\min \sigma(\mathfrak{A}_0) \leqslant \min \sigma(\mathfrak{A}_1) \leqslant \cdots \leqslant \min \sigma(\mathfrak{A}_k) \leqslant \cdots.$$

(ii) If the spectrum of $\mathfrak{A}_0$ is discrete, then the same is true for any operator $\mathfrak{A}_k$, $k \in \mathbb{N}$.

(iii) If the spectrum of $\mathfrak{A}_0$ is discrete, then

$$\min \sigma(\mathfrak{A}_k) \to \infty \qquad \text{as } k \to \infty. \tag{3.14}$$

Note that theorem 3.6(iii) does not follow from the above construction and needs a separate proof. It will be given in section 4 (see (4.3)) for trees of finite height and at the end of section 5.3 for trees of infinite height.

It follows from theorem 3.2 that

$$\sigma_p(\boldsymbol{\Delta}) = \bigcup_{k=0}^{\infty} \sigma_p(\mathfrak{A}_k); \qquad \sigma(\boldsymbol{\Delta}) = \overline{\bigcup_{k=0}^{\infty} \sigma(\mathfrak{A}_k)}. \tag{3.15}$$

Together with theorem 3.6, this leads to the following result.

Corollary 3.7.

(i) The Laplacian $\boldsymbol{\Delta}$ on a regular tree is positive definite if and only if the operator $\mathfrak{A}_0$ is positive definite. Moreover,

$$\min \sigma(\boldsymbol{\Delta}) = \min \sigma(\mathfrak{A}_0).$$

(ii) The spectrum of $\boldsymbol{\Delta}$ is discrete if and only if the spectrum of $\mathfrak{A}_0$ is discrete.

4. Laplacian on regular trees of finite height

Here we assume $h_\Gamma < \infty$.

Theorem 4.1.

(i) *Let Γ be a regular tree and $h_\Gamma < \infty$. Then the spectrum of $\mathbf{\Delta}$ is discrete. For each operator $\mathfrak{A}_k$, $k \geqslant 0$ the Weyl asymptotic formula for its eigenvalue counting function is satisfied:*

$$N(\lambda; \mathfrak{A}_k) = \pi^{-1}(h_\Gamma - t_k)\lambda^{1/2}(1 + \mathrm{o}(1)), \qquad \lambda \to \infty. \tag{4.1}$$

(ii) *Suppose in addition that $|\Gamma| < \infty$. Then the Weyl asymptotic formula is satisfied for the operator $\mathbf{\Delta}$:*

$$N(\lambda; \mathbf{\Delta}) = \pi^{-1}|\Gamma|\lambda^{1/2}(1 + \mathrm{o}(1)), \qquad \lambda \to \infty. \tag{4.2}$$

This is basically theorem 5.3 of paper [15]; for this reason we only outline the proof of the first statement in (i). Note also that statement (ii) is a particular case of theorem 7.2 below and one more proof can be derived from [5].

Proof. We have to show that $\sigma(\mathfrak{A}_0)$ is discrete and (3.14) holds. The change of variables

$$s(t) = \int_0^t \frac{\mathrm{d}\tau}{g_\Gamma(\tau)}$$

shows that $\mathfrak{A}_0 \sim \mathfrak{B}_0$ where $\mathfrak{B}_0$ is the operator in $\mathsf{L}^2((0, L_\Gamma); W)$ generated by the quadratic form $\int_0^{L_\Gamma} |\psi'(s)|^2\, \mathrm{d}s$, with the domain

$$\mathsf{H}^{1,\bullet}(0, L_\Gamma) = \{\psi \in \mathsf{H}^1(0, L_\Gamma) : \psi(0) = 0\};$$

