EIGENVALUE ESTIMATES FOR THE WEIGHTED LAPLACIAN ON METRIC TREES

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1. *Introduction*

1.1. Spectral theory on graphs, in particular on trees, is a fairly popular field; see, for example, the recent book [**6**]. In the standard setting, a graph is a combinatorial object. In contrast to this, what we consider are *metric graphs*. This means that we regard each edge of Γ as a non-degenerate line segment of finite length. This difference is important because it affects the nature of functions on a graph: a function on a combinatorial graph is actually defined on the set of its vertices, while a function on a metric graph is a family of functions, defined on its edges and usually subject to some compatibility conditions at the vertices.

The Laplacian on Γ reflects both the metric and the combinatorial nature of the graph: on the edges we have just $\Delta u = u''$ but the description of $\mathrm{Dom}(\Delta)$ involves compatibility (Kirchhoff) conditions at each vertex, whose origin is rather combinatorial.

In this paper we study the eigenvalue problem on a rooted tree Γ:

$$-\lambda \Delta u = Vu \ \text{ on } \Gamma, \quad u(o) = 0. \tag{1.1}$$

In (1.1), V is a given non-negative 'weight function' on Γ, and o is the root of Γ. A rigorous statement of the problem is described in § 3.1; it uses the techniques of quadratic forms.

Our attention to the problem (1.1) was attracted by the paper [**7**] by W. D. Evans and D. J. Harris. The spectral properties of the Neumann Laplacian Δ_N in $L_2(\Omega)$, where Ω is a domain in $\mathbb{R}^d$, are determined by the nature of the embedding of the Sobolev space $W^{1,2}(\Omega)$ into $L_2(\Omega)$. The compactness of this embedding, and hence the discreteness of the spectrum of Δ_N, heavily depends on the regularity of the boundary $\partial\Omega$. In [**7**], the authors studied the properties of the embedding of $W^{1,p}(\Omega)$ into $L_p(\Omega)$ for special domains with non-smooth (in particular, fractal) boundary. A characteristic feature of these domains is that they have a 'ridge' or a 'skeleton', this being a metric tree. It was shown in [**7**] that the study of the mentioned embedding can be reduced to the investigation of a weighted Volterra operator on this tree. For $p = 2$, such an operator is closely related to the problem (1.1). This is discussed in § 3.3.

A boundedness criterion for weighted Volterra operators in $L_p(\Gamma)$ was established in [**8**]. The next natural question concerns quantitative characteristics of these operators, in particular the behaviour of their approximation numbers. Practically nothing was known about this subject until now. The results presented

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in our paper give some answers for the case $p = 2$, when the approximation numbers coincide with the singular numbers.

Let us describe the structure of the paper in more detail. In § 2 some pieces of auxiliary material are presented. We define the Neumann–Schatten operator classes $\mathscr{C}_p$ and their weak analogues $\mathscr{C}_{p,\infty}$, and discuss triangle inequalities in the quasi-normed spaces $\mathscr{C}_p$, $\mathscr{C}_{p,\infty}$, with $p \leqslant 1$. We describe the technique of quadratic forms which we will need further for the precise statement of our problem. We also cite some results for eigenvalue problems similar to (1.1) on an interval; we will use these results later, when we reduce our problem to a family of 1-dimensional ones.

In § 3 the rigorous statement of the problem (1.1) is given. Namely, we associate with (1.1) the self-adjoint operator $A_{\Gamma,V}$ on the Dirichlet space $\mathscr{H}(\Gamma)$; actually, this is a realization of $(-\Delta)^{-1}V$.

Spectral properties of $A_{\Gamma,V}$ depend on V and on the geometry of Γ. In principle, $A_{\Gamma,V}$ may be of any of the classes $\mathscr{C}_p$ with $p > \frac{1}{2}$, or $\mathscr{C}_{p,\infty}$ with $p \geqslant \frac{1}{2}$, and our results concern the whole of this scale. In § 3.4 we present the boundedness criterion from [**8**], prove an elementary but very useful Trace class ($\mathscr{C}_1$) criterion (Theorem 3.3), and then interpolate between the two criteria. This leads to the upper estimates for $A_{\Gamma,V}$ in $\mathscr{C}_p$ and $\mathscr{C}_{p,\infty}$ for $1 < p < \infty$.

The case $p < 1$ is investigated in much more detail. In particular, one of the key questions analysed in the paper concerns the membership of $A_{\Gamma,V}$ in the class $\mathscr{C}_{1/2,\infty}$ and the validity of the Weyl type asymptotics. The corresponding eigenvalue behaviour is $\lambda_n(A_{\Gamma,V}) = O(n^{-2})$ and $\lambda_n(A_{\Gamma,V}) \sim (\pi^{-1}\int_\Gamma \sqrt{V}\,dx)^2 n^{-2}$, respectively, which is typical for second-order differential operators in dimension 1. We suggest two different ways to treat the case $p < 1$. In both of them, the problem is reduced to a family of 1-dimensional problems.

The well-studied case $\Gamma = \mathbb{R}_+$ (see [**3**, **15**] and references there) serves us as a prototype for our first approach (§ 4). The 1-dimensional problems arise as a result of decomposition of the tree into a family of segments. Theorem 4.1 gives the upper estimates for $A_{\Gamma,V}$ in $\mathscr{C}_p$, with $\frac{1}{2} < p < 1$ and in $\mathscr{C}_{p,\infty}$, with $\frac{1}{2} \leqslant p < 1$. However, the results for general trees are not as exhaustive as the ones for $\mathbb{R}_+$. The nature of this difference is discussed in § 4.2. The converse of Theorem 4.1 holds only under additional conditions on Γ; the corresponding result is given in Theorem 4.6.

The second approach (§ 5) is based upon a useful orthogonal decomposition of $\mathscr{H}(\Gamma)$, described in Theorem 5.1. This decomposition is of quite a general nature, related to the geometry of trees; it has analogues for combinatorial trees (with or without conductances). For the simplest case of a binary tree without conductances one such analogue was used recently in [**1**].

In § 6 we introduce a special class of trees, the so-called regular trees. We will make full use of their properties later, in § 7. Here we show how the structure of the space $\mathscr{H}(\Gamma)$ can be analysed for such trees.

The most complete results can be obtained if the tree is regular and the weight is symmetric, that is, $V(x)$ depends only on the distance between x and the root o. This is done in § 7. Here the operator $A_{\Gamma,V}$ admits a decomposition into the orthogonal sum of second-order operators on intervals (Theorem 7.2). This leads to the complete spectral analysis of the problem.

In § 8 we discuss several examples. All of them fall into the situation described in § 7, and our approach allows us to exploit the self-similar structure of the problem and on this basis to describe its spectral asymptotics. Sometimes it is of the standard Weyl type, and sometimes it is of the same type as for many other problems with self-similarity; see, for example, [**9**, **11**, **12**, **14**].

We think it is important to obtain some general picture in the whole scale of the classes $\mathscr{C}_p$, $\mathscr{C}_{p,\infty}$; this is done in § 3.4 and § 4. In addition to Theorems 3.3, 4.1 and 4.6, we would like to draw attention to the discussion in § 4.2. We present there several examples and counter-examples, which demonstrate the strengths and weaknesses of the general theorems.

We believe that the idea of the second approach applied to the regular trees with symmetric weights is of particular interest. In this direction, the main results are Theorems 5.1 and 7.2 and their illustration in the examples of § 8.

As a rule, the notation we use is standard, other notation is introduced in the course of our presentation. Notation such as $C_{2.4}$ for a repeated constant means that this constant appears first in the equation (2.4).

1.2. *Geometry of a tree*

Our understanding of a metric tree is the same as in [**7**, § 2]. Let Γ be a rooted tree with the set of vertices $\mathscr{V}(\Gamma)$ and the set of edges $\mathscr{E}(\Gamma)$. We assume that $\#\mathscr{V}(\Gamma) = \#\mathscr{E}(\Gamma) = \infty$. We regard every edge e as a non-degenerate closed line segment, so its length $|e|$ is well defined. We assume that any two points $y, z \in \Gamma$ are connected by a unique polygonal path. (This excludes the pathological situation of infinitely many vertices lying between y and z.) Then the distance between y and z is defined as the length of this path. Endowed with this distance, Γ becomes a metric space. The Lebesgue measure on Γ is introduced in a natural way.

We denote by o the root of Γ. Given two points $y, z \in \Gamma$, we write $y \prec z$ if y lies on the path connecting o with z. The relation $\prec$ defines on Γ a partial ordering. If $y \prec z$, we denote

$$\langle y, z \rangle := \{x \in \Gamma \colon y \preccurlyeq x \preccurlyeq z\}.$$

Given a point $y \in \Gamma$, $|y|$ stands for the length of the path $\langle o, y \rangle$.

The *branching number* $b(y)$ of a vertex y is defined as the number of edges emanating from y. We suppose that $b(y) < \infty$ for any y. We also suppose that $b(o) = 1$ and denote by e_0 the only edge emanating from o. For any vertex $y \neq o$ the degree of y, that is the quantity of all vertices z adjacent to y (notation $z \sim y$), is $b(y) + 1$. We consider only trees with no leaves, that is $b(y) > 0$ for any $y \in \mathscr{V}(\Gamma)$.

The semiaxis $\mathbb{R}_+$ becomes a tree if we choose a positive sequence $y_k \nearrow \infty$ and call the points y_k vertices. Their choice is indifferent.

Given a subtree $T \subset \Gamma$, we denote its root by o_T. We say that T is a *W-subtree* ($T \in \mathscr{W}(\Gamma)$) if it satisfies the following property:

$$\text{if } o_T \neq x \in T \text{ and } y \succ x, \text{ then } y \in T.$$

We call two subtrees disjoint if they have no points in common except, may be, for their common root. Any two W-subtrees either are disjoint, or one of them is a subtree of the other. We say that T is a W_0*-subtree* ($T \in \mathscr{W}_0(\Gamma)$) if T is a W-subtree and contains only one edge originating at o_T. In particular, $\Gamma \in \mathscr{W}_0(\Gamma)$.

For any $z \in \mathscr{V}(\Gamma)$ put

$$T_z = \{x \in \Gamma \colon x \succcurlyeq z\}.$$

The subtree T_z belongs to $\mathscr{W}(\Gamma)$ and splits into $b(z)$ disjoint W_0-subtrees rooted at z. We call all of them *daughter subtrees* of T_z and denote them by T_z^j, for $j = 1, \dots, b(z)$. Any subtree $T \in \mathscr{W}(\Gamma)$ rooted at z is the union of some of the daughter subtrees of T_z. It follows that the set $\mathscr{W}(\Gamma)$ is countable.

2. *Preliminary information*

2.1. *The spaces l_p and $l_{p,\infty}$*

We begin with notation concerning countable families of complex numbers.

Let $\mathbf{h} = \{h_j\}$ be such a family, indexed by elements j running over a finite or a denumerable set $\mathscr{J}$. The distribution function of $\mathbf{h}$ is defined as

$$n(\lambda, \mathbf{h}) = \#\{j \in \mathscr{J}\colon |h_j| > \lambda\}, \quad \text{for } \lambda > 0.$$

Given $0 < p \leqslant \infty$, we denote $\|\mathbf{h}\|_p := \|\mathbf{h}\|_{l_p}$ and, for $0 < p < \infty$,

$$\|\mathbf{h}\|_{p,\infty} = \sup_{\lambda > 0} \lambda (n(\lambda, \mathbf{h}))^{1/p}. \tag{2.1}$$

The space $l_{p,\infty} = l_{p,\infty}(\mathscr{J})$ is defined as

$$l_{p,\infty} = \{\mathbf{h}\colon \|\mathbf{h}\|_{p,\infty} < \infty\}.$$

This is a complete linear quasinormed space with respect to the quasinorm (2.1).

2.2. *Operator classes $\mathscr{C}_p$ and $\mathscr{C}_{p,\infty}$*

Let $\mathfrak{H}$ be a Hilbert space. By $\mathscr{B} = \mathscr{B}(\mathfrak{H})$ and $\mathscr{C} = \mathscr{C}(\mathfrak{H})$ we denote the spaces of all bounded and all compact linear operators in $\mathfrak{H}$. The sequence of singular numbers (counting multiplicities) of an operator $A \in \mathscr{C}$ is denoted by $\mathbf{s}(A) = \{s_k(A)\}$; if $A \in \mathscr{C}$ is non-negative, its singular numbers coincide with its eigenvalues. The corresponding distribution function is denoted by

$$n(\lambda, A) := n(\lambda, \mathbf{s}(A)), \quad \text{for } A \in \mathscr{C}.$$

The classes $\mathscr{C}_p$ and $\mathscr{C}_{p,\infty}$ are defined for an arbitrary $p > 0$ as

$$\mathscr{C}_p = \{A \in \mathscr{C}\colon \mathbf{s}(A) \in l_p\}, \quad \mathscr{C}_{p,\infty} = \{A \in \mathscr{C}\colon \mathbf{s}(A) \in l_{p,\infty}\}. \tag{2.2}$$

The quasinorms in $\mathscr{C}_p$ and $\mathscr{C}_{p,\infty}$ are induced by the definition (2.2):

$$\|A\|_p := \|\mathbf{s}(A)\|_p = \left(\sum_k s_k^p(A)\right)^{1/p}$$

and

$$\|A\|_{p,\infty} := \|\mathbf{s}(A)\|_{p,\infty} = \sup_{\lambda > 0} \lambda (n(\lambda, A))^{1/p} = \sup_k k^{1/p} s_k(A).$$

2.3. *Triangle inequalities in $\mathscr{C}_p$, $\mathscr{C}_{p,\infty}$*

Here we present some inequalities for the quasinorms in $\mathscr{C}_p$ and $\mathscr{C}_{p,\infty}$, with $p \leqslant 1$.

PROPOSITION 2.1. (i) *The following triangle inequality is valid in $\mathscr{C}_p$ for $0 < p \leqslant 1$:*

$$\left\| \sum_j A_j \right\|_p^p \leqslant \sum_j \|A_j\|_p^p. \tag{2.3}$$

(ii) *For any* $0 < p < 1$ *there exists a constant* $C_{2.4}(p)$ *such that for any family of operators* $\{A_j\}$*, with* $j = 1, 2, \ldots,$ *of the class* $\mathscr{C}_{p,\infty}$ *one has*

$$\left\| \sum_j A_j \right\|_{p,\infty}^p \leq C_{2.4}(p) \sum_j \|A_j\|_{p,\infty}^p. \tag{2.4}$$

The triangle inequality (2.3) for $p < 1$ was proven independently by McCarthy [**13**] and by Rotfeld [**16**]. For the proof of (2.4) see [**17**, § 3]. More exactly, it was shown in [**17**] that the expression

$$\sup_{t>0} \left(t^{1-p} \sum_{k=1}^{\infty} \frac{s_k(A)}{1 + t s_k(A)} \right)$$

is equivalent to $\|A\|_{p,\infty}^p$ and meets the usual triangle inequality. This immediately leads to (2.4). Moreover, the inequality (3.4) in [**17**] shows that one can take $C_{2.4}(p) = 2\pi p / \sin \pi p$.

2.4. *Quadratic forms*

Here we recall some elementary facts concerning bounded quadratic forms in a Hilbert space $\mathfrak{H}$. Let $Q[u, v]$ be a sesqui-linear Hermitian form on $\mathfrak{H} \times \mathfrak{H}$ and $Q[u] := Q[u, u]$ be the corresponding quadratic form on $\mathfrak{H}$. Note that $Q[u, v]$ can be recovered from $Q[u]$ by the polarization formula, and is Hermitian if and only if $Q[u]$ is real-valued. Boundedness of $Q[u]$ means that $\sup\{|Q[u]|: \|u\| = 1\} < \infty$. With any real-valued bounded quadratic form Q, a unique self-adjoint bounded operator, say A_Q, is associated by the following rule:

$$A_Q u = f \iff Q[u, v] = (f, v) \text{ for any } v \in \mathscr{H}. \tag{2.5}$$

If necessary, we use for A_Q more detailed notation, like $A(\mathfrak{H}, Q)$, and if $A_Q \in \mathscr{C}$, write $n(\lambda; \mathfrak{H}, Q)$ instead of $n(\lambda, A_Q)$. Recall that, according to the variational principle, the non-zero eigenvalues of A_Q coincide with the critical values of the ratio (Rayleigh quotient)

$$\mathscr{R}[u] = \frac{Q[u]}{\|u\|^2}, \quad \text{for } u \in \mathfrak{H},\ u \neq 0.$$

Now we present a version of Proposition 2.1 which is convenient when dealing with quadratic forms. Suppose that a Hilbert space $\mathfrak{H}$ is decomposed into the orthogonal sum of its subspaces,

$$\mathfrak{H} = \sum_j{}^{\oplus} \mathfrak{H}_j.$$

Let P_j, for $j = 1, 2, \ldots,$ stand for the corresponding orthogonal projections. Suppose also that a bounded non-negative quadratic form $Q[u]$ is given on $\mathfrak{H}$, and the operator A_Q is compact. Along with A_Q introduce the operators $A_{Q,j} = A(\mathfrak{H}_j, Q)$ on $\mathfrak{H}_j$.

LEMMA 2.2. *Under the above conditions, one has*

$$\|A_Q\|_p^p \leq \sum_j \|A_{Q,j}\|_p^p, \quad \textit{for } 0 < p \leq 1, \tag{2.6}$$

$$\|A_Q\|_{p,\infty}^p \leq C_{2.4}(p) \sum_j \|A_{Q,j}\|_{p,\infty}^p, \quad \textit{for } 0 < p < 1. \tag{2.7}$$

Proof. Relations (2.6) and (2.7) are proved similarly. For definiteness, we give the proof of (2.7).

We have

$$\|A_Q\|_{p,\infty} = \|A_Q^{1/2}\|_{2p,\infty}^2 = \Big\| \sum_j P_j A_Q^{1/2} \Big\|_{2p,\infty}^2$$
$$= \Big\| \Big(\sum_j P_j A_Q^{1/2} \Big)^* \sum_j P_j A_Q^{1/2} \Big\|_{p,\infty} = \Big\| \sum_j A_Q^{1/2} P_j A_Q^{1/2} \Big\|_{p,\infty}. \qquad (2.8)$$

Now, consider the operators $P_j A_Q P_j$. Clearly each of them coincides with $A_{Q,j}$ on $\mathfrak{H}_j$ and is zero on its orthogonal complement, so that $\mathbf{s}(P_j A_Q P_j) = \mathbf{s}(A_{Q,j})$. Therefore, for any $p > 0$,

$$\|A_Q^{1/2} P_j A_Q^{1/2}\|_{p,\infty} = \|P_j A_Q^{1/2} A_Q^{1/2} P_j\|_{p,\infty} = \|P_j A_Q P_j\|_{p,\infty} = \|A_{Q,j}\|_{p,\infty}.$$

Taking this into account and applying Proposition 2.1 to (2.8), we finally obtain (2.7):

$$\|A_Q\|_{p,\infty}^p \leqslant C_{2.4}(p) \sum_j \|A_Q^{1/2} P_j A_Q^{1/2}\|_{p,\infty}^p = C_{2.4}(p) \sum_j \|A_{Q,j}\|_{p,\infty}^p.$$

This completes the proof of Lemma 2.2.

It is worth observing that the relations (2.6), (2.7) are valid in spite of the fact that, in general, $A_Q \neq \sum_j P_j A_Q P_j$.

2.5. *Eigenvalue problems on an interval*

We will reduce our problem on the tree to a family of 1-dimensional problems. The latter are well studied; here we present some results for them which we will need later; see [**3**, **15**].

Consider the eigenvalue problem

$$-\lambda u'' = Vu, \quad u(a) = 0, \quad u'(b) = 0, \qquad (2.9)$$

on a finite interval $I = (a, b)$ of length $|I| = b - a$. The corresponding Rayleigh quotient is

$$\frac{\int_I V|u|^2\,dx}{\int_I |u'|^2\,dx}, \quad \text{where } u \in H^1(I),\ u \neq 0,\ u(a) = 0. \qquad (2.10)$$

The formulation of the following result is borrowed essentially from [**15**]; for proof and further references see [**3**].

PROPOSITION 2.3. *There is a constant $C_{2.11} < \infty$ such that for any finite interval $I = (a, b) \subset \mathbb{R}$, any non-negative function $V \in L_1(I)$ and any $\lambda > 0$ the following estimate for the eigenvalue distribution function of the problem* (2.9) *is valid:*

$$\lambda^{1/2} n(\lambda) \leqslant C_{2.11} \Big(|I| \int_I V\,dx \Big)^{1/2}. \qquad (2.11)$$

In [**15**] the result was stated for the problem with the Dirichlet boundary condition at both ends of I. Our Proposition 2.3 reduces to this case by passing to the Dirichlet problem on the 'doubled' interval $(a, 2b-a)$ with the weight function V_1 obtained from V by reflection about the point b. This passage does not change the eigenvalues. The important point in Proposition 2.3 is independence of the constant factor $C_{2.11}$ of the interval and weight function. Actually, $C_{2.11} = 1$.

We will also need results on the eigenvalue problem for the equation

$$-\lambda(gu')' = Vu, \quad u(a) = 0, \quad u'(b) = 0, \tag{2.12}$$

on a finite or infinite interval $I = (a, b)$ with $-\infty < a < b \leqslant \infty$. More exactly, let g be a piecewise continuous and positive function on I. Introduce the weighted Sobolev space

$$\mathscr{H}(I, g) = \left\{ u \in H^1_{\mathrm{loc}}(I) \colon \|u\|^2_{\mathscr{H}(I,g)} := \int_I |u'|^2 g\, dx < \infty,\ u(a) = 0 \right\}. \tag{2.13}$$

We are interested in the operator generated by the quadratic form $\int_I V|u|^2\, dx$ in this space.

Corollary 2.4. *Suppose that*

$$l := \int_I \frac{dx}{g(x)} < \infty.$$

Then for the eigenvalue distribution function of the problem (2.12) *one has*

$$\lambda^{1/2} n(\lambda) \leqslant C_{2.11} \left(\int_I \frac{dx}{g(x)} \int_I V\, dx \right)^{1/2}. \tag{2.14}$$

Proof. The Rayleigh quotient for our problem is

$$\frac{\int_I V|u|^2\, dx}{\int_I g|u'|^2\, dx}, \quad \text{where } u \in \mathscr{H}(I, g),\ u \neq 0.$$

The standard substitution $s = s(x) = \int_a^x g^{-1}(t)\, dt$ reduces it to the ratio (2.10) for the interval $(0, l)$ and weight function $g(x(s))V(x(s))$, where $x(s)$ is the function inverse to $s(x)$. Applying Proposition 2.3 to this new problem, we obtain (2.14).

3. *Setting of the problem. Estimates in the classes $\mathscr{C}_p$ with $p \geqslant 1$, and $\mathscr{C}_{p,\infty}$ with $p > 1$*

3.1. *The space $\mathscr{H}(\Gamma)$ and the operator $A_{\Gamma,V}$*

The Dirichlet space $\mathscr{H}(\Gamma)$ is defined as follows: we say that a function u on Γ belongs to $\mathscr{H}(\Gamma)$ if it is absolutely continuous on Γ, $u(o) = 0$, and $\int_\Gamma |u'|^2\, dx < \infty$. The latter integral is taken as the metric form on $\mathscr{H}(\Gamma)$. Equipped with it, $\mathscr{H}(\Gamma)$ becomes a Hilbert space.

Now let $V \in L_{1,\mathrm{loc}}(\Gamma)$ be a given non-negative function (weight) on Γ. On $\mathscr{H}(\Gamma)$ we consider the quadratic functional

$$Q_{\Gamma,V}[u] = \int_\Gamma V|u|^2\, dx. \tag{3.1}$$

Suppose that this functional is bounded, that is,

$$\int_\Gamma V|u|^2\,dx \leqslant C\int_\Gamma |u'|^2\,dx, \quad \text{for } u\in\mathscr{H}(\Gamma).$$

Then we call V a *Hardy weight*; see § 3.4.1 for the complete description (borrowed from [**8**]) of Hardy weights on trees. Here we only note that any Hardy weight must be integrable away from the root, since for arbitrary $\delta > 0$ the space $\mathscr{H}(\Gamma)$ contains functions which are equal to 1 for $|x| > \delta$. The required behaviour of V near the root is not related to the geometry of the tree and is the same as for the well-studied case of an interval (see [**3**]). For simplicity, and without essential loss of generality, we often assume $V\in L_1(\Gamma)$.

Our main object is the operator $A_{\Gamma,V} := A(\mathscr{H}(\Gamma), Q_{\Gamma,V})$; cf. § 2.4. According to (2.5), for a given $u\in\mathscr{H}(\Gamma)$ the function $f = A_{\Gamma,V}u$ is the unique element from $\mathscr{H}(\Gamma)$ such that

$$\int_\Gamma Vu\overline{\phi}\,dx = \int_\Gamma f'\overline{\phi}'\,dx \quad \text{for any } \phi\in\mathscr{H}(\Gamma). \tag{3.2}$$

The assumption $V\geqslant 0$ is not necessary in order to define the bounded operator $A_{\Gamma,V}$ by (3.2): it is enough to assume that $|V|$ is a Hardy weight. Thus admissible weights form a linear space, where interpolation is possible. We use such interpolation as a technical tool in some proofs. However, if not stated otherwise, everywhere in this paper we assume the weight to be non-negative.

In order to represent $A_{\Gamma,V}u$ in a more explicit way, we need some more notation. Let $z\neq o$ be a vertex of Γ and $b = b(z)$. Given a function $f\in\mathscr{H}(\Gamma)$, denote by f_- its restriction to the edge terminating at z and by $f_1,\dots,f_b$ its restrictions to the edges originating at z. Finally, denote

$$[f'](z) = \sum_{j=1}^{b} f_j'(z) - f_-'(z).$$

Standard reasonings show that (3.2) corresponds to the differential equation

$$-f'' = Vu \quad \text{on each edge } e\in\mathscr{E}(\Gamma), \tag{3.3}$$

with the following conditions at the vertices:

$$f(o) = 0, \tag{3.4}$$

$$f_-(z) = f_1(z) = \ldots = f_b(z), \quad [f'](z) = 0, \quad \text{for } z\in\mathscr{V}(\Gamma),\ z\neq o. \tag{3.5}$$

Our main goal is the investigation of the eigenvalue behaviour of $A_{\Gamma,V}$. According to (3.3)–(3.5), any eigenpair $\{\lambda, u\}$ satisfies the equation $-\lambda u'' = Vu$ on each edge $e\in\mathscr{E}(\Gamma)$ and the conditions (3.4) and (3.5) at the vertices. It is useful to write down the Rayleigh quotient corresponding to this eigenvalue problem:

$$\mathscr{R}[u] = \frac{\int_\Gamma V|u|^2\,dx}{\int_\Gamma |u'|^2\,dx}, \quad \text{for } u\in\mathscr{H}(\Gamma),\ u\neq 0. \tag{3.6}$$

3.2. *Harmonic functions from* $\mathscr{H}(\Gamma)$

The set of compactly supported functions from $\mathscr{H}(\Gamma)$ is not necessarily dense in $\mathscr{H}(\Gamma)$. We denote by $\mathscr{H}^\circ(\Gamma)$ its closure and by $\mathbb{H}(\Gamma)$ the orthogonal complement of $\mathscr{H}^\circ(\Gamma)$. The functions from $\mathbb{H}(\Gamma)$ are called *harmonic*. It follows from the definition that any harmonic function from $\mathscr{H}(\Gamma)$ is linear on each edge of Γ, and hence determined by its values at the vertices. In other words, the set of harmonic functions on the metric tree Γ is the same as on the combinatorial tree with weights (conductances), when the weight of the edge e is $|e|^{-1}$. Recall that the tree Γ is called *recurrent* if $\mathbb{H}(\Gamma)$ is trivial; otherwise, it is called *transient* (see, for example, [**18**]). In particular, $\mathbb{R}_+$ is recurrent.

Along with $A_{\Gamma,V}$ we also consider the operators

$$A^{\circ}_{\Gamma,V} := A(\mathscr{H}^{\circ}(\Gamma), Q_{\Gamma,V}), \quad A^{\mathbb{H}}_{\Gamma,V} := A(\mathbb{H}(\Gamma), Q_{\Gamma,V}). \tag{3.7}$$

Evidently any estimate for $n(\lambda, A_{\Gamma,V})$ yields the same estimate for these two operators. Note that $A^{\circ}_{\Gamma,V}$ can be referred to as the inverse to the weighted Dirichlet Laplacian on Γ.

We pay special attention to the following question. The space $\mathbb{H}(\Gamma)$ is rather 'poor', so one may expect that $n(\lambda, A^{\mathbb{H}}_{\Gamma,V}) = o(n(\lambda, A_{\Gamma,V}))$ as $\lambda \to 0$. In Theorem 6.3 we point out a condition under which this relation holds, and in § 8 we give examples showing that this is not always the case.

3.3. *Formulation in terms of Volterra operators*

An equivalent description of the problem can be given in terms of the singular number behaviour for a weighted Volterra operator in $L_2(\Gamma)$:

$$(K_{\Gamma,W} v)(x) = W(x) \int_{\langle o,x \rangle} v(t)\, dt.$$

Indeed, assume that $K_{\Gamma,W}$ is a compact operator and recall that the squared singular numbers $s_n^2(K_{\Gamma,W})$ are the stationary values of the ratio

$$\frac{\| K_{\Gamma,W} v\|^2_{L_2(\Gamma)}}{\| v\|^2_{L_2(\Gamma)}}, \quad \text{for } v \in L_2(\Gamma),\ v \neq 0. \tag{3.8}$$

The mapping $v(x) \mapsto u(x) = \int_{\langle o,x \rangle} v(t)\, dt$ defines the natural isometry between the spaces $L_2(\Gamma)$ and $\mathscr{H}(\Gamma)$. In terms of u the ratio (3.8) turns into (3.6) for $V = |W|^2$. So we arrive at the following conclusion.

LEMMA 3.1. *The operator $K_{\Gamma,W}$ is compact in $L_2(\Gamma)$ if and only if $A_{\Gamma,|W|^2}$ is compact in $\mathscr{H}(\Gamma)$. Moreover,*

$$s_n^2(K_{\Gamma,W}) = \lambda_n(A_{\Gamma,|W|^2}), \quad \textit{for } n = 1, 2, \dots .$$

3.4. *Estimates in $\mathscr{C}_p$ for $p \geqslant 1$, and $\mathscr{C}_{p,\infty}$ for $p > 1$*

Locally, any tree is 1-dimensional; therefore $n(\lambda, A_{\Gamma,V})$ grows at least as $O(\lambda^{-1/2})$ as $\lambda \to 0$. We are interested in the conditions on V guaranteeing $A_{\Gamma,V} \in \mathscr{C}_p$ or $A_{\Gamma,V} \in \mathscr{C}_{p,\infty}$, for $p \geqslant \frac{1}{2}$. In this subsection we treat the cases $p = \infty$ and $p = 1$ and then interpolate between them. The more complicated case $p < 1$ is investigated in §§ 4–5.

3.4.1. *Boundedness criterion.* The main result on the boundedness of $A_{\Gamma,V}$ is due to Evans, Harris and Pick [**8**]. It is given there in terms of Volterra operators. We present it as applied to our situation.

Let K be a subtree of Γ containing o, and denote by ∂K the set of boundary points of K. We mean here the boundary points in the topology of Γ, so they are all a finite distance from o. A point $t \in \partial K$ is called *maximal* if every $x \succ t$ lies in $\Gamma \setminus K$. Denote by $\mathscr{M}$ the set of all subtrees $K \subset \Gamma$ containing o whose boundary points are all maximal. For $K \in \mathscr{M}$ define

$$\alpha_K = \inf\left\{ \|u\|_{L_2(\Gamma)} : \int_0^t |u|\, dx = 1 \text{ for all } t \in \partial K \right\}.$$

The following statement gives a boundedness criterion for $A_{\Gamma,V}$ (equivalently, for V being a Hardy weight); see [**8**, Theorem 3.1].