cf (3.10), (3.11). The operator $\mathfrak{B}_0^{-1}$ can be identified with the operator in $\mathsf{H}^{1,\bullet}(0, L_\Gamma)$, generated by the quadratic form $\int_0^{L_\Gamma} |\psi(s)|^2 W(s)\, \mathrm{d}s$. Since $\int_0^{L_\Gamma} W(s)\, \mathrm{d}s = |\Gamma| < \infty$, the operator $\mathfrak{B}_0^{-1}$ is compact which is equivalent to the discreteness of $\sigma(\mathfrak{A}_0)$. Besides, for $\psi \in \mathsf{H}^{1,\bullet}(0, L_\Gamma)$ we have $|\psi(s)|^2 \leqslant L_\Gamma \int_0^{L_\Gamma} |\psi'(s)|^2\, \mathrm{d}s$ which implies

$$\int_0^{L_\Gamma} |\psi(s)|^2 W(s)\, \mathrm{d}s \leqslant L_\Gamma|\Gamma| \int_0^{L_\Gamma} |\psi'(s)|^2\, \mathrm{d}s.$$

This shows that

$$\min \sigma(\mathfrak{A}_0) = \min \sigma(\mathfrak{B}_0) \geqslant (L_\Gamma|\Gamma|)^{-1}.$$

Applying the same argument to the operators $\mathfrak{A}_k$ with $k > 0$, we find that

$$\min \sigma(\mathfrak{A}_k) \geqslant (L_{T_e}|T_e|)^{-1}, \qquad \mathrm{gen}(e) = k. \tag{4.3}$$

This gives (3.14) and, hence, justifies (i). □

Note that the Weyl formula (4.2) also remains valid for the Laplacian with the boundary condition $\lim_{|x|\to h_\Gamma} f(x) = 0$.

5. Laplacian on regular trees of infinite height

5.1. Trees with arbitrarily long edges

Our next result is quite elementary and its proof is standard. The result applies to arbitrary metric graphs rather than to trees only, see [15]. Still, below we formulate only the particular case in which we are interested in this paper.

Theorem 5.1. *Let $\mathcal{G}$ be a regular metric tree and* $\sup_{e\in\mathcal{E}(\Gamma)}|e| = \infty$. *Then* $\sigma(\boldsymbol{\Delta}) = [0,\infty)$.

Proof. It is enough to show that for any $r > 0$ the point $\lambda = r^2$ belongs to the spectrum. For this purpose we fix a non-negative function $\zeta \in C_0^\infty(-1,1)$ such that $\zeta(t) = 1$ on $(-1/2, 1/2)$. Further, choose an edge $e \in \mathcal{E}(\Gamma)$. In an appropriate coordinate system, e can be identified with the interval $(-l, l)$ where $l = |e|/2$. The function f on Γ,

$$f(t) = \zeta(t/l)\sin rt \qquad \text{on } e, \qquad f(t) = 0 \text{ otherwise},$$

belongs to $\mathrm{Dom}(\boldsymbol{\Delta})$. An elementary calculation shows that

$$\|\boldsymbol{\Delta} f - r^2 f\| \leqslant \varepsilon(l)\|f\|, \qquad \varepsilon(l) \to 0 \text{ as } l \to \infty.$$

Choosing a sequence of edges e such that $|e| \to \infty$, we obtain a Weyl sequence for the operator $\boldsymbol{\Delta}$ and the point $\lambda = r^2$. This implies that $\lambda \in \sigma(\boldsymbol{\Delta})$. □

The assumption of theorem 5.1 does not exclude the embedded eigenvalues. This will be shown in the example 6.2.

5.2. Criterion of positive definiteness of the Laplacian

Theorem 5.2. *Let Γ be a regular tree and $h_\Gamma = \infty$. Then the Laplacian on Γ is positive definite if and only if $L_\Gamma < \infty$ and*

$$B(\Gamma) = B(g_\Gamma) := \sup_{t>0}\left(\int_0^t g_\Gamma(\tau)\mathrm{d}\tau \int_t^\infty \frac{\mathrm{d}\tau}{g_\Gamma(\tau)}\right) < \infty. \tag{5.1}$$

Moreover,

$$(4B(g_\Gamma))^{-1} \leqslant \min\sigma(\boldsymbol{\Delta}) \leqslant B(g_\Gamma)^{-1}. \tag{5.2}$$

Proof. According to corollary 3.7, we have to establish positive definiteness of the operator $\mathfrak{A}_0$. Let $c_0 := \min\sigma(\mathfrak{A}_0)$. Taking (3.6) into account, we come to the inequality

$$c_0\int_0^\infty |\varphi(t)|^2 g_\Gamma(t)\,\mathrm{d}t \leqslant \int_0^\infty |\varphi'(t)|^2 g_\Gamma(t)\,\mathrm{d}t, \qquad \forall\varphi \in \mathsf{H}^{1,\bullet}(\mathbb{R}_+; g_\Gamma). \tag{5.3}$$

It is enough to have this inequality for functions with compact support.