THEOREM 3.2. *The operator $A_{\Gamma,V}$ is bounded if and only if*

$$\sup_{K \in \mathscr{M}} \left(\alpha_K^{-2} \int_{\Gamma \setminus K} V\, dx \right) < \infty. \tag{3.9}$$

REMARK. Condition (3.9) is hard to verify in practice. This fact also affects the applicability of Theorem 3.4 below, which depends on choosing a 'model' Hardy weight. For a special class of trees, we can give a simpler description of *symmetric* Hardy weights (that is, $V = V(|x|)$); see the Remark after Theorem 7.2.

3.4.2. *Trace class criterion.* Here we give a simple necessary and sufficient Trace class condition for the operator $A_{\Gamma,V}$.

THEOREM 3.3. *The operator $A_{\Gamma,V}$ belongs to $\mathscr{C}_1$ if and only if*

$$\int_\Gamma |x|\, V(x)\, dx < \infty, \tag{3.10}$$

and moreover

$$\|A_{\Gamma,V}\|_1 = \operatorname{Tr} A_{\Gamma,V} = \int_\Gamma |x|\, V(x)\, dx. \tag{3.11}$$

Proof. Based upon Lemma 3.1, we get directly

$$\|A_{\Gamma,V}\|_1 = \|K_{\Gamma,\sqrt{V}}\|_2^2 = \int_\Gamma dx \int_{\langle o,x \rangle} V(x)\, dy = \int_\Gamma |x|\, V(x)\, dx.$$

The relation (3.11) is an extension to trees of a well-known result for the semi-axis.

3.4.3. *Interpolation: $p > 1$.* Based on Theorems 3.2 and 3.3, we obtain estimates in the classes $\mathscr{C}_p$ and $\mathscr{C}_{p,\infty}$ with $p > 1$, by means of interpolation; see [**2**] for an exposition of interpolation techniques. The results we shall obtain are of a 'parametric type': they involve an auxiliary Hardy weight as a parameter. For operators on $\mathbb{R}^d$, estimates of a similar character were found in [**5**].

Let $\Psi > 0$ be a *normalized* Hardy weight, that is,

$$\int_\Gamma \Psi |u|^2\, dx \leqslant \int_\Gamma |u'|^2\, dx, \quad \text{for } u \in \mathscr{H}(\Gamma). \tag{3.12}$$

Assume that the function V satisfies $|V| \leqslant C\Psi$; then $A_{\Gamma,V}$ is well defined and $\|A_{\Gamma,V}\| \leqslant C$. Interpolation between this boundedness condition and the Trace class criterion (3.10) leads to the following result. In its formulation V is assumed to be non-negative, though in the proof we consider arbitrary weights in order to use interpolation.

THEOREM 3.4. *Let $\Psi > 0$ be a normalized Hardy weight. Then for $1 < p < \infty$,*

$$\|A_{\Gamma,V}\|_p^p = \sum_k \lambda_k^p(A_{\Gamma,V}) \leqslant \int_\Gamma |x| V^p \Psi^{1-p}\, dx \tag{3.13}$$

and

$$\|A_{\Gamma,V}\|_{p,\infty}^p = \sup_{\lambda>0}(\lambda^p n(\lambda, A_{\Gamma,V})) \leqslant C_{3.14}(p) \sup_{t>0}\left(t^p \int_{x\in\Gamma:\, V(x)/\Psi(x)>t} |x|\Psi\, dx\right). \tag{3.14}$$

Proof. It follows from (3.12) that for any real function V one has

$$\|A_{\Gamma,V}\|_{\mathscr{B}} \leqslant \|V\Psi^{-1}\|_{L_\infty(\Gamma)}. \tag{3.15}$$

On the other hand, the equality (3.11) after the passage to sign-indefinite V turns into the inequality

$$\|A_{\Gamma,V}\|_1 \leqslant \int_\Gamma |x|\,|V(x)|\, dx. \tag{3.16}$$

Let us write (3.15) and (3.16) in a consistent way. Denote $\widetilde{V} = V\Psi^{-1}$. Consider the family of weighted spaces $L_p^* = L_p(\Gamma; |x|\Psi)$ and the mapping

$$\Pi\colon \widetilde{V} \mapsto A_{\Gamma,V}.$$

Now (3.15) and (3.16) can be interpreted as

$$\|\Pi\widetilde{V}\|_{\mathscr{B}} \leqslant \|\widetilde{V}\|_{L_\infty^*} \quad \text{and} \quad \|\Pi\widetilde{V}\|_1 \leqslant \|\widetilde{V}\|_{L_1^*}.$$

The complex interpolation between these two inequalities gives $\|A_{\Gamma,V}\|_p \leqslant \|\widetilde{V}\|_{L_p^*}$, for $1 < p < \infty$, which is exactly (3.13). The real interpolation with the functor $\mathscr{K}_{1/p,\infty}$ gives $\|A_{\Gamma,V}\|_{p,\infty} \leqslant c(p)\|\widetilde{V}\|_{L_{p,\infty}^*}$. (The weak spaces $L_{p,\infty}$ are defined similarly to the weak spaces $l_{p,\infty}$; for more details, see, for example, [**2**, § 4.2].) Replacing the quasinorms in $\mathscr{C}_{p,\infty}$ and in $L_{p,\infty}^*$ by their explicit expressions, we come to the inequality (3.14) with $C_{3.14}(p) = c^p(p)$.

4. *Estimates in $\mathscr{C}_p$ and $\mathscr{C}_{p,\infty}$ with $p < 1$: first approach*

It follows from Theorem 3.3 that under the assumption (3.10), $n(\lambda, A_{\Gamma,V}) = o(\lambda^{-1})$ as $\lambda \to 0$. For the weight functions decaying fast enough, $n(\lambda, A_{\Gamma,V})$ should grow more slowly, the limiting case being $n(\lambda, A_{\Gamma,V}) = O(\lambda^{-1/2})$, which is typical for 1-dimensional problems. We suggest two general ways to obtain estimates of the order $n(\lambda, A_{\Gamma,V}) = O(\lambda^{-p})$ with $\frac{1}{2} \leqslant p < 1$, the first is presented in this section, and the second in § 5.

In the first approach we choose a partition Ξ of the tree Γ into a denumerable union of segments $\langle y_j, z_j\rangle$, whose interiors do not intersect. It is not necessary (and not always convenient) to assume that $y_j, z_j \in \mathscr{V}(\Gamma)$. Given a partition Ξ, with any function $V \geqslant 0$ on Γ we associate the sequence $\boldsymbol{\eta} = \boldsymbol{\eta}(V) = \boldsymbol{\eta}(V, \Xi) = \{\eta_j(V)\}$, where

$$\eta_j(V) = |z_j| \int_{\langle y_j,\, z_j \rangle} V\, dx.$$

4.1. *The upper estimates*

Denote by χ_j the characteristic function of the segment $\langle y_j, z_j \rangle$, and let $V_j := V\chi_j$. If the operator $A_{\Gamma,V}$ is bounded in $\mathscr{H}(\Gamma)$, then the operators A_{Γ,V_j} are also bounded and, moreover,

$$A_{\Gamma,V} = \sum_j A_{\Gamma,V_j};$$

the series converges at least in the weak sense. Indeed, for any $u, v \in \mathscr{H}(\Gamma)$ one has

$$\sum_j (A_{\Gamma,V_j} u, v)_{\mathscr{H}(\Gamma)} = \sum_j \int_{\langle y_j,\, z_j \rangle} V u \bar{v}\, dx = \int_\Gamma V u \bar{v}\, dx = (A_{\Gamma,V} u, v)_{\mathscr{H}(\Gamma)}.$$

Consider the operator A_{Γ,V_j}. We have $Q_{\Gamma,V_j}[u] = 0$ on the subspace $\mathscr{F}_j \subset \mathscr{H}(\Gamma)$, constituted by the functions u vanishing on $\langle o, z_j \rangle$. Hence the non-zero spectrum of A_{Γ,V_j} does not change if we restrict Q_{Γ,V_j} to the orthogonal complement $\mathscr{F}_j{}^\perp$ which consists of functions $u \in \mathscr{H}(\Gamma)$ such that $u' = 0$ outside of $\langle o, z_j \rangle$. Identifying $\langle o, z_j \rangle \subset \Gamma$ with the segment $[0, |z_j|] \subset \mathbb{R}$, we see that for $u \in \mathscr{F}_j{}^\perp$ the ratio (3.6) turns into the ratio (2.10) for $I = (0, |z_j|)$. Proposition 2.3 applies to the corresponding problem (2.9) and gives the estimate

$$\lambda^{1/2} n(\lambda, A_{\Gamma,V_j}) \leqslant C_{2.11} \left(|z_j| \int_{\langle o, z_j \rangle} V\, dx \right)^{1/2}.$$

Therefore,

$$\begin{aligned} \|A_{\Gamma,V_j}\|_{1/2,\infty} &= \sup_{\lambda > 0} (\lambda n^2(\lambda, A_{\Gamma,V_j})) \\ &\leqslant C_{2.11}^2 |z_j| \int_{\langle y_j,\, z_j \rangle} V\, dx = C_{2.11}^2 \eta_j(V). \end{aligned} \tag{4.1}$$

Now we are in a position to prove the main result of this section.

THEOREM 4.1. *Let Ξ be an arbitrary partition of Γ.*

(i) *The following holds:*

$$\|A_{\Gamma,V}\|_{1/2,\infty}^{1/2} = \sup_{\lambda > 0} (\lambda^{1/2} n(\lambda, A_{\Gamma,V})) \leqslant C_{4.2} \sum_j \eta_j^{1/2}(V) = C_{4.2} \|\boldsymbol{\eta}(V)\|_{1/2}^{1/2}. \tag{4.2}$$

If the expression on the right-hand side of (4.2) *is finite, then the Weyl type asymptotic formula is also valid:*

$$\lim_{\lambda \to 0} \lambda^{1/2} n(\lambda, A_{\Gamma,V}) = \frac{1}{\pi} \int_\Gamma V^{1/2}\, dx. \tag{4.3}$$

(ii) *For $\frac{1}{2} < p \leqslant 1$,*

$$\|A_{\Gamma,V}\|_p^p = \sum_k \lambda_k^p(A_{\Gamma,V}) \leqslant C_{4.4}(p) \sum_j \eta_j^p(V) = C_{4.4}(p) \|\boldsymbol{\eta}(V)\|_p^p. \tag{4.4}$$

(iii) *For* $\frac{1}{2} < p < 1$,

$$\|A_{\Gamma,V}\|_{p,\infty}^p = \sup_{\lambda>0}(\lambda^p n(\lambda, A_{\Gamma,V}))$$
$$\leqslant C_{4.5}(p) \sup_{t>0}(t^p \#\{j: \eta_j(V) > t\})$$
$$= C_{4.5}(p)\|\boldsymbol{\eta}(V)\|_{p,\infty}^p. \tag{4.5}$$

The constant factors $C_{4.2}$, $C_{4.4}$, $C_{4.5}$ *do not depend on the choice of* Ξ.

Proof. Proposition 2.1(ii) and (4.1) imply (4.2) with $C_{4.2} = C_{2.4}(\frac{1}{2})C_{2.11}$. Further, it follows from (3.16) that

$$\|A_{\Gamma,V}\|_1 \leqslant \sum_j \int_{\langle y_j, z_j\rangle} |x|\,|V(x)|\,dx \leqslant \sum_j \eta_j(|V|),$$

which gives (4.4) for $p = 1$ with $C_{4.4}(1) = 1$.

We use interpolation in order to derive the other estimates. Denote by $\mathscr{X}_j$ the space $L_1(\langle y_j, z_j\rangle)$ endowed with the norm $\|f\|_{\mathscr{X}_j} = \eta_j(f)$. The estimates just obtained show that the mapping Π: $V \mapsto A_{\Gamma,V}$ is continuous when acting between the spaces

$$\Pi: l_{1/2}(\{\mathscr{X}_j\}) \mapsto \mathscr{C}_{1/2,\infty}(\mathscr{H}(\Gamma))$$

and

$$\Pi: l_1(\{\mathscr{X}_j\}) \mapsto \mathscr{C}_1(\mathscr{H}(\Gamma)).$$

Apply to Π the interpolation functor $\mathscr{K}_{\theta,r}$ (see, for example, [**2**]) with $\theta = 2 - p^{-1}$, so that $\theta \in (0, 1)$ for $\frac{1}{2} < p < 1$. We obtain (4.4) taking $r = p$ and (4.5) taking $r = \infty$.

Finally, the set of all compactly supported weight functions $0 \leqslant V \in L_1(\Gamma)$ is dense in the quasi-Banach space defined by the condition that the right-hand side of (4.2) is finite. For such V the asymptotic formula (4.3) is clearly valid. The result extends for all V from the above space by the well-known fact on the continuity of the asymptotic coefficients, see for example, [**4**, Lemma 1.18].

4.2. *Discussion*

REMARK 1. Certainly the estimates (4.4) and (4.5) are meaningful only if the series on the right-hand side of (4.2) diverges.

REMARK 2. The sequence $\boldsymbol{\eta}(V, \Xi)$ depends heavily on the partition, and for an inappropriate choice of Ξ the estimates (4.2), (4.4) and (4.5) can be rather rough. Moreover, the quasinorms $\|\boldsymbol{\eta}(V, \Xi)\|_p$ and $\|\boldsymbol{\eta}(V, \Xi)\|_{p,\infty}$ for different choices of Ξ may be non-equivalent.

EXAMPLE 4.2. Let $\Gamma = \mathbb{R}_+$ and

$$V(t) = \begin{cases} c_k & \text{for } t \in (2^{2^k}, 2^{2^k} + 1) \text{ with } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then for the partition Ξ_1: $\mathbb{R}_+ = [0, 1] \cup \left(\bigcup_{j=1}^{\infty}[2^{j-1}, 2^j]\right)$ we find (ignoring zero terms) that

$$\boldsymbol{\eta}(V, \Xi_1) = \{2^{2^k+1}c_k\}, \quad \text{where } k \in \mathbb{N},$$

and for the partition Ξ_2: $\mathbb{R}_+ = [0, 2] \cup \left(\bigcup_{j=1}^{\infty}[2^{2^{j-1}}, 2^{2^j}]\right)$,

$$\boldsymbol{\eta}(V, \Xi_2) = \{2^{2^{k+1}} c_k\}, \quad \text{where } k \in \mathbb{N}.$$

It is clear that $\boldsymbol{\eta}(V, \Xi_2)$ cannot be estimated through $\boldsymbol{\eta}(V, \Xi_1)$ either in l_p or in $l_{p,\infty}$ for any $p > 0$.

EXAMPLE 4.3. Again let $\Gamma = \mathbb{R}_+$ and

$$V(t) = \begin{cases} 0 & \text{if } t \in (0, 1), \\ c_k & \text{if } t \in (2^{k-1}, 2^k) \text{ with } k \in \mathbb{N}. \end{cases}$$

For the partition Ξ_1 from Example 4.2,

$$\boldsymbol{\eta}(V, \Xi_1) = \{2^{2j-1} c_j\}, \quad \text{where } j \in \mathbb{N}, \tag{4.6}$$

and for the partition Ξ_2: $\mathbb{R}_+ = \bigcup_{j=1}^{\infty}[j-1, j]$,

$$\boldsymbol{\eta}(V, \Xi_2) = \{jc_k\}, \quad \text{where } k \in \mathbb{N},\ 2^{k-1} < j \leqslant 2^k. \tag{4.7}$$

It is easy to see that for any $0 < p < 1$, $\boldsymbol{\eta}(V, \Xi_1)$ can be estimated through $\boldsymbol{\eta}(V, \Xi_2)$ in both l_p and $l_{p,\infty}$, and the converse is wrong.

We see from these examples that taking 'too long' or 'too short' segments makes the estimates (4.2), (4.4), (4.5) rough. It is always reasonable to take partitions which satisfy the condition

$$\mu_1 \leqslant |z_j|/|y_j| \leqslant \mu_2, \quad 1 < \mu_1 \leqslant \mu_2. \tag{4.8}$$

In particular, suppose that for a given tree Γ each edge $\langle y_j, z_j \rangle \in \mathscr{E}(\Gamma)$ except for e_0 satisfies (4.8). Then it makes sense to include these edges in the partition. Additional division points can be put on e_0 to satisfy (4.8) as well.

Actually, for any $q > 1$ it is possible to choose Ξ in such a way that $|z_j| = q|y_j|$, for $j \in \mathbb{Z}$. Indeed, order the vertices of Γ by their distance from the root:

$$0 = |o| < |y_1| < |y_2| \leqslant |y_3| \leqslant \dots .$$

In this construction, we mean by 'path' an infinite path starting at o. Such a path J can be viewed as a copy of $\mathbb{R}_+$ with the t-coordinate corresponding to the distance along J, the notation like $[t, qt] \subset J$ becoming clear.

Take first any path J_1 and introduce the first set of segments by

$$J_1 = \bigcup_{i \in \mathbb{Z}} [q^{i-1}|y_1|, q^i|y_1|].$$

Next, take the vertex y_1 and choose the paths $J_2, \dots, J_{b(y_1)}$ so that $J_k \cap J_l = \langle o, y_1 \rangle$ for any $1 \leqslant k < l \leqslant b(y_1)$. The segment $\langle o, y_1 \rangle$ is already taken into account; the new set of segments is obtained by dividing

$$J_k \backslash \langle o, y_1 \rangle = \bigcup_{i=1}^{\infty} [q^{i-1}|y_1|, q^i|y_1|], \quad \text{for } k = 2, \dots, b(y_1).$$

Now, proceed by induction. Assume that the paths corresponding to the vertices up to y_{n-1} are already taken into account. Take the next vertex y_n; note that at the moment only one path (call it $\widetilde{J}_1$) passes through it. Choose paths $\widetilde{J}_2, \dots, \widetilde{J}_{b(y_n)}$ so that $\widetilde{J}_k \cap \widetilde{J}_l = \langle o, y_n \rangle$ for any $1 \leqslant k < l \leqslant b(y_n)$. Now get the new set of segments by dividing

$$\widetilde{J}_k \setminus \langle o, y_n \rangle = \bigcup_{i=1}^{\infty} [q^{i-1}|y_n|, q^i|y_n|], \quad \text{for } k = 2, \dots, b(y_n).$$

Clearly, as a result of this inductive procedure, we get a required partition. We call such Ξ a *q-partition*; we have for it $\mu_1 = \mu_2 = q$. However, for different q-partitions (with the same q) $\|\boldsymbol{\eta}(V, \Xi)\|_p$ and $\|\boldsymbol{\eta}(V, \Xi)\|_{p,\infty}$ still can be non-equivalent.

EXAMPLE 4.4. Let Γ be binary tree (that is, $b(z) = 2$ for any $z \in \mathscr{V}(\Gamma)$, $z \neq o$), with all edges having the same length 1. Choose any infinite path starting at o; denote it by J. Let V be a weight function on Γ such that $\mathrm{supp}(V) \subset J$, and on J take V to be the same as in Example 4.3. We now describe two 2-partitions, Ξ_1 and Ξ_2, of the tree Γ. Following our construction, we note that segments which do not intersect J give zero terms in $\boldsymbol{\eta}$, which we ignore.

First partition. This way is natural: choose the first path J_1 in our construction to be exactly J, and write $J = [0, 1] \cup \left(\bigcup_{i \in \mathbb{N}} [2^{i-1}, 2^i] \right)$. Then

$$\boldsymbol{\eta}(V, \Xi_1) = \{2^{2j-1} c_j\}, \quad \text{for } j \in \mathbb{N},$$

which coincides with (4.6).

Second partition. Again, we describe only the segments which intersect J. Take paths J_n such that $J_n \cap J = [0, n+1]$, and which are represented as $J_n = [0, n] \cup \left(\bigcup_{i=1}^{\infty} [2^{i-1}n, 2^i n] \right)$. Contribution of J_n comes from only one of its segments, namely, $[n, n+1] \subset [n, 2n] \subset J_n$. So,

$$\boldsymbol{\eta}(V, \Xi_2) = \{2nc_k\}, \quad \text{where } k \in \mathbb{N}, \, 2^{k-1} \leqslant n < 2^k,$$

which is equivalent to (4.7).

We come now to the same conclusions as in Example 4.3.

REMARK 3. For the semiaxis $\mathbb{R}_+$ the results of Theorem 4.1 are well known; see [**3**] where further references can be found. In [**3**], the 2-partition of $\mathbb{R}_+$ was taken. Estimates (4.4) and (4.5) in the case of $\mathbb{R}_+$ hold for all $p \in (\frac{1}{2}, \infty)$ and are in a sense invertible.

Invertibility here means that the finiteness of the right-hand side of (4.4), (4.5) is not only sufficient but also necessary for the finiteness of the corresponding left-hand side. The estimate (4.2) is 'weakly invertible': the finiteness of its left-hand side implies that $\boldsymbol{\eta}(V) \in l_{1/2,\infty}$. The lack of such invertibility for general trees is clear from Remark 2; in § 4.3 we give the corresponding inverse results for a special class of trees.

Technically, the validity of (4.4), (4.5) for all $p \in (\frac{1}{2}, \infty)$ in the case $\Gamma = \mathbb{R}_+$ is due to the fact that $\boldsymbol{\eta}(V) \in l_\infty$ ensures the boundedness of $A_{\mathbb{R}_+ V}$, and it is possible to interpolate between the results for $p = \frac{1}{2}$ and $p = \infty$. This boundedness result does not extend to general trees, and for $1 < p < \infty$ we gave estimates of a different sort in Theorem 3.4.

EXAMPLE 4.5. Let Γ be a binary tree such that $|e_0| = 1$ and $|z| = 2|y|$ for any edge $\langle y, z \rangle \neq e_0$. Take

$$V(x) = \begin{cases} 0 & \text{if } x \in e_0, \\ |x|^{-2} & \text{otherwise.} \end{cases}$$

Then $V \notin L_1(\Gamma \setminus e_0)$ and for this reason cannot be a Hardy weight. At the same time, take the natural partition of Γ into the union of its edges. The initial edge e_0 does not contribute to $\|\boldsymbol{\eta}(V, \Xi)\|_\infty$. All the other edges satisfy (4.8) and contribute term 1 to $\boldsymbol{\eta}(V, \Xi)$; thus $\boldsymbol{\eta}(V, \Xi) \in l_\infty$.

REMARK 4. The estimate (4.4) for $p = 1$ is implied by (3.11), and, in general, is weaker than (3.11). However, if (4.8) is satisfied, then η_j is equivalent to $\int_{\langle y_j, z_j \rangle} |x| V\, dx$, and (4.4) for $p = 1$ becomes invertible. For $p \neq 1$, the results of Theorem 4.1 can be inverted only under additional assumptions on Γ.

4.3. *The lower estimates*

Here we prove the result converse to Theorem 4.1 for a specific class of trees. The reasonings are basically the same as for $\mathbb{R}_+$ (see [**3**]), with some technical complications arising.

Let all the edges $\langle y_j, z_j \rangle \in \mathscr{E}(\Gamma)$, except for e_0, meet the condition (4.8). Denote $e_0 = \langle o, y_{1,1} \rangle$ and choose $y_{k,1} \in e_0$ so that $\langle y_{k,1}, y_{k+1,1} \rangle$, for $k = 0, -1, -2, \dots$, satisfy (4.8). Define the extended set of vertices by $\mathscr{V}'(\Gamma) = \mathscr{V}(\Gamma) \cup \{y_{k,1}\}_{k=-\infty}^{0}$, and let $\mathscr{E}'(\Gamma)$ be the corresponding extended set of edges.

THEOREM 4.6. *Let Γ satisfy two assumptions:*

(a) *there exists $B \geqslant 1$ such that $b(z) \leqslant B$ for all $z \in \mathscr{V}(\Gamma)$;*

(b) *any edge $e_0 \neq \langle y_j, z_j \rangle \in \mathscr{E}(\Gamma)$ meets the condition* (4.8).

Let Ξ be defined by $\Gamma = \bigcup_{e \in \mathscr{E}'(\Gamma)} e$, where $\mathscr{E}'(\Gamma)$ is as above. Then there exists a constant $C > 0$, depending only on μ_1, μ_2 and B, such that

$$n(\lambda, A_{\Gamma,V}) \geqslant (3B)^{-1} n(C\lambda, \boldsymbol{\eta}(V, \Xi)). \tag{4.9}$$

In particular, for some $C_1, C_2 > 0$,

$$\|A_{\Gamma,V}\|_p \geqslant \|\boldsymbol{\eta}(V, \Xi)\|_p, \quad \text{for } \tfrac{1}{2} < p < \infty,$$
$$\|A_{\Gamma,V}\|_{p,\infty} \geqslant C_2 \|\boldsymbol{\eta}(V, \Xi)\|_{p,\infty}, \quad \text{for } \tfrac{1}{2} \leqslant p < \infty.$$

Proof. First, let us introduce convenient notation for the vertices and edges of Γ. We say that $y \in \mathscr{V}(\Gamma)$ is of generation k if the closed segment $\langle o, y \rangle$ contains exactly $k+1$ vertices from $\mathscr{V}(\Gamma)$. Denote all the vertices of generation k by $y_{k,m}$, where $1 \leqslant m \leqslant n(k)$ and $n(k) \leqslant B^{k-1}$ is the number of such vertices. This is consistent with the notation $y_{1,1}$, and if for $k = 0, -1, -2, \dots$ we put $n(k) = 1$, then $\mathscr{V}'(\Gamma) = \{y_{k,m}\}$ for $k \in \mathbb{Z}$ and $1 \leqslant m \leqslant n(k)$. The edge from $\mathscr{E}'(\Gamma)$ whose endpoint is $y_{k,m}$ is denoted by $e_{k,m}$, and its length by $|e_{k,m}|$.

Now, for each pair (k, m) let the function $v_{k,m} \in \mathscr{H}(\Gamma)$ be linear on each edge $e \in \mathscr{E}'(\Gamma)$ and such that $v_{k,m}$ equals 1 at $y_{k,m}$ and at all of its adjacent descendants, and equals 0 at all other vertices from $\mathscr{V}'(\Gamma)$. It is not difficult to see that $\int_\Gamma |v_{k,m}'|^2\, dx \leqslant C|e_{k,m}|^{-1}$; here C depends only on μ_1, μ_2 and B.

Fix $l \in \{0, 1, 2\}$; then the functions $v_{3k+l,m}$ for different k and m are disjointly supported. Denote $\mathscr{F}_l = \operatorname{Span}\{v_{3k+l,m}\}$, for $l = 0, 1, 2$.

Consider the operator A_0 corresponding to the restriction $a_{\Gamma,V} \restriction \mathscr{F}_0$. By the variational principle, $n(\lambda, A_{\Gamma,V}) \geqslant n(\lambda, A_0)$.

Denote by $\mathscr{D}_{k,m}$ the union of edges emanating from $y_{k,m}$. For $u \in \mathscr{F}_0$, with

$u = \sum c_{3k,m} v_{3k,m}$, we have

$$a_{\Gamma,V}[u] \geqslant \sum_{k,m} |c_{3k,m}|^2 \int_{\mathscr{D}_{3k,m}} V\,dx, \qquad \int_\Gamma |u'|^2\,dx \leqslant C \sum_{k,m} |c_{3k,m}|^2 |e_{k,m}|^{-1}.$$

Therefore,

$$n(\lambda, A_{\Gamma,V}) \geqslant n\left(C\lambda, \left\{|e_{3k,m}| \int_{\mathscr{D}_{3k,m}} V\,dx\right\}_{k,m}\right).$$

Similar relations hold for $\mathscr{F}_1$ and $\mathscr{F}_2$, so we get

$$3n(\lambda, A_{\Gamma,V}) \geqslant n\left(C\lambda, \left\{|e_{k,m}| \int_{\mathscr{D}_{k,m}} V\,dx\right\}_{k,m}\right). \tag{4.10}$$

By the assumption (a), $\mathscr{D}_{k,m}$ consists of at most B edges. Let e be one of them; then $|e| \leqslant c|e_{k,m}|$, where $c = \mu_2(\mu_2 - 1)(\mu_1 - 1)^{-1}$. Thus,

$$n\left(\lambda, \left\{|e_{k,m}| \int_{\mathscr{D}_{k,m}} V\,dx\right\}_{k,m}\right) \geqslant B^{-1} n\left(c^{-1}\lambda, \left\{|e| \int_e V\,dx\right\}_{e \in \mathscr{E}'(\Gamma)}\right). \tag{4.11}$$

For $e = \langle y_j, z_j\rangle \in \mathscr{E}'(\Gamma)$, we have

$$\eta_j(V, \Xi) = |z_j| \int_e V\,dx,$$

and

$$|e| = |z_j| - |y_j| \geqslant |z_j| - \mu_1^{-1}|z_j| = (1 - \mu_1^{-1})|z_j|,$$

so

$$\eta_j(V, \Xi) \leqslant \mu_1(\mu_1 - 1)^{-1} |e| \int_e V\,dx.$$

Therefore,

$$n\left(\lambda, \left\{|e| \int_e V\,dx\right\}_{e \in \mathscr{E}'(\Gamma)}\right) \geqslant n((1 - \mu_1^{-1})\lambda, \boldsymbol{\eta}). \tag{4.12}$$

Finally, (4.9) follows from (4.10)–(4.12).

Combining the results of Theorems 4.1 and 4.6, we arrive at the following conclusion.

COROLLARY 4.7. *Under the assumptions of Theorem* 4.6, *one has*

$$c\|\boldsymbol{\eta}(V, \Xi)\|_{1/2,\infty} \leqslant \|A_{\Gamma,V}\|_{1/2,\infty} \leqslant C\|\boldsymbol{\eta}(V, \Xi)\|_{1/2},$$
$$c_1\|\boldsymbol{\eta}(V, \Xi)\|_p \leqslant \|A_{\Gamma,V}\|_p \leqslant C_1\|\boldsymbol{\eta}(V, \Xi)\|_p, \quad \text{for } \tfrac{1}{2} < p \leqslant 1,$$
$$c_2\|\boldsymbol{\eta}(V, \Xi)\|_{p,\infty} \leqslant \|A_{\Gamma,V}\|_{p,\infty} \leqslant C_2\|\boldsymbol{\eta}(V, \Xi)\|_{p,\infty}, \quad \text{for } \tfrac{1}{2} < p < 1.$$

REMARK. If V is integrable at o, it is convenient to deal with the 'natural' sequence $\widetilde{\boldsymbol{\eta}}(V, \widetilde{\Xi})$ rather than with $\boldsymbol{\eta}(V, \Xi)$:

$$\widetilde{\boldsymbol{\eta}} = \widetilde{\boldsymbol{\eta}}(V, \widetilde{\Xi}) = \left\{|z_j| \int_{\langle y_j, z_j\rangle} V\,dx\right\}_{\langle y_j, z_j\rangle \in \mathscr{E}(\Gamma)}, \quad \widetilde{\Xi}\colon \Gamma = \bigcup_{e \in \mathscr{E}(\Gamma)} e. \tag{4.13}$$

Exactly as in the case of $\mathbb{R}_+$ (see [**3**]), this sequence can replace $\boldsymbol{\eta}$ in the upper estimates (Theorem 4.1), but not in the lower ones (Theorem 4.6). However, Corollary 4.7 implies qualitative results for $V \in L_1(\Gamma)$:

$$\widetilde{\boldsymbol{\eta}} \in l_{1/2} \quad \Longrightarrow \quad A_{\Gamma,V} \in \mathscr{C}_{1/2,\infty} \quad \Longrightarrow \quad \widetilde{\boldsymbol{\eta}} \in l_{1/2,\infty},$$

and

$$\begin{aligned} A_{\Gamma,V} \in \mathscr{C}_p \quad &\Longleftrightarrow \quad \widetilde{\boldsymbol{\eta}} \in l_p, \qquad \text{for } \tfrac{1}{2} < p \leqslant 1, \\ A_{\Gamma,V} \in \mathscr{C}_{p,\infty} \quad &\Longleftrightarrow \quad \widetilde{\boldsymbol{\eta}} \in l_{p,\infty}, \quad \text{for } \tfrac{1}{2} < p < 1. \end{aligned} \tag{4.14}$$

(One should take into account the fact that according to Proposition 2.3 the contribution of e_0 into $n(\lambda, A_{\Gamma,V})$ is of the order $O(\lambda^{-1/2})$.)