The inequality (5.3) with $c_0 > 0$ is a special case of the Hardy inequality with two weights. Necessary and sufficient conditions for such inequalities to be satisfied (Muckenhoupt conditions) are well known; see e.g. [8], section 1.3.1. Since $g_\Gamma(t) \geqslant 1$, they are never satisfied if $L_\Gamma = \infty$, so that the condition $L_\Gamma < \infty$ is necessary for positive definiteness of the Laplacian.

Let $\varphi \in \mathsf{H}^{1,\bullet}(\mathbb{R}_+; g_\Gamma)$ be a function with compact support. Then its derivative $\omega = \varphi'$ lies in $\mathsf{L}^2(\mathbb{R}_+; g_\Gamma)$ and also has compact support. Besides, $\int_{\mathbb{R}_+}\omega\,\mathrm{d}t = 0$. Denote by Ω the class of all such functions ω. For any $\omega \in \Omega$ the function $\varphi(t) = -\int_t^\infty \omega(\tau)\,\mathrm{d}\tau$ lies in $\mathsf{H}^{1,\bullet}(\mathbb{R}_+; g_\Gamma)$ and has compact support. For this reason, (5.3) is equivalent to

$$c_0\int_0^\infty \left|\int_t^\infty \omega(\tau)\,\mathrm{d}\tau\right|^2 g_\Gamma(t)\,\mathrm{d}t \leqslant \int_0^\infty |\omega(t)|^2 g_\Gamma(t)\,\mathrm{d}t, \qquad \forall\omega \in \Omega. \tag{5.4}$$

Since $\mathbf{1} \notin \mathsf{L}^2(\mathbb{R}_+; g_\Gamma)$, the set Ω is dense in the whole of $\mathsf{L}^2(\mathbb{R}_+; g_\Gamma)$. Hence, (5.4) is valid on $\mathsf{L}^2(\mathbb{R}_+; g_\Gamma)$ if and only if it is valid on Ω. The condition (5.1) is exactly the Muckenhoupt condition for (5.4) to be satisfied with some constant $c_0 > 0$, see [8], theorem 1.3.1/3. The inequality (5.2) is also a part of this theorem. □

It follows from theorem 5.2 that in the case $L_\Gamma = \infty$ the point $\lambda = 0$ lies in $\sigma(\boldsymbol{\Delta})$. A straightforward calculation shows that $\lambda = 0$ is not an eigenvalue and, hence, $0 \in \sigma_{\mathrm{ess}}(\boldsymbol{\Delta})$.

5.3. *Discreteness of* $\sigma(\boldsymbol{\Delta})$

Theorem 5.3. *Let Γ be a regular tree and $h_\Gamma = \infty$. Then the Laplacian on Γ has discrete spectrum if and only if $L_\Gamma < \infty$, $B(\Gamma) < \infty$ and*

$$\lim_{t\to\infty}\left(\int_0^t g_\Gamma(\tau)\,\mathrm{d}\tau \int_t^\infty \frac{\mathrm{d}\tau}{g_\Gamma(\tau)}\right) = 0. \tag{5.5}$$