5. *Estimates in $\mathscr{C}_p$ and $\mathscr{C}_{p,\infty}$ for $p < 1$: second approach*

5.1. *An orthogonal decomposition of $\mathscr{H}(\Gamma)$*

The approach we develop here is based upon a special decomposition of the space $\mathscr{H}(\Gamma)$.

In order to avoid some technical complications, we restrict ourselves to the trees meeting the following additional condition. Denote

$$L(\Gamma) := \sup\{|y|: y \in \Gamma\} \leqslant \infty.$$

Then we require

$$\#\{y \in \mathscr{V}(\Gamma): |y| \leqslant l\} < \infty \quad \text{for any } l < L(\Gamma). \tag{5.1}$$

A path in Γ is called *maximal* if it is not a part of another path. It follows from (5.1) and from the assumed absence of leaves that all the maximal paths in Γ have the same length $L(\Gamma)$.

With any subtree $T \in \mathscr{W}(\Gamma)$ we associate a subspace $\mathscr{F}_T$ of $\mathscr{H}(\Gamma)$. Namely, a function $u \in \mathscr{H}(\Gamma)$ belongs to $\mathscr{F}_T$ if and only if

$$u = 0 \text{ outside } T \tag{5.2}$$

and

$$u(x) = u(y) \text{ provided } x, y \in T \text{ and } |x| = |y|. \tag{5.3}$$

Clearly, for disjoint W-subtrees these subspaces are mutually orthogonal.

Recall that given a vertex $z \in \mathscr{V}(\Gamma)$, we set $T_z = \{x \in \Gamma: x \succcurlyeq z\} \in \mathscr{W}(\Gamma)$, and denote by T_z^j, for $j = 1, \dots, b(z)$, the daughter subtrees of T_z. Set

$$\mathscr{F}_z = \sum_{j=1}^{b(z)}{}^{\oplus} \mathscr{F}_{T_z^j}.$$

The reader should not mix the notations $\mathscr{F}_z$ and $\mathscr{F}_{T_z}$. Evidently $\mathscr{F}_{T_z} \subset \mathscr{F}_z$, and we define

$$\mathscr{G}_z = \mathscr{F}_z \ominus \mathscr{F}_{T_z}. \tag{5.4}$$

Note that $T_o = \Gamma$ coincides with its only daughter subtree; thus $\mathscr{G}_o = \{0\}$.

THEOREM 5.1. *For an arbitrary tree Γ satisfying* (5.1)*, all the subspaces $\mathscr{G}_z$ $(z \in \mathscr{V}(\Gamma))$ and $\mathscr{F}_\Gamma$ are mutually orthogonal; in fact, if u and v belong to different such subspaces, then*

$$\sum_{|x|=t} u'(x)\overline{v'(x)} = 0 \quad \textit{for almost all } t \geqslant 0. \tag{5.5}$$

Moreover,

$$\mathscr{H}(\Gamma) = \mathscr{F}_\Gamma \oplus \sum_{z\in\mathscr{V}(\Gamma)}^{\oplus} \mathscr{G}_z. \tag{5.6}$$

In order to prove this theorem, note that the mapping

$$\mathscr{I}\colon u(x) \mapsto f(x) = u'(x)$$

defines the natural isometry between the spaces $\mathscr{H}(\Gamma)$ and $L_2(\Gamma)$, so it is enough to prove a similar statement for $L_2(\Gamma)$. Namely, for any $T\in\mathscr{W}(\Gamma)$ and any $z\in\mathscr{V}(\Gamma)$ define the respective subspaces of $L_2(\Gamma)$: $\mathscr{F}'_T = \mathscr{I}(\mathscr{F}_T)$, $\mathscr{G}'_z = \mathscr{I}(\mathscr{G}_z)$. Clearly, a function $f\in L_2(\Gamma)$ belongs to $\mathscr{F}'_T$ if and only if it satisfies (5.2) and (5.3). Also, analogously to (5.4),

$$\mathscr{G}'_z = \mathscr{F}'_z \ominus \mathscr{F}'_{T_z}. \tag{5.7}$$

Again, $\mathscr{G}'_o = \{0\}$. A function $f\in\mathscr{F}'_z$ belongs to $\mathscr{G}'_z$ if and only if

$$\sum_{x\succ z:\,|x|=t} f(x) = 0 \quad \text{a.e.} \tag{5.8}$$

It is easy to describe explicitly the orthogonal projections $\mathscr{P}'_T$, Π'_z onto the subspaces $\mathscr{F}'_T$, $\mathscr{G}'_z$. On the interval

$$I_T = (|o(T)|, L(\Gamma)) \tag{5.9}$$

consider the piecewise-constant function

$$g_T(t) = \#\{x\in T\colon |x| = t\};$$

it may be called 'the branching function of the subtree T'. It is always finite, in view of condition (5.1). Then clearly

$$(\mathscr{P}'_T f)(x) = \begin{cases} \dfrac{1}{g_T(|x|)} \displaystyle\sum_{y\in T:\,|y|=|x|} f(y) & \text{for } x\in T,\\ 0 & \text{for } x\notin T. \end{cases} \tag{5.10}$$

Let us analyse the formula (5.10) for a particular case when $T=\widetilde{T}_z$ is one of the daughter subtrees of a given subtree T_z. Let $\langle z,\tilde{z}\rangle$ be the initial edge of $\widetilde{T}_z$. Then, according to (5.10),

$$\mathscr{P}'_{\widetilde{T}_z} f = \begin{cases} f & \text{on } \langle z,\tilde{z}\rangle,\\ \mathscr{P}'_{T_{\tilde{z}}} f & \text{on } \widetilde{T}_z\setminus\langle z,\tilde{z}\rangle. \end{cases} \tag{5.11}$$

In particular, let $T_z = \widetilde{T}_z = \Gamma$; denote $e_0 = \langle o, z_0\rangle$. Then

$$\mathscr{P}'_\Gamma f = \begin{cases} f & \text{on } e_0,\\ \mathscr{P}'_{T_{z_0}} f & \text{on } \Gamma\setminus e_0. \end{cases} \tag{5.12}$$

In view of some general formulas from the geometry of Hilbert space, we derive, from (5.7),

$$\Pi'_z f = \sum_{j=1}^{b(z)} \mathscr{P}'_{T_z^j} f - \mathscr{P}'_{T_z} f. \tag{5.13}$$

Now we see that Theorem 5.1 is an immediate consequence of the following result.

LEMMA 5.2. *For an arbitrary tree* Γ *satisfying* (5.1), *all the subspaces* $\mathscr{G}'_z$ $(z \in \mathscr{V}(\Gamma))$ *and* $\mathscr{F}'_\Gamma$ *are mutually orthogonal; in fact, if* f *and* g *belong to different such subspaces, then*

$$\sum_{|x|=t} f(x)\overline{g(x)} = 0 \quad \text{a.e.} \tag{5.14}$$

Moreover,

$$L_2(\Gamma) = \mathscr{F}'_\Gamma \oplus \sum_{z \in \mathscr{V}(\Gamma)}^{\oplus} \mathscr{G}'_z. \tag{5.15}$$

Proof. First, for the subspaces $\mathscr{F}'_\Gamma$ and any $\mathscr{G}'_z$, (5.14) is implied by (5.3) with $T = \Gamma$ and (5.8). Further, let $y, z \in \mathscr{V}(\Gamma)$. If $y \not\succ z$ and $z \not\succ y$, then the subtrees T_y and T_z are disjoint, which implies (5.14) for $\mathscr{G}'_y$ and $\mathscr{G}'_z$. Now, assume that $z \succ y$ and let $f \in \mathscr{G}'_y$ and $g \in \mathscr{G}'_z$. The function f satisfies (5.3) on all the daughter subtrees of T_y. The function g vanishes outside T_z (which is completely contained in some daughter subtree of T_y) and satisfies (5.8). So, (5.14) also follows for $\mathscr{G}'_y$ and $\mathscr{G}'_z$ with $z \succ y$.

In order to verify (5.15), fix an edge e. Let z stand for its endpoint. Let $o, z_0, \dots, z_n$ be all the vertices lying on the path $\langle o, z\rangle$, so $z_n = z$ and $e = \langle z_{n-1}, z_n\rangle$. We suppose first that $e \neq e_0$. Fix a function $f \in L_2(\Gamma)$ and consider its projections $\mathscr{P}_\Gamma f, \Pi'_{z_0} f, \Pi'_{z_1} f, \dots, \Pi'_{z_{n-1}} f$, restricted to the edge e. To simplify notation, we drop the sign of this restriction. According to (5.13), (5.11), and (5.12) we get:

$$\begin{aligned}
\mathscr{P}'_\Gamma f &= \mathscr{P}'_{T_{z_0}} f,\\
\Pi'_{z_0} f &= \mathscr{P}'_{T_{z_1}} f - \mathscr{P}'_{T_{z_0}} f,\\
&\dots\\
\Pi'_{z_{n-2}} f &= \mathscr{P}'_{T_{z_{n-1}}} f - \mathscr{P}'_{T_{z_{n-2}}} f,\\
\Pi'_{z_{n-1}} f &= f - \mathscr{P}'_{T_{z_{n-1}}} f.
\end{aligned}$$

It follows that if f is orthogonal to each component of the sum on the right-hand side of (5.15), then $f = 0$ on each edge $e \neq e_0$. The same is true for e_0, which follows directly from (5.12). So $f \equiv 0$ on Γ.

5.2. *Realization of* $\mathscr{F}_T$ *as a weighted Sobolev space*

Let T be a W-subtree and I_T be the interval (5.9). Introduce the Hilbert space $\mathscr{H}(I_T, g_T)$ (cf. (2.13)) consisting of functions v such that

$$\|v\|^2_{\mathscr{H}(I_T, g_T)} := \int_{I_T} |v'(t)|^2 g_T(t)\, dt < \infty, \quad v(|o_T|) = 0. \tag{5.16}$$

For a given $u \in \mathscr{F}_T$, let v be the function on I_T such that $u(x) = v(|x|)$ for $x \in T$; it is well defined in view of (5.3). Then the operator

$$\mathscr{U}_T\colon u \mapsto v$$

is an isometry (identification operator) between the spaces $\mathscr{F}_T$ and $\mathscr{H}(I_T, g_T)$.

The definition of $\mathscr{G}_z$ and the above realization of $\mathscr{F}_T$ imply that $\mathscr{G}_z$ can be identified with an appropriate subspace of the orthogonal sum

$$\sum_{j=1}^{b(z)}{}^{\oplus}\mathscr{H}(I_T, g_{T_z^j}).$$

More precisely, let $\{v_1, \ldots, v_{b(z)}\}$ be an element of this orthogonal sum. In view of (5.8), it belongs to the subspace in question if and only if

$$\sum_{j=1}^{b(z)} v_j'(t)\, g_{T_z^j}(t) = 0 \quad \text{a.e. on } (|z|, L(\Gamma)). \tag{5.17}$$

Let us discuss the structure of $\mathscr{H}(I_T, g_T)$. Denote by $\mathscr{H}^\circ(I_T, g_T)$ the closure of $C_0^\infty(I_T)$ in the metric of $\mathscr{H}(I_T, g_T)$, and by $\mathbb{H}(I_T, g_T)$ its orthogonal complement. Clearly, $\mathscr{H}^\circ(I_T, g_T) = \mathscr{U}_T(\mathscr{F}_T \cap \mathscr{H}^\circ(\Gamma))$ and $\mathbb{H}(I_T, g_T) = \mathscr{U}_T(\mathscr{F}_T \cap \mathbb{H}(\Gamma))$.

Any function $v \in \mathbb{H}(I_T, g_T)$ is harmonic with respect to the metric form (5.16), that is, is continuous on I, linear on each interval where $g_T(t)$ is constant and satisfies the boundary condition $v(|o_T|) = 0$ and the standard transmission condition at the points where g_T has jumps. The only possible solutions are $c \cdot v_T$ where

$$v_T(t) = \int_{|o_T|}^{t} \frac{ds}{g_T(s)}. \tag{5.18}$$

This function belongs to $\mathscr{H}(I_T, g_T)$ if and only if

$$\ell(T) := \int_{I_T} \frac{ds}{g_T(s)} < \infty. \tag{5.19}$$

We arrive at the following result.

LEMMA 5.3. *Let* (5.1) *be satisfied. Then for a given W_0-subtree T the subspace $\mathscr{F}_T \cap \mathbb{H}(\Gamma)$ is non-trivial if and only if the condition* (5.19) *is fulfilled. In this case $\mathscr{F}_T \cap \mathbb{H}(\Gamma)$ has dimension* 1 *and is generated by the function $u_T := \mathscr{U}_T^{-1} v_T$.*

5.3. *A preliminary estimate*

We start with a statement of a rather general nature, which is implied by decomposition (5.6) and Lemma 2.2. Then in the next subsection we derive from it somewhat more detailed information.

In what follows the notation $A(\mathscr{F}_T, V)$ stands for the operator in the subspace $\mathscr{F}_T \subset \mathscr{H}(\Gamma)$, generated by the quadratic functional (3.1) (more exactly, by its restriction to this subspace). Similar notation will also be used in other cases.

LEMMA 5.4. *Let Γ be a tree satisfying* (5.1). *Then*

$$\|A_{\Gamma,V}\|_{1/2,\infty}^{1/2} \leqslant C_{2.4}(\tfrac{1}{2}) \sum_{T \in \mathscr{W}_0(\Gamma)} \|A(\mathscr{F}_T, V)\|_{1/2,\infty}^{1/2}, \tag{5.20}$$

$$\|A_{\Gamma,V}\|_p^p \leqslant C_{5.21}(p) \sum_{T \in \mathscr{W}_0(\Gamma)} \|A(\mathscr{F}_T, V)\|_{1/2,\infty}^p, \quad \textit{for } \tfrac{1}{2} < p \leqslant 1, \tag{5.21}$$

$$\|A_{\Gamma,V}\|_{p,\infty}^p \leqslant C_{5.22}(p) \sup_{l>0} (\lambda^p \#\{T \in \mathscr{W}_0(\Gamma) \colon \|A(\mathscr{F}_T, V)\|_{1/2,\infty} > \lambda\}), \quad \textit{for } \tfrac{1}{2} < p < 1. \tag{5.22}$$

Proof. It follows immediately from (5.6) and (2.7) for $p = \frac{1}{2}$ that

$$\|A_{\Gamma,V}\|_{1/2,\infty}^{1/2} \leqslant C_{2.4}(\tfrac{1}{2})\left(\|A(\mathscr{F}_\Gamma, V)\|_{1/2,\infty}^{1/2} + \sum_{z\in\mathscr{V}(\Gamma)} \|A(\mathscr{G}_z, V)\|_{1/2,\infty}^{1/2}\right).$$

Note also that $A(\mathscr{F}_z, V)$ is the orthogonal sum of the operators $A(\mathscr{F}_{T_z^j}, V)$. By the variational principle

$$n(\lambda; \mathscr{G}_z, V) \leqslant n(\lambda; \mathscr{F}_z, V) = \sum_{j=1}^{b(z)} n(\lambda; \mathscr{F}_{T_z^j}, V), \tag{5.23}$$

which implies that

$$\|A(\mathscr{G}_z, V)\|_{1/2,\infty}^{1/2} \leqslant \sum_{j=1}^{b} \|A(\mathscr{F}_{T_z^j}, V)\|_{1/2,\infty}^{1/2}. \tag{5.24}$$

Summing up the inequalities (5.24) over all $z \in \mathscr{V}(\Gamma)$, we arrive at (5.20).

The same reasoning but based on (2.6) for $p = 1$ rather than on (2.7) gives (5.21) for $p = 1$:

$$\|A_{\Gamma,V}\|_1 \leqslant \sum_{T\in\mathscr{W}_0(\Gamma)} \|A(\mathscr{F}_T, V)\|_1 \leqslant C \sum_{T\in\mathscr{W}_0(\Gamma)} \|A(\mathscr{F}_T, V)\|_{1/2,\infty}. \tag{5.25}$$

The last inequality is just a coarsening of the first one.

Now we can interpolate between (5.20) and (5.25). This is done exactly as in the proof of Theorem 4.1, the only difference being that instead of $\mathscr{X}_j$ we take $\mathscr{X}_T = \mathscr{C}_{1/2,\infty}(T)$, for $T \in \mathscr{W}_0(\Gamma)$.

5.4. *The basic estimate of the second approach*

Let T be a W_0-subtree of Γ. Realize $\mathscr{F}_T$ as the space $\mathscr{H}(I_T, g_T)$; see (5.16). Then, using a standard change of variables (cf. Corollary 2.4), we see that the operator $A(\mathscr{F}_T, V)$ corresponds to the Rayleigh quotient (2.10) on the (finite or infinite) interval $I = (0, \ell(T))$. The case $\ell(T) = \infty$ will be touched upon in §§ 7–8. Here we restrict ourselves to a simpler situation when $\ell(T) < \infty$ for any $T \in \mathscr{W}_0(\Gamma)$.

From now on, we always assume $V \in L_1(\Gamma)$. Set

$$\sigma_T(V) = \ell(T)\int_T V(x)\,dx = \int_{I_T} \frac{ds}{g_T(s)} \int_T V(x)\,dx, \quad \text{with } T \in \mathscr{W}_0(\Gamma). \tag{5.26}$$

Here these numbers play the same role as the numbers $\eta_j(V)$ in the first approach. We denote $\boldsymbol{\sigma}(V) = \{\sigma_T(V) : T \in \mathscr{W}_0(\Gamma)\}$. The following result is parallel to Theorem 4.1.

THEOREM 5.5. *Suppose that a tree* Γ *satisfies* (5.1)*, and, in addition,*

$$\ell(T) = \int_{I_T} \frac{ds}{g_T(s)} < \infty, \quad \textit{for any } T \in \mathscr{W}_0(\Gamma). \tag{5.27}$$

(i) *If* $\boldsymbol{\sigma}(V)\in l_{1/2}$, *then*

$$\|A_{\Gamma,V}\|^{1/2}_{1/2,\infty}=\sup_{\lambda>0}(\lambda^{1/2}n(\lambda,A_{\Gamma,V}))\leqslant C_{5.28}\sum_{T\in\mathscr{W}_0(\Gamma)}\sigma_T^{1/2}(V)$$

$$=C_{5.28}\|\boldsymbol{\sigma}(V)\|^{1/2}_{1/2},\tag{5.28}$$

and the asymptotic formula (4.3) *is valid.*

(ii) *For* $\frac12<p\leqslant 1$,

$$\|A_{\Gamma,V}\|^p_p=\sum_k\lambda_k^p(A_{\Gamma,V})\leqslant C_{5.29}(p)\sum_{T\in\mathscr{W}_0(\Gamma)}\sigma_T^p(V)$$

$$=C_{5.29}(p)\|\boldsymbol{\sigma}(V)\|^p_p.\tag{5.29}$$

(iii) *For* $\frac12<p<1$,

$$\|A_{\Gamma,V}\|^p_{p,\infty}=\sup_{\lambda>0}(\lambda^p n(\lambda,A_{\Gamma,V}))\leqslant C_{5.30}(p)\sup_{\lambda>0}(\lambda^p n(\lambda,\boldsymbol{\sigma}(V)))$$

$$=C_{5.30}(p)\|\boldsymbol{\sigma}(V)\|^p_{p,\infty}.\tag{5.30}$$

Proof. Fix a subtree $T\in\mathscr{W}_0(\Gamma)$ and realize $\mathscr{F}_T$ as the space $\mathscr{H}(I_T,g_T)$; see (5.16). The operator $A(\mathscr{F}_T,V)$ can be identified with $A(\mathscr{H}(I_T,g_T),V_T)$ where the weight function V_T on I_T is defined as

$$V_T(t)=\sum_{x\in T,\,|x|=t}V(x).$$

Under the assumption (5.27), Corollary 2.4 applies and we obtain

$$\sup_{\lambda>0}(\lambda^{1/2}n(\lambda;\mathscr{H}(I_T,g_T),V_T))\leqslant C_{2.11}(\tfrac12)\left(\ell(T)\int_{I_T}V_T(s)\,ds\right)^{1/2}$$

$$=C_{2.11}(\tfrac12)\sigma_T^{1/2}(V).$$

The inequalities (5.28)–(5.30) follow from here and Lemma 5.4. The asymptotic formula (4.3) is justified in exactly the same way as in Theorem 4.1.

The estimates given by Theorem 5.5 may turn out to be rather rough. First, the passage from the subspace $\mathscr{G}_z$ to the subspace $\mathscr{F}_z$ in (5.23) increases the corresponding function $n(\lambda)$. However, a coarsening of another origin is more important: the sequence $\boldsymbol{\sigma}(V)$ poorly reflects behaviour of V when V is 'strongly non-symmetric'. To show this, let us return to Example 4.3. It is clear that $\sigma_T(V)\neq 0$ only for subtrees $T\in\mathscr{W}_0(\Gamma)$ with $o_T\in J$. In what follows T_n, with $n\in\mathbb{N}$, stands for the single such subtree with $|o_{T_n}|=n$. We have $g_{T_n}(t)=2^{k-n}$ on $(k,k+1)$, with $k\geqslant n$, so all the integrals (5.27) are equal to 1. Thus for $n\in[2^{k-1},2^k)$,

$$\sigma_{T_n}(V)=\int_n^\infty V\,dt=(2^k-n)c_k+\sum_{j>k}2^{j-1}c_j.$$

We leave it to the reader to check that the quasinorms of this sequence in l_p and in $l_{p,\infty}$ are equivalent to those of the sequence $\boldsymbol{\eta}(V,\Xi_2)$ from Example 4.3, and hence the estimates (5.28)–(5.30) are coarse.

This shows that Theorem 4.1 can give better estimates than Theorem 5.5, provided that the partition Ξ is chosen in an appropriate way, co-ordinated with the behaviour of V.

However, under additional assumptions on Γ and V the usage of the decomposition (5.6) gives much more information than the results of § 4. This will be shown in the next three sections.

6. *Regular trees*

6.1. We call a tree Γ *regular* if all the vertices of the same generation k have equal branching numbers b_k, and all the edges of the same generation have equal lengths. More precisely, with any regular tree a monotone number sequence $\{t_k\}_{k=0}^{\infty}$ is associated, so that for any edge $e = \langle y, z\rangle$ there is an index k such that $|y| = t_k$, $|z| = t_{k+1}$ and $b(y) = b_k$; we say that e is an edge of generation k, and y is a vertex of generation k. We also say that a W_0-subtree T is of generation k if $|o_T| = t_k$; sometimes we write simply $T = T_k$. Condition (5.1) is clearly satisfied for any regular tree.

Note that in order to be consistent with previous notation we should have $t_0 = 0$. However, it is not always convenient, so we slightly modify our notation to allow $t_0 \neq 0$. We believe that such a modification is clear; one may view it as a shift of the initial sequence $\{t_k\}$ by t_0.

Many objects introduced earlier for a general tree are considerably simplified if Γ is regular. In particular, for the subspaces $\mathscr{H}^\circ(\Gamma)$ and $\mathbb{H}(\Gamma)$ a decomposition similar to (5.6) holds. This enables us to analyse the operators $A^\circ_{\Gamma,V}$ and $A^{\mathbb{H}}_{\Gamma,V}$, defined in (3.7); see Theorem 6.3 and also §§ 7–8.

Later, we consider only regular trees. Let us introduce some convenient notation; we present it for increasing $\{t_k\}$, though in general the opposite case is not excluded. Denote

$$L = L(\Gamma) = \sup_k t_k;$$

this agrees with the notation from § 5. Further, we denote

$$I_k = (t_k, L) \ \ (= I_{T_k}).$$

For any subtree T_k,

$$g_k(t) := g_{T_k}(t) = b_{k+1} \dots b_j \quad \text{on } (t_j, t_{j+1}), \quad \text{where } j \geqslant k. \tag{6.1}$$

The family of requirements (5.27) reduces to the single assumption

$$\sum_{k=0}^{\infty} \frac{|t_{k+1} - t_k|}{b_1 \dots b_k} < \infty. \tag{6.2}$$

The subspaces $\mathscr{F}_T$, $\mathscr{G}_z$ of $\mathscr{H}(\Gamma)$, which were defined in (5.2), (5.3) and (5.4), can be described in more detail when Γ is regular. Namely, denote by $\mathscr{P}_T$ and Π_z the orthogonal projections onto these subspaces. Then an explicit formula similar to (5.10) may be written for $\mathscr{P}_T$:

$$(\mathscr{P}_T u)(x) = \begin{cases} \dfrac{1}{g_T(|x|)} \displaystyle\sum_{y \in T: |y| = |x|} u(y) - u(o_T) & \text{if } x \in T, \\ 0 & \text{if } x \notin T. \end{cases} \tag{6.3}$$

Indeed, it is easy to see that the right-hand side of (6.3) is a continuous function whose derivative coincides with $\mathscr{P}'_T u'$.

For Π_z one may write an analogue of (5.13):

$$\Pi_z u = \sum_{j=1}^{b(z)} \mathscr{P}_{T_z^j} u - \mathscr{P}_{T_z} u \tag{6.4}$$

(this is true in general and not only for regular trees). What is specific for regular trees is the analogue of (5.8). Namely, a function $u \in \mathscr{F}_z$ belongs to $\mathscr{G}_z$ if and only if

$$\sum_{x > z:\,|x|=t} u(x) = 0. \tag{6.5}$$

Indeed, let z be of generation k. As it was done in § 5.2, identify u with the element $\{v_1, \dots, v_{b_k}\}$ of the space $(\mathscr{H}(I_T, g_k))^{b_k}$. Since in (5.17), $g_{T_z^j} = g_k$ for any $j = 1, \dots, b_k$, u belongs to $\mathscr{G}_z$ if and only if

$$\sum_{j=1}^{b_k} v'_j(t) = 0 \quad \text{a.e.} \tag{6.6}$$

Further,

$$\sum_{x > z:\,|x|=t} u(x) = \sum_{j=1}^{b_k} v_j(t) g_k(t) = g_k(t) \sum_{j=1}^{b_k} \int_{t_k}^{t} v'_j(\tau)\, d\tau$$

$$= g_k(t) \int_{t_k}^{t} \left(\sum_{j=1}^{b_k} v'_j(\tau) \right) d\tau,$$

and so (6.6) is equivalent to (6.5).

In addition to Theorem 5.1, the following analogue of (5.14) holds.

Lemma 6.1. *Under the assumptions of Theorem* 5.1, *let* Γ *also be regular. Then along with* (5.5) *one has*

$$\sum_{|x|=t} u(x)\overline{v(x)} = 0, \quad \textit{for } t \geqslant 0. \tag{6.7}$$

The proof of (6.7) completely imitates the proof of (5.14) in Lemma 5.2; one uses (6.5) instead of (5.8).

6.2. *Decomposition of the subspaces* $\mathscr{H}^\circ$ *and* $\mathbb{H}$

Since $u(o) = 0$, it follows immediately from (6.3) that if a function $u \in \mathscr{H}(\Gamma)$ is compactly supported, then its projection $\mathscr{P}_\Gamma u$ is also compactly supported. The same formula shows that this is no longer true for $\mathscr{P}_T$ with $T \neq \Gamma$. However, substituting (6.3) into (6.4), one sees that the constant terms $u(o_T)$ cancel. So, if $u \in \mathscr{H}(\Gamma)$ is compactly supported, then this is true for $\Pi_z u$.

Thus the subspaces $\mathscr{H}^\circ(\Gamma)$ and $\mathbb{H}(\Gamma)$ are invariant with respect to $\mathscr{P}_\Gamma$ and Π_z for any $z \in \mathscr{V}(\Gamma)$.

Introduce the notation

$$\mathscr{F}_T^\circ = \mathscr{F}_T \cap \mathscr{H}^\circ(\Gamma), \quad \mathscr{F}_T^{\mathbb{H}} = \mathscr{F}_T \cap \mathbb{H}(\Gamma), \quad \text{for } T \in \mathscr{W}(\Gamma),$$

and

$$\mathscr{G}_z^\circ = \mathscr{G}_z \cap \mathscr{H}^\circ(\Gamma), \quad \mathscr{G}_z^{\mathbb{H}} = \mathscr{G}_z \cap \mathbb{H}(\Gamma), \quad \text{for } z \in \mathscr{V}(\Gamma).$$

In view of Lemma 5.3, (6.2) is the criterion of non-triviality of all the subspaces $\mathscr{F}_T^{\mathbb{H}}$, with $T \in \mathscr{W}(\Gamma)$. If (6.2) is fulfilled, then $\dim \mathscr{F}_\Gamma^{\mathbb{H}} = 1$ and if $z \in \mathscr{V}(\Gamma)$ is of generation k, then $\dim \mathscr{G}_z^{\mathbb{H}} = b_k - 1$. Indeed, any harmonic function $u \in \mathscr{F}_z$ is a linear combination $u = \sum_{j=1}^{b_k} u_j$, where $u_j \in \mathscr{F}_{T_z^j}^{\mathbb{H}}$, and it belongs to $\mathscr{G}_z^{\mathbb{H}}$ if and only if (6.5) is satisfied.

The next proposition is an immediate consequence of Theorem 5.1.

Lemma 6.2. *Let Γ be a regular tree. Then there are orthogonal decompositions*

$$\mathscr{H}^\circ(\Gamma) = \mathscr{F}_\Gamma^\circ \oplus \sum_{z \in \mathscr{V}(\Gamma)}^{\oplus} \mathscr{G}_z^\circ$$

and

$$\mathbb{H}(\Gamma) = \mathscr{F}_\Gamma^{\mathbb{H}} \oplus \sum_{z \in \mathscr{V}(\Gamma)}^{\oplus} \mathscr{G}_z^{\mathbb{H}}. \tag{6.8}$$

If (6.2) *is fulfilled, then* (6.8) *is an orthogonal sum of finite-dimensional subspaces; namely,* $\dim \mathscr{F}_\Gamma^{\mathbb{H}} = 1$ *and if* $z \in \mathscr{V}(\Gamma)$ *is of generation* k, *then* $\dim \mathscr{G}_z^{\mathbb{H}} = b_k - 1$. *If* (6.2) *is violated, then* $\mathscr{H}^\circ(\Gamma) = \mathscr{H}(\Gamma)$ *and* $\mathbb{H}(\Gamma)$ *is trivial.*

6.3. *The operator* $A_{\Gamma,V}^{\mathbb{H}}$

The subspace $\mathbb{H}(\Gamma) \subset \mathscr{H}(\Gamma)$ is comparatively poor. Indeed, according to Lemma 5.3 its component in each $\mathscr{F}_T$, with $T \in \mathscr{W}_0(\Gamma)$, is not more than 1-dimensional. It follows from (5.18) that the single non-zero eigenvalue of the operator $A(\mathscr{F}_T^{\mathbb{H}}, V)$ is

$$\lambda^{\mathbb{H}}(T, V) = \left(\int_{I_T} \frac{ds}{g_T(s)} \right)^{-1} \int_T V(x) \left(\int_{|o_T|}^{|x|} \frac{d\tau}{g_T(\tau)} \right)^2 dx. \tag{6.9}$$

Replacing the inner integral in the second factor by its majorant $\int_{I_T} g_T^{-1}(s)\,ds$, we obtain the inequality

$$\lambda^{\mathbb{H}}(T, V) \leqslant \sigma_T(V), \quad \text{for } T \in \mathscr{W}_0(\Gamma).$$

Now we are in a position to prove the following.