Proof. The necessity of the assumption $L_\Gamma < \infty$ is clear from theorem 5.2. Under this assumption, the condition $B(\Gamma) < \infty$ is necessary and sufficient for the boundedness of the operator $\mathfrak{A}_0^{-1}$. It follows in a standard way that the condition (5.5) is necessary and sufficient for the compactness of this operator. By theorem 3.6(ii), each operator $\mathfrak{A}_k^{-1}$ is also compact, and it remains for us to show that $\|\mathfrak{A}_k^{-1}\| \to 0$ as $k \to \infty$. For this purpose we apply theorem 5.2 to the function g_k, cf (2.1). Taking (2.2) into account, we find

$$\begin{aligned} B(g_k) &= \sup_{t>t_k}\left(\int_{t_k}^t g_k(\tau)\,\mathrm{d}\tau \int_t^\infty \frac{\mathrm{d}\tau}{g_k(\tau)}\right) \\ &= \sup_{t>t_k}\left(\int_{t_k}^t g_\Gamma(\tau)\,\mathrm{d}\tau \int_t^\infty \frac{\mathrm{d}\tau}{g_\Gamma(\tau)}\right) \leqslant \sup_{t>t_k}\left(\int_0^t g_\Gamma(\tau)\,\mathrm{d}\tau \int_t^\infty \frac{\mathrm{d}\tau}{g_\Gamma(\tau)}\right). \end{aligned}$$

In view of (5.5), $B(g_k) \to 0$ as $k \to \infty$. By (5.2), $\min \sigma(\mathfrak{A}_k) \to \infty$ and we are done. □

6. Examples

In our first example we show that for a tree Γ of finite height but infinite volume the eigenvalues of the Laplacian may have quite an unusual behaviour; see [15] for the proof.

Example 6.1. Fix the numbers $q \in (0, 1)$ and $b \in \mathbb{N}$. Consider the tree Γ defined by the sequences $t_n = 1 - q^n$, $n = 0, 1, \ldots$ and $b_n = b =$ constant, $n = 1, 2, \ldots$. Then $h_\Gamma = 1$, so that the spectrum of the Laplacian on Γ is always discrete. Further, $g_\Gamma(t) = b^n$ for $t_n < t \leqslant t_{n+1}$. The total length of Γ is

$$|\Gamma| = 1 - q + \sum_{n=1}^{\infty} b^n(q^n - q^{n+1}) = (1 - q)\sum_{n=0}^{\infty}(bq)^n.$$

Hence, $|\Gamma| = \frac{1-q}{1-bq} < \infty$ if $bq < 1$ and $|\Gamma| = \infty$ otherwise. In the first case, theorem 4.1 shows that for the eigenvalues of $\boldsymbol{\Delta}$ the Weyl law (4.2) holds.

If $bq = 1$, then

$$N(\lambda; \boldsymbol{\Delta}) = \frac{1-q}{2\pi \ln b}\sqrt{\lambda}(\ln\lambda + \mathrm{O}(1)), \qquad \lambda \to \infty,$$

and if $bq > 1$, then there exists a bounded and bounded away from zero periodic function ψ with the period $\ln(q^{-2})$ such that

$$N(\lambda; \boldsymbol{\Delta}) = \lambda^{\beta/2}(\psi(\ln\lambda) + \mathrm{o}(1)), \qquad \lambda \to \infty$$

where $\beta = -\log_q b > 1$.

In our next example we show that under the assumptions of theorem 5.1 the spectrum $\sigma(\mathbf{\Delta})$ may have a dense set of embedded eigenvalues.

Example 6.2. Consider the tree Γ with $t_n = 2^{n-1}\pi, n \in \mathbb{N}$. The sequence b_n such that $b_0 = 1$ and $b_n > 1$ for $n \geqslant 1$ can be arbitrary. A direct inspection shows that for any integer l the function $u_l(t) = (g_\Gamma(t))^{-1}\sin(lt)$ is an eigenfunction of the operator $\mathfrak{A}_0$, with the eigenvalue $\lambda_l = l^2$. In the same way, the function $u_{l,k}(t) = (g_\Gamma(t))^{-1}\sin(2^{-n}lt), t \geqslant t_k$ is an eigenfunction of any operator $\mathfrak{A}_k$ with $k > n$. The corresponding eigenvalue is $\lambda_{l,n} = 2^{-2n}l^2$, and the result follows from (3.15).

Note that in this example the spectrum of $(\mathbf{\Delta})$ is not pure point, since for each k we have $\sigma(\mathfrak{A}_k) \neq \sigma_p(\mathfrak{A}_k)$.