Theorem 6.3. *Let Γ be a regular tree. Under the assumptions of Theorem* 5.5(i) *one has*

$$A_{\Gamma,V}^{\mathbb{H}} \in \mathscr{C}_{1/2}.$$

In particular, $n(\lambda, A_{\Gamma,V}^{\mathbb{H}}) = o(n(\lambda, A_{\Gamma,V}))$.

Proof. Using Lemma 6.2 (decomposition (6.8)) and (2.6) with $p = \frac{1}{2}$, we find that

$$\|A_{\Gamma,V}^{\mathbb{H}}\|_{1/2}^{1/2} \leqslant \|A(\mathscr{F}_\Gamma^{\mathbb{H}}, V)\|_{1/2}^{1/2} + \sum_{z \in \mathscr{V}(\Gamma)} \|A(\mathscr{G}_z^{\mathbb{H}}, V)\|_{1/2}^{1/2}.$$

In its turn, by the variational principle and again by (2.6),

$$\|A(\mathscr{G}_z^{\mathbb{H}}, V)\|_{1/2}^{1/2} \leqslant \|A(\mathscr{F}_z^{\mathbb{H}}, V)\|_{1/2}^{1/2} = \sum_{j=1}^{b(z)} (\lambda^{\mathbb{H}}(T_z^j, V))^{1/2} \leqslant \sum_{j=1}^{b(z)} (\sigma_{T_z^j}(V))^{1/2}.$$

The result follows immediately.

Theorem 6.3 implies in particular that under the assumptions of Theorem 5.5(i) the operators $A_{\Gamma,V}$ and $A^{\circ}_{\Gamma,V}$ have the same spectral asymptotics (4.3).

7. *Symmetric weights*

We call a function V on a tree Γ *symmetric* if

$$x, y \in \Gamma,\ |x| = |y| \quad \Longrightarrow \quad V(x) = V(y),$$

and write unambiguously $V = V(t)$.

7.1. *Symmetric weights on regular trees*

For regular trees and symmetric weights the subspaces $\mathscr{F}_\Gamma$ and $\mathscr{G}_z$, appearing in the decomposition (5.6), are mutually orthogonal in the norm generated by the quadratic form $Q_{\Gamma,V}$; this follows from (6.7). Therefore, $A_{\Gamma,V}$ decomposes into the orthogonal sum

$$A_{\Gamma,V} = A(\mathscr{F}_\Gamma, V) \oplus \sum_{z \in \mathscr{V}(\Gamma)}^{\oplus} A(\mathscr{G}_z, V). \tag{7.1}$$

Further, each operator $A(\mathscr{F}_{T_k}, V)$ can be identified with

$$A_k = A(\mathscr{H}(I_k, g_k), g_k V).$$

In a similar way, we denote

$$A^{\circ}_k = A(\mathscr{H}^{\circ}(I_k, g_k), g_k V), \quad A^{\mathbb{H}}_k = A(\mathbb{H}(I_k, g_k), g_k V).$$

If (6.2) is satisfied, then the latter is an operator of rank 1; according to (6.9), its single non-zero eigenvalue is

$$\lambda_k(V) = \left(\int_{I_k} \frac{dt}{g_k(t)} \right)^{-1} \int_{I_k} g_k(t) V(t) \left(\int_{t_k}^{t} \frac{ds}{g_k(s)} \right)^2 dt. \tag{7.2}$$

Note that if (6.2) is violated, then $A^{\circ}_k = A_k$ and $A^{\mathbb{H}}_k = 0$ for all $k \geqslant 0$; hence $A^{\circ}_{\Gamma,V} = A_{\Gamma,V}$ and $A^{\mathbb{H}}_{\Gamma,V} = 0$ also.

The following simple result on the structure of $A(\mathscr{G}_z, V)$ is the key observation for the analysis which follows.

LEMMA 7.1. *Let Γ be a regular tree, and let V be a symmetric function on Γ. Then for any $z \in \mathscr{V}(\Gamma)$, with $z \neq o$ and $|z| = t_k$, the operator $A(\mathscr{G}_z, V)$ is unitarily equivalent to the orthogonal sum of $b_k - 1$ copies of the operator A_k. In particular,*

$$n(\lambda, A(\mathscr{G}_z, V)) = (b_k - 1) n(\lambda, A_k), \quad \textit{where } |z| = t_k,\ k > 0.$$

If (6.2) *is satisfied, then similar statements are valid for the operators $A(\mathscr{G}^{\circ}_z, V)$ and $A(\mathscr{G}^{\mathbb{H}}_z, V)$.*

Proof. Indeed, $A(\mathscr{F}_z, V)$ is the orthogonal sum of b_k copies of the operator A_k. The passage to the operator $A(\mathscr{G}_z, V)$ corresponds to the withdrawal of one of the copies; this is due to (5.5) and (6.7), which mean orthogonality at each 'level' $|x| = t$.

THEOREM 7.2. *Let the assumptions of Lemma* 7.1 *be satisfied.*

(i) *The operator* $A_{\Gamma,V}$ *is bounded if and only if all the operators* A_k, *for* $k = 0, 1, \dots,$ *are bounded, and*

$$\|A_{\Gamma,V}\| = \sup_k \|A_k\|. \tag{7.3}$$

The operator $A_{\Gamma,V}$ *is compact if and only if all the* A_k *are compact and for any* $\lambda > 0$ *the series* $\sum_k b_1 \dots b_{k-1}(b_k - 1) n(\lambda, A_k)$ *converges. Moreover,*

$$n(\lambda, A_{\Gamma,V}) = n(\lambda, A_0) + \sum_{k=1}^{\infty} b_1 \dots b_{k-1}(b_k - 1) n(\lambda, A_k). \tag{7.4}$$

(ii) *Similar statements are valid for the operator* $A^{o}_{\Gamma,V}$.

(iii) *The operator* $A^{\mathbb{H}}_{\Gamma,V}$ *is compact if and only if* $\lambda_k(V) \to 0$, *where* $\lambda_k(V)$ *are given by the equality* (7.2). *Under this assumption, the non-zero spectrum of* $A^{\mathbb{H}}_{\Gamma,V}$ *consists of these eigenvalues; the multiplicity of* $\lambda_0(V)$ *is* 1, *and for* $k \geqslant 1$ *the multiplicity of* $\lambda_k(V)$ *is* $b_1 \dots b_{k-1}(b_k - 1)$.

Proof. Under the assumptions of Lemma 7.1 the orthogonal decomposition (7.1) is valid. To prove (i), it is sufficient to realize $A(\mathscr{G}_z, V)$ according to the result of Lemma 7.1, and the operator $A(\mathscr{F}_\Gamma, V)$ as A_0. The proof of (ii) and (iii) is the same.

REMARK. The boundedness of A_k is equivalent to the weighted Hardy inequality

$$\int_{I_k} g_k V |v|^2 \, dt \leqslant C \int_{I_k} g_k |v'|^2 \, dt.$$

Such inequalities are well studied, and so (7.3) enables one to obtain a complete description of symmetric Hardy weights on regular trees.

COROLLARY 7.3. *Let* Γ *be a regular tree satisfying* (6.2), *and let* V *be symmetric. Then the estimates* (5.29) *and* (5.30) *are valid for all* $p \in (\frac{1}{2}, \infty)$.

Proof. The expansion (7.4) yields the corresponding expansion of $\|A_{\Gamma,V}\|_p^p$ into a series in $\|A_k\|_p^p$. For any $p \in (\frac{1}{2}, \infty)$, $\|A_k\|_p \leqslant \|A_k\|_{1/2,\infty}$. By Corollary 2.4, $\|A_k\|_{1/2,\infty} \leqslant C^2_{2.11} \sigma_k$, and we get (5.29) for all $p \in (\frac{1}{2}, \infty)$. Similar argument leads to (5.30).

7.2. *Trees of type* (b, q)

A regular tree Γ is called *b-regular* if $b_k = b$ for any $k \in \mathbb{N}$. We say that a b-regular tree Γ is *of type* (b, q) if $t_k = q^k$, for $k \geqslant 0$.

Let Γ be a tree of type (b, q) with $q > 1$ and $b > q$; the latter assumption is equivalent to (6.2). Denote $\beta = \log_q b$ and $\gamma = q/b$, so that $\beta > 1$ and $\gamma < 1$. Also let the weight V on Γ be symmetric. Then, in addition to the results from the previous subsection, we can compare our two approaches which were described in § 4 and § 5 respectively.

Let us analyse more carefully the two sequences $\boldsymbol{\sigma} = \boldsymbol{\sigma}(V)$ (see (5.26)) and $\widetilde{\boldsymbol{\eta}} = \widetilde{\boldsymbol{\eta}}(V, \Xi)$ (see (4.13)) appearing in these approaches.

The sequence $\widetilde{\boldsymbol{\eta}}$ consists of the numbers

$$\widetilde{\eta}_k = q^{k+1} \int_{q^k}^{q^{k+1}} V(t) \, dt, \quad \text{for } k = 0, 1, \dots,$$

each of them repeated b^k times, and $\boldsymbol{\sigma}$ consists of the numbers

$$\sigma_k = \int_{q^k}^{\infty} \frac{dt}{g_k(t)} \int_{q^k}^{\infty} g_k(t) V(t)\, dt, \quad \text{for } k = 0, 1, \dots,$$

each of them also repeated b^k times.

Now, taking (6.1) into account, rewrite σ_k as

$$\begin{aligned}\sigma_k &= \left(\sum_{j=k}^{\infty} q^j (q-1) b^{k-j} \right) \left(\sum_{j=k}^{\infty} b^{j-k} \int_{q^j}^{q^{j+1}} V(t)\, dt \right) \\ &= \left((1 - q^{-1}) b^k \sum_{j=k}^{\infty} \gamma^j \right) \left(b^{-k} \sum_{j=k}^{\infty} \gamma^{-j} \widetilde{\eta}_j \right) \\ &= K \sum_{j=k}^{\infty} \gamma^{k-j} \widetilde{\eta}_j, \end{aligned} \tag{7.5}$$

where $K = (q-1)/q(1-\gamma)$. We see immediately that $\sigma_k \geqslant K \widetilde{\eta}_k$, so for any $p > 0$,

$$K \| \widetilde{\boldsymbol{\eta}} \|_p \leqslant \| \boldsymbol{\sigma} \|_p, \quad K \| \widetilde{\boldsymbol{\eta}} \|_{p,\infty} \leqslant \| \boldsymbol{\sigma} \|_{p,\infty}.$$

Pass on to the inverse estimates. Set $\alpha = \gamma b^{1/p}$ and note that

$$\alpha > 1 \quad \text{for } 0 < p < \beta(\beta - 1)^{-1}. \tag{7.6}$$

Consider first the case $p \leqslant 1$. Using (7.5), we get

$$\begin{aligned}\| \boldsymbol{\sigma} \|_p^p = \sum_{k=0}^{\infty} b^k \sigma_k^p &\leqslant K^p \sum_{k=0}^{\infty} b^k \sum_{j=k}^{\infty} \gamma^{(k-j)p} \widetilde{\eta}_j^p \\ &= K^p \sum_{j=0}^{\infty} \widetilde{\eta}_j^p b^j \sum_{k=0}^{j} \alpha^{-p(j-k)} \\ &\leqslant c(p) \| \widetilde{\boldsymbol{\eta}} \|_p^p, \quad \text{for } 0 < p < 1. \end{aligned} \tag{7.7}$$

Here $c(p) = K^r (1 - \alpha^p)^{-1}$.

Now let $p \geqslant 1$. Denote $\boldsymbol{x} = \{x_k\}_{k=0}^{\infty}$, with $x_k = b^{k/p} \widetilde{\eta}_k$; then $\| \widetilde{\boldsymbol{\eta}} \|_p = \| \boldsymbol{x} \|_{l_p}$. Using (7.5) again, write

$$\begin{aligned}\| \boldsymbol{\sigma} \|_p^p = \sum_{k=0}^{\infty} b^k \sigma_k^p &= K^p \sum_{k=0}^{\infty} b^k \left(\sum_{j=k}^{\infty} \gamma^{k-j} b^{-j/p} x_j \right)^p \\ &= K^p \sum_{k=0}^{\infty} \left(\sum_{j=k}^{\infty} \alpha^{k-j} x_j \right)^p. \end{aligned}$$

Consider the mapping

$$\Pi_\alpha \colon \{x_k\}_{k=0}^{\infty} \mapsto \left\{ \sum_{j=k}^{\infty} \alpha^{k-j} x_j \right\}_{k=0}^{\infty}$$

and show that, for $\alpha > 1$,

$$\| \Pi_\alpha \|_{l_r \to l_r} \leqslant \alpha / (\alpha - 1), \quad \text{for } 1 \leqslant r \leqslant \infty. \tag{7.8}$$

Indeed, for $r = 1$ and $r = \infty$ this can be seen immediately, and the rest is just interpolation; it is also possible to use the result of [**10**, Problem 275].

Taking into account (7.6) and combining (7.8) and (7.7), we find that for some constant $c(p) > 0$,

$$\|\boldsymbol{\sigma}\|_p \leqslant c(p)\|\widetilde{\boldsymbol{\eta}}\|_p, \quad \text{for } 0 < p < \beta(\beta-1)^{-1}. \tag{7.9}$$

Interpolating (7.9) between $p = \frac{1}{2}$ and $p = \beta(\beta-1)^{-1} - \varepsilon$ for arbitrarily small ε, we also get

$$\|\boldsymbol{\sigma}\|_{p,\infty} \leqslant c'(p)\|\widetilde{\boldsymbol{\eta}}\|_{p,\infty}, \quad \text{for } 0 < p < \beta(\beta-1)^{-1}$$

for some $c'(p) > 0$.

So, we have proved the following statement.

LEMMA 7.4. *Let Γ be a (b, q)-tree with $1 < q < b$, and let V on Γ be symmetric. Then there are constants $C(p)$, $C'(p) > 0$ such that*

$$C(p)\|\boldsymbol{\sigma}\|_p \leqslant \|\widetilde{\boldsymbol{\eta}}\|_p \leqslant K^{-1}\|\boldsymbol{\sigma}\|_p, \quad \text{for } 0 < p < \beta(\beta-1)^{-1},$$

$$C'(p)\|\boldsymbol{\sigma}\|_{p,\infty} \leqslant \|\widetilde{\boldsymbol{\eta}}\|_{p,\infty} \leqslant K^{-1}\|\boldsymbol{\sigma}\|_{p,\infty}, \quad \text{for } 0 < p < \beta(\beta-1)^{-1}.$$

We see that in the indicated region of p the usage of sequences $\widetilde{\boldsymbol{\eta}}$ and $\boldsymbol{\sigma}$ is equivalent. Thus, $\|A_{\Gamma,V}\|_p$, $\|A_{\Gamma,V}\|_{p,\infty}$ may be estimated from above using any of these two sequences, and the results of Theorem 4.1 are actually valid in a wider range of p. Similarly, the equivalence relations in (4.14) are valid in this wider region, $\frac{1}{2} < p < \beta(\beta-1)^{-1}$. Example 4.5 shows that there is no chance of expecting such results for large values of p.

8. *Examples*

We shall analyse three examples. In all of them the tree Γ is (b, q)-regular and the function V is symmetric, and we use the notation and formulas from §§ 6–7. In our examples the self-similarity arguments apply and we obtain not only estimates for $n(\lambda)$, but also its asymptotics, including the non-Weylian case. We leave it to the reader to verify that the results of our analysis agree with the general theorems of §§ 3–7.

In each example we study three operators: $A_{\Gamma,V}$, $A^{\circ}_{\Gamma,V}$ and $A^{\mathbb{H}}_{\Gamma,V}$.