Now we present an example (borrowed from [13]) of a tree for which the Laplacian is positive definite.

Example 6.3. Consider the tree $\Gamma = \Gamma_b$ with $b_n = b =$ constant, $n \in \mathbb{N}$ and $t_n = n$; so, all the edges of Γ_b are of the same length 1. We have $g_{\Gamma_b}(t) \sim \exp(\beta t), \beta = \ln b$, so that the conditions of theorem 5.2 are satisfied, which yields positive definiteness of the Laplacian.

For the tree Γ_b all the operators $\mathfrak{A}_k$, $k \in \mathbb{N}$ can be identified with $\mathfrak{A}_0$, and the equality (3.13) takes the form

$$\mathbf{\Delta} \sim \mathfrak{A}_0^{[\infty]}.$$

The spectrum of $\mathfrak{A}_0$ can be calculated explicitly. Define

$$\theta = \arccos \frac{2}{b^{1/2} + b^{-1/2}}.$$

It turns out that $\sigma(\mathfrak{A}_0)$ consists of the bands $[(\pi(l-1)+\theta)^2, (\pi l - \theta)^2]$ and the eigenvalues $\lambda_l = (\pi l)^2$, $l = 1, 2, \ldots$. So, the spectrum has the bandgap structure typical for periodic problems. The spectrum of $\mathbf{\Delta}$ is geometrically the same but has infinite multiplicity.

For comparison, consider the discrete Laplacian $\mathbf{\Delta}_d$ on the combinatorial rooted tree, with the branching numbers as for our tree Γ_b. It is well known (and can be easily calculated, see e.g. [1] where this was done for $b = 2$) that $\sigma(\mathbf{\Delta}_d) = [(b^{1/2}-1)^2, (b^{1/2}+1)^2]$.

We conclude this section with an example of a tree for which the Laplacian has discrete spectrum.

Example 6.4. Consider the tree Γ with $b_n = b =$ constant, $n \in \mathbb{N}$ and $t_n = n^{1/\alpha}, \alpha > 1$. Then $g_\Gamma(t) = b^n$ for $n^{1/\alpha} < t \leqslant (n+1)^{1/\alpha}$, which implies $g_\Gamma(t) \sim \exp(\beta t^\alpha)$ as $t \to \infty$. It is easy to check that condition (5.5) is satisfied. Hence, $\sigma(\mathbf{\Delta})$ is discrete.

An alternative way to construct a similar example is to take $t_n = n$ and b_n growing fast enough.

7. Concluding remarks

7.1. The Neumann boundary condition at the root

By definition, the Neumann Laplacian $\mathbf{\Delta}_N$ is the self-adjoint operator in $\mathsf{L}^2(\Gamma)$, associated with the quadratic form $\int_\Gamma |f'|^2\,\mathrm{d}x$ considered on the form domain

$$\mathrm{Quad}(\mathbf{\Delta}_N) = \mathsf{H}^1(\Gamma),$$

cf the definition of $\mathbf{\Delta}$ in section 3.2. The codimension of the subspace $\mathsf{H}^{1,0}(\Gamma)$ in $\mathsf{H}^1(\Gamma)$ is one, which implies that the qualitative properties of the operators $\mathbf{\Delta}_N$ and $\mathbf{\Delta}$ are the same. But as a matter of fact, much more can be said about the relations between these two operators.

Let us return to the orthogonal decomposition (3.5) of the space $\mathsf{L}^2(\Gamma)$. If $f \in \mathcal{F}_v^{\langle s\rangle} \cap \mathsf{H}^1(\Gamma)$ and $\operatorname{gen}(v) = k > 0, s < b_k$, then $f(v) = 0$. Indeed, f is continuous on Γ and $f(x) = 0$ outside the subtree T_v. It follows that

$$\boldsymbol{\Delta}_N \restriction \mathcal{F}_v^{\langle s\rangle} = \boldsymbol{\Delta} \restriction \mathcal{F}_v^{\langle s\rangle}.$$

Therefore, the analogue of the decomposition (3.13) for the operator $\boldsymbol{\Delta}_N$ takes the form

$$\boldsymbol{\Delta}_N \sim \mathfrak{A}_0' \oplus \sum_{k=1}^{\infty} \oplus \mathfrak{A}_k{}^{[b_0 \cdots b_{k-1}(b_k-1)]} \tag{7.1}$$

where only the first term differs from the one in (3.13): namely, in (3.12) for $k = 0$ the condition $u(0) = 0$ is replaced by $u'(0) = 0$.