EXAMPLE 8.1. Let $q > 1$. Take $t_k = q^k$, for $k \geqslant 0$ (so $t_0 = 1$), and $V(t) = t^{-\alpha}$. Denote $\beta = \log_q b$ (>0). The condition (6.2) corresponds to $\beta > 1$. We have $g_k(t) = b^{r-k}$ for $t_r < t < t_{r+1}$, $r \geqslant k$; thus

$$g_k(t) = g_0(q^{-k}t). \tag{8.1}$$

Clearly

$$b^{-1}t^{\beta} \leqslant g_0(t) \leqslant t^{\beta}. \tag{8.2}$$

We start with the study of the operator $A^{\circ}_{\Gamma,V}$. Consider the corresponding operator A°_0; its Rayleigh quotient is

$$\mathscr{R}^{\circ}_0[v] = \frac{\int_1^{\infty} g_0(t)t^{-\alpha}|v|^2\,dt}{\int_1^{\infty} g_0(t)|v'|^2\,dt}, \quad \text{for } v \in \mathscr{H}^{\circ}(I_0, g_0). \tag{8.3}$$

It is more convenient to deal with the operator $\widetilde{A}_0^\circ$ whose Rayleigh quotient is

$$\widetilde{\mathscr{R}}_0^\circ[v] = \frac{\int_1^\infty t^{\beta-\alpha}|v|^2\,dt}{\int_1^\infty t^\beta |v'|^2\,dt}. \tag{8.4}$$

It follows from (8.2) that the spectral estimates of the operators A_0° and $\widetilde{A}_0^\circ$ are equivalent. By the Hardy inequality, $\widetilde{A}_0^\circ$ is bounded if $\beta \neq 1$ and $\alpha \geqslant 2$, or $\beta = 1$ and $\alpha > 2$; it is compact for all $\beta > 0$, $\alpha > 2$. After the substitution $s = s(t) = \int_1^t \tau^{-\beta}\,d\tau$, the quotient (8.4) reduces to the quotient which corresponds to the Dirichlet boundary value problem for the equation $-\lambda v'' = W(s)v$ on the interval $\widetilde{I}_0 = (0, \int_1^\infty \tau^{-\beta}\,d\tau)$, where $W(s) = t^{2\beta-\alpha}(s)$; here $t(s)$ is the function inverse to $s(t)$. So,

$$\widetilde{I}_0 = \begin{cases} \mathbb{R}_+ & \text{if } \beta \leqslant 1, \\ (0, (\beta-1)^{-1}) & \text{if } \beta > 1, \end{cases}$$

and

$$W(s) = \begin{cases} (1 + s(1-\beta))^{-(\alpha-2\beta)/(1-\beta)} & \text{if } \beta \neq 1, \\ e^{-s(\alpha-2)} & \text{if } \beta = 1. \end{cases}$$

If $\beta > 0$ and $\alpha > 2$, then W is monotone and

$$\int_{\widetilde{I}_0} \sqrt{W(s)}\,ds = \int_1^\infty t^{-\alpha/2}\,dt < \infty.$$

This guarantees the standard Weyl type eigenvalue estimate and asymptotics for the operator $\widetilde{A}_0^\circ$ and, consequently, for A_0° (see [**3**, Corollary 6.2]):

$$n_0(\lambda) := n_0(\lambda, A_0^\circ) \leqslant M\lambda^{-1/2}, \quad \text{where } M = M(q, b, \alpha), \tag{8.5}$$

and

$$\lim_{\lambda\to 0} \lambda^{1/2} n_0(\lambda) = c_\alpha. \tag{8.6}$$

Further, it follows from (8.1) that the operator A_k° is unitarily equivalent to $q^{k(2-\alpha)}A_0^\circ$. Thus, when A_0° is bounded (and thus $\alpha \geqslant 2$), the operators A_k° are uniformly bounded and by Theorem 7.2, $A_{\Gamma,V}^\circ$ is bounded too. For $\beta > 0$ and $\alpha > 2$, A_k° is compact and

$$n(\lambda, A_k^\circ) = n(\lambda q^{k(\alpha-2)}, A_0^\circ), \quad \text{for } k = 1, 2, \ldots. \tag{8.7}$$

According to (7.4) and (8.7),

$$n(\lambda, A_{\Gamma,V}^\circ) = n_0(\lambda) + (1 - b^{-1}) \sum_{k=1}^\infty q^{k\beta} n_0(\lambda q^{k(\alpha-2)}). \tag{8.8}$$

If $\alpha > 2(\beta+1)$, then replacing each term in the last series by its majorant $q^{k\beta}M(\lambda q^{k(\alpha-2)})^{-1/2}$, we get a convergent series. In this case we obtain $A_{\Gamma,V}^\circ \in \mathscr{C}_{1/2,\infty}$, and the asymptotic formula (4.3) is also valid.

If $\alpha \leqslant 2(\beta+1)$, we derive from (8.8) and (8.5),

$$n(\lambda, A_{\Gamma,V}^\circ) \leqslant M\lambda^{-1/2} \sum_{k:\, \lambda q^{k(\alpha-2)} \leqslant M^2} q^{k(\beta-(\alpha/2)+1)}. \tag{8.9}$$

If $\alpha < 2(\beta+1)$, then the sum in (8.9) does not exceed its maximal (last) term multiplied by an unimportant constant. This gives

$$n(\lambda, \mathring{A}_{\Gamma,V}) \leqslant C\lambda^{-1/2}(M^2\lambda^{-1})^{(\beta-(\alpha/2)+1)/(\alpha-2)} = C'\lambda^{-\delta}, \quad \delta = \frac{\beta}{\alpha-2} > \frac{1}{2}.$$

Actually, much more can be said on the behaviour of $n(\lambda, \mathring{A}_{\Gamma,V})$ in this case. Consider the functions $\varphi(\lambda) = \lambda^{\delta} n_0(\lambda)$ and $\Phi(\lambda) = \lambda^{\delta} n(\lambda, \mathring{A}_{\Gamma,V})$. Multiplying both sides of (8.8) by λ^{δ}, we obtain

$$\Phi(\lambda) = \varphi(\lambda) + (1 - b^{-1}) \sum_{k=1}^{\infty} \varphi(\lambda q^{k(\alpha-2)}), \quad \text{for } \lambda > 0,$$

which implies that

$$\Phi(\lambda) - \Phi(\lambda q^{\alpha-2}) = \varphi(\lambda) - b^{-1}\varphi(\lambda q^{\alpha-2}). \tag{8.10}$$

Here set $\lambda = e^{-\mu}$ and denote $F(\mu) = \Phi(\lambda)$, $f(\mu) = \varphi(\lambda)$ and $\gamma = (\alpha - 2)\ln q$. Then (8.10) reduces to the Renewal Equation,

$$F(\mu) - F(\mu - \gamma) = f_0(\mu) := f(\mu) - b^{-1} f(\mu - \gamma), \quad \text{for } -\infty < \mu < \infty.$$

The function f_0 is zero at $-\infty$ and, in view of (8.5), decays exponentially as $\mu \to \infty$. So, the Renewal Theorem applies; see [**12**] where the result is given in a form convenient for our purposes. We find that $F(\mu)$ behaves asymptotically as a γ-periodic function, say $\psi(\mu)$, which is bounded and bounded away from zero. So for $\delta > \frac{1}{2}$ we have

$$n(\lambda, \mathring{A}_{\Gamma,V}) = \lambda^{-\delta}\psi(\ln(\lambda^{-1})) + o(\lambda^{-\delta}), \quad \text{as } \lambda \to 0. \tag{8.11}$$

For $\alpha = 2(\beta + 1)$ we derive from (8.9),

$$n(\lambda, \mathring{A}_{\Gamma,V}) \leqslant M\lambda^{-1/2}\#\{k\colon q^{k(\alpha-2)}\lambda \leqslant M^2\} \leqslant C\lambda^{-1/2}\ln(c\lambda^{-1}).$$

It is possible to show that the product $\lambda^{1/2}(\ln(1/\lambda))^{-1} n(\lambda, \mathring{A}_{\Gamma,V})$ tends to a finite limit as $\lambda \to 0$. There is no need for the Renewal Equation in this case.

The results for the operator $A_{\Gamma,V}$ are basically the same. For $\beta \leqslant 1$ the condition (6.2) is violated and we simply have $A_{\Gamma,V} = \mathring{A}_{\Gamma,V}$; see § 7.1. For $\beta > 1$ the only difference is that the operator A_0 is well defined and satisfies the relations (8.5) and (8.6) in a restricted region of the parameters α and β, namely for $\alpha > \beta + 1$; otherwise, the Rayleigh quotient (8.3) is unbounded on the space $\mathscr{H}(I_0, g_0)$. As a consequence, we find that $n(\lambda, A_{\Gamma,V})$ has the Weyl type behaviour in the same region $\alpha > 2(\beta + 1)$ as above, and for $\beta > 1$ and $\beta + 1 < \alpha < 2(\beta + 1)$ its asymptotic behaviour is described by the formula (8.11); however, the functions ψ for the operators $A_{\Gamma,V}$ and $\mathring{A}_{\Gamma,V}$ are different because they are expressed in terms of the functions $n(\lambda, A_0)$ and $n(\lambda, \mathring{A}_0)$ rather than of their asymptotics.

We pass on to the operator $A^{\mathbb{H}}_{\Gamma,V}$. Its spectrum can be not only estimated but explicitly calculated. Indeed, let λ_0 be the single eigenvalue of the 1-dimensional operator $A^{\mathbb{H}}_0$; it can be found by the formula (7.2). The non-zero spectrum of $A^{\mathbb{H}}_{\Gamma,V}$ consists of the eigenvalues $\lambda_k = q^{k(2-\alpha)}\lambda_0$, with $k = 0, 1, \ldots$. The multiplicity of λ_0 is 1 and the multiplicity of λ_k with $k \geqslant 1$ is $(1 - b^{-1})b^k$ (see Theorem 7.2(iii)). An elementary calculation shows that for $\alpha > 2(\beta + 1)$ we have $A^{\mathbb{H}}_{\Gamma,V} \in \mathscr{C}_{1/2}$; this agrees with the result of Theorem 6.3. For $\alpha \leqslant 2(\beta + 1)$ we get $n(\lambda, A^{\mathbb{H}}_{\Gamma,V}) = O(\lambda^{-\delta})$. The asymptotic behaviour of $n(\lambda, A^{\mathbb{H}}_{\Gamma,V})$ can be described by the formula (8.11) with a function ψ having jumps at the points $-\ln\lambda_k$. These jumps appear because of the high multiplicity of the eigenvalues.

So for $\alpha = 2(\beta+1)$ (that is, $\delta = \frac{1}{2}$) we still have $n(\lambda, A^{\mathbb{H}}_{\Gamma,V}) = o(n(\lambda, A_{\Gamma,V}))$. For $\alpha < 2(\beta+1)$ these functions are of equal order at $\lambda = 0$, which means that the contribution of the 'poor' subspace $\mathbb{H}(\Gamma)$ into the spectrum of $A_{\Gamma,V}$ in this case is 'rich enough'.

Other examples are analysed in a similar way and we only outline the calculations.

EXAMPLE 8.2. Here we take $q < 1$ and $t_k = q^k$, for $k = 0, 1, \dots,$ so again $t_0 = 1$. However, this time the tree grows downward. Denote $\beta = -\log_q b$, so

$$q^{-\beta} = b, \quad \beta > 0.$$

The condition (6.2) is always satisfied. We have $g_k(t) = b^{r-k}$ for $t_{r+1} < t < t_r$ and $r \geqslant k$, so again $g_k(t) = g_0(q^{-k}t)$.

As in Example 8.1, we consider $V(t) = t^{-\alpha}$. The operator A°_0 is compact if and only if $-\infty < \alpha < 2$, and its eigenvalue behaviour is described by the relations (8.5) and (8.6). Writing down the Rayleigh quotients for A°_k, we come to the same equality (8.7). An analogue of (8.8) holds:

$$n(\lambda, A^\circ_{\Gamma,V}) = n_0(\lambda) + (1 - b^{-1}) \sum_{k=1}^{\infty} q^{-k\beta} n_0(\lambda q^{k(\alpha-2)}).$$

As a result, we find that if $\alpha < 2(1-\beta)$, then $n(\lambda, A^\circ_{\Gamma,V}) = O(\lambda^{-1/2})$ and the asymptotics formula (4.3) holds. For $\alpha > 2(1-\beta)$ one has $n(\lambda, A^\circ_{\Gamma,V}) = O(\lambda^{-\delta})$ with $\delta = \beta(2-\alpha)^{-1}$ and the asymptotic formula (8.11) with this value of δ is satisfied. For $\alpha = 2(1-\beta)$ the function $n(\lambda, A^\circ_{\Gamma,V})$ has the asymptotic behaviour of the order $O(\lambda^{-1/2}\ln(1/\lambda))$.

The same is true for the operator $A_{\Gamma,V}$, but in a restricted region $\alpha < 1-\beta$. Namely, the corresponding function $n(\lambda)$ behaves in a standard way for $\alpha < 2(1-\beta)$; its behaviour is described by the formula (8.11) (with $\delta = \beta(2-\alpha)^{-1}$) for $\alpha > 2(1-\beta)$; in the borderline case $\alpha = 2(1-\beta)$ one has $n(\lambda) = O(\lambda^{-1/2}\ln(1/\lambda))$.

The results for the operator $A^{\mathbb{H}}_{\Gamma,V}$ are parallel to the ones in Example 8.1 and we do not write them down.

This example includes, in particular, the case $\alpha = 0$ which corresponds to the usual (non-weighted) Laplacian on the tree. We see that for this case the function $n(\lambda, A_{\Gamma,1})$ always has the standard Weyl type asymptotics. For the operator $A^\circ_{\Gamma,1}$, all the three types of the asymptotic behaviour are possible.

EXAMPLE 8.3. Take $t_k = k$, for $k = 0, 1, \dots,$ so all the edges of Γ are of length 1. We have $g_k(t) = b^{r-k}$ on $(r, r+1)$ with $r \geqslant k$; thus $g_k(t) = g_0(t-k)$. We take $V(t) = e^{-\alpha t}$, with $\alpha > 0$. The Rayleigh quotient for A°_0 is

$$\mathscr{R}^o_0[v] = \frac{\int_0^\infty g_0(t) e^{-\alpha t} |v|^2\, dt}{\int_0^\infty g_0(t) |v'|^2\, dt}, \quad \text{for } v \in \mathscr{H}^0(\mathbb{R}_+, g_0).$$

Denote $\beta = \ln b$; then $e^{\beta(t-1)} \leqslant g_0(t) \leqslant e^{\beta t}$. For any $\alpha, \beta > 0$, the operator A°_0 is bounded and has standard eigenvalue behaviour.

Further, each operator A_k° is unitarily equivalent to $e^{-k\alpha}A_0^\circ$. The formula (7.4) turns into

$$n(\lambda, A_{\Gamma,V}^\circ) = n_0(\lambda) + (1 - b^{-1}) \sum_{k=1}^{\infty} e^{k\beta} n_0(\lambda e^{k\alpha}).$$

The further analysis required follows the same line as in Examples 8.1 and 8.2. If $\alpha > 2\beta$, then the behaviour of $n(\lambda, A_{\Gamma,V}^\circ)$ is standard. For $\alpha < 2\beta$ we find that $n(\lambda, A_{\Gamma,V}^\circ) = O(\lambda^{-\delta})$ with $\delta = \beta/\alpha$; the asymptotics of the type (8.11) with this value of δ is valid, and the period of ψ is α. For $\alpha = 2\beta$, $n(\lambda, A_{\Gamma,V}^\circ)$ behaves asymptotically as $c\lambda^{-1/2}\ln 1/\lambda$.

For the operator $A_{\Gamma,V}$ the results are similar, but, as usual, are valid for the narrower domain $\alpha, \beta > 0$ with $\alpha > \beta$. The results for $A_{\Gamma,V}^{\mathbb{H}}$ are of the same type as in Example 8.1.

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