The only point where this difference might be important is theorem 5.2, where a bound for $\min \sigma(\boldsymbol{\Delta})$ was found. However, even here the inequality (5.2) remains valid for the Neumann Laplacian. Indeed, the condition $\varphi(t) = 0$ for large t was used when justifying (5.4), rather than the condition $\varphi(0) = 0$.

7.2. Regular trees without boundary

Let Γ be a general metric tree. Choose a vertex $o \in \mathcal{E}(\Gamma)$ and suppose that there are $b_0 > 1$ edges of Γ adjacent to o. Then Γ splits into b_0 rooted subtrees $\Gamma_1, \ldots, \Gamma_{b_0}$ having the common root o. We say that the tree Γ is regular if and only if all the subtrees Γ_j are regular in the sense of definition 2.1 and the corresponding sequences $\{t_n\}$ and $\{b_n\}$ are the same for all $j = 1, \ldots, b_0$. Note that this definition is not invariant with respect to the choice of the vertex o.

The definition of the Laplacian $\boldsymbol{\Delta}$ extends to the trees without boundary in a natural way. The only difference is that now we have no boundary condition at o. Instead, the functions from $\operatorname{Quad}(\boldsymbol{\Delta})$ are required to be continuous at o; the functions $f \in \operatorname{Dom}(\boldsymbol{\Delta})$ also satisfy (3.2) for $v = o$.

Theorems 3.2 and 3.5 extend to the new situation, with small changes appearing due to the fact that now $b_0 > 1$. As a result, the subspace $\mathcal{F}_o'$ is no longer trivial, and the operator $\boldsymbol{\Delta} \restriction \mathcal{F}_o'$ is unitary equivalent to the orthogonal sum of $(b_0 - 1)$ copies of the operator $\mathfrak{A}_0$ described in section 3.2 (for $k = 0$). The analogue of the decomposition (3.13) now takes the form

$$\boldsymbol{\Delta} \sim \mathfrak{A}_0' \oplus \mathfrak{A}_0^{b_0-1} \oplus \mathfrak{A}_1^{b_0(b_1-1)} \oplus \mathfrak{A}_2^{b_0 b_1(b_2-1)} \ldots$$

where $\mathfrak{A}_0'$ is the same operator as in (7.1).

7.3. Harmonic functions on Γ

Material of section 5 shows that for the trees with $h_\Gamma = \infty$ there is a big difference between the spectral properties of the operator $\boldsymbol{\Delta}$ on Γ for the cases $L_\Gamma < \infty$ and $L_\Gamma = \infty$. In this section we discuss the difference between these two cases from another point of view.

Let Γ be a rooted tree. Along with the space $\mathsf{H}^{1,\bullet}(\Gamma)$, let us introduce *the homogeneous Sobolev space*

$$\mathcal{H}^1 = \mathcal{H}^1(\Gamma) = \left\{ f \in \mathsf{H}^{1,\bullet}_{\mathrm{loc}}(\Gamma) : \int_\Gamma |f'(x)|^2 \,\mathrm{d}x < \infty \right\}. \tag{7.2}$$

We consider $\mathcal{H}^1$ as the Hilbert space with respect to the scalar product

$$(f_1, f_2)_{\mathcal{H}^1} = \int_\Gamma f_1'(x) \overline{f_2'(x)} \,\mathrm{d}x.$$

Functions $u \in \mathcal{H}^1$ do not necessarily lie in L^2 and it is clear that $\mathsf{H}^{1,\bullet} = \mathcal{H}^1 \cap \mathsf{L}^2$.

The class $\mathsf{H}^{1,\bullet}(\Gamma)$ is not always dense in $\mathcal{H}^1(\Gamma)$. The functions from $\mathcal{H}^1$ which are orthogonal to $\mathsf{H}^{1,\bullet}$ are called *harmonic*. Any such function is linear on each edge and satisfies the boundary condition $f(o) = 0$ and the Kirchhoff matching conditions (3.2) at each vertex $v \neq o$. Harmonic functions play an important part in the theory of Brownian motion on trees. The above definition makes sense for arbitrary (not necessarily regular) rooted trees.

The tree Γ is called recurrent if $\mathsf{H}^{1,\bullet}(\Gamma)$ is dense in $\mathcal{H}^1(\Gamma)$ and is called transient otherwise. It turns out that for the regular trees the quantity L_Γ is the distinguishing parameter between these two cases.

Theorem 7.1. *Let Γ be a regular rooted tree. Then Γ is transient if and only if $L_\Gamma < \infty$.*

This result is well known in the theory of combinatorial trees, see e.g. [11], example 8.3. For metric trees, the result was proved in [9] and [10]. Theorem 7.1 can also be easily derived from its counterpart for combinatorial trees.

7.4. Eigenvalue problems for the weighted Laplacian on metric graphs of finite total length

In conclusion, we discuss the 'eigenvalue problem with weight'

$$\lambda \boldsymbol{\Delta} f = V f \tag{7.3}$$

on the metric graphs (not necessarily trees) of finite total length. Here $V \in \mathsf{L}^1(\Gamma)$ is an arbitrary real-valued weight function. For technical reasons, it is convenient to put the spectral parameter in front of the Laplacian rather than in the right-hand side.

As in section 3, we use the variational approach to the problem. Let $\mathcal{G}$ be a metric graph, and let a point $x_0 \in \mathcal{G}$ be singled out. Consider the Sobolev space

$$\mathsf{H}^1(\mathcal{G}, x_0) := \{f \in \mathsf{H}^1(\mathcal{G}) : f(x_0) = 0\},$$

with the metric form $\int_{\mathcal{G}} |f'|^2 \, \mathrm{d}x$. For any function $V \in \mathsf{L}^1(\mathcal{G})$ the quadratic form $\int_{\mathcal{G}} V|f|^2 \, \mathrm{d}x$ is bounded in $\mathsf{H}^1(\mathcal{G})$, and therefore generates a bounded compact operator, say $\boldsymbol{B}_V$, in this space. If V is real valued, the operator $\boldsymbol{B}_V$ is self-adjoint. Its spectrum can be identified, in a natural way, with the spectrum of the equation (7.3), under the boundary conditions

$$f(x_o) = 0; \qquad f'(v) = 0 \text{ for all } v \in \partial\mathcal{G}, v \neq x_0. \tag{7.4}$$

We explain that the boundary $\partial\mathcal{G}$ of the graph $\mathcal{G}$ consists of all vertices v having only one neighboring vertex. Note that the case $\partial\mathcal{G} = \emptyset$ is not excluded.

The following result was established in [14].

Theorem 7.2. *Let $\mathcal{G}$ be a connected graph of finite total length, $x_0 \in \mathcal{G}$ be its arbitrary point, and let $V = \overline{V} \in \mathsf{L}^1(\mathcal{G})$. Then the positive eigenvalues λ_n^+ and the negative eigenvalues $-\lambda_n^-$ of the problem (7.3) under the boundary conditions (7.4) satisfy the inequality*

$$\lambda_n^\pm \leqslant \frac{|\mathcal{G}| \int_{\mathcal{G}} V_\pm \, \mathrm{d}x}{n^2}, \qquad \forall n \in \mathbb{N} \tag{7.5}$$

where $2V_\pm = |V| \pm V$. Along with the estimate (7.5), the Weyl-type asymptotics holds:

$$n\sqrt{\lambda_n^\pm} \to \pi^{-1} \int_{\mathcal{G}} \sqrt{V_\pm(x)} \, \mathrm{d}x, \qquad n \to \infty.$$

The estimate (7.5) is sharp for any $n \in \mathbb{N}$, including the value of the constant factor; this was shown in [16]. The statement (ii) of theorem 4.1 is a particular case of theorem 7.2.

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