Can One Hear the Shape of a Network?

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1 INTRODUCTION

Already in 1892, in a book about spectrocospy, A. Shuster raised the question "... how to find a shape of a bell by means of the sounds which it is capable of sending out." [15] The inverse spectral problem hidden behind the physical context was first formulated in a mathematical setting by H. Weyl indirectly in 1911 and by S. Bochner in 1950, cf. [15]. Finally, in 1966, M. Kac published the famous paper [16] intitled "Can one hear the shape of a drum?" After a separation ansatz of the wave equation, the problem reads in mathematical terms as follows. Suppose that two bounded domains Ω_1 and Ω_2 in $\mathbb{R}^n$ are *isospectral* i.e. the spectra of their Laplacian under the homogeneous Dirichlet boundary condition (D) or under the Neumann boundary condition (N) coincide counting multiplicities. Does this imply that Ω_1 and Ω_2 are isometric, i.e. do they coincide up to rotations, reflections or translations?

$$\sigma\left(\Delta_{\Omega_1}^{D/N}\right) \overset{\mathrm{m}}{=} \sigma\left(\Delta_{\Omega_2}^{D/N}\right) \quad \overset{?}{\Longrightarrow} \quad \Omega_1 \cong \Omega_2$$

Evidently, for $n = 1$ the answer is affirmative, as well as the inverse implication is true. In fact, a negative answer to the question has been given earlier on compact riemannian manifolds. In 1964 J. Milnor

[19] already gave examples of isospectral nonisometric 16–dimensional tori, while in 1971 M. Berger, P. Gauduchon and E. Mazet [9] showed that two–dimensional isospectral tori are isometric. Finally, for any $n \geq 2$, M.–F. Vignéras [26] (1980) constructed n–dimensional compact isospectral nonisometric manifolds. For euclidian domains, H. Urakawa [25] (1982) gave a counterexample in $\mathbb{R}^4$. But for plane domains the problem remained unsolved until 1992, when C. Gordon, D. Webb and S. Wolpert [13], [14] and P. Bérard [8] showed that one cannot determine the shape of a domain by the spectrum of its Laplacian, cf. also [11]. A distinctive feature of their counterexamples is that the domains involved are all polygonal. As it stands, the problem seems not yet to be solved for smooth domains.

For ramified spaces of dimension greater than 1, as treated e.g. in [2], [7], [18], [22], [23], the inverse spectral problem is settled by the negative answer for higher dimensional domains. But, it still remains to solve the problem on ramified networks with one–dimensional branches. As basic multistructures, these networks enjoyed an increasing interest during the last twenty years, see e.g. [1], [3]–[6], [17], [18], [21], [22] and [24]. In the present paper, we show that, in contrast to the one-dimensional domain case, one cannot recover the shape of a network from the spectrum of its Laplacian under the continuity condition at ramification nodes and the Kirchhoff condition at all vertices. Thus, in that regard, networks behave like higher dimensional objects. Moreover, we shall discuss the eigenvalue asymptotics as well as the distinction of network immanent eigenvalues from those stemming from single branches.

2 NETWORKS AND VERTEX TRANSITION

All networks in this paper are supposed to be finite $\mathcal{C}^2$-networks in the sense of [3] Chapter 2. By definition, a finite $\mathcal{C}^2$-network G is the union of the edges k_j of a topological graph Γ in $\mathbb{R}^m$ with finite sets of vertices $V = \{v_i | 1 \leq i \leq n\}$ and edges $K = \{k_j | 1 \leq j \leq N\}$. Moreover, the arc length parametrizations π_j are supposed to belong to $\mathcal{C}^2([0, l_j], \mathbb{R}^m)$. The arc length parameter of an edge k_j is denoted by x_j. The topological graph Γ belonging to G is assumed to be simple and connected, i.e. $\Gamma = (V, K)$ consists in a collection of the supports of N Jordan curves k_j with the following properties: Each k_j has its endpoints in the set V, any two vertices in V can be connected by a path with arcs in K, and any two edges $k_j \neq k_h$ satisfy $k_j \cap k_h \subset V$ and $|k_j \cap k_h| \leq 1$. Endowed with the induced topology G is a connected and compact space in $\mathbb{R}^m$. The valency of each vertex is denoted by $\gamma_i = \gamma(v_i) = |\{k \in K | v_i \in k\}|$. We distinguish the ramification nodes $V_r = \{v_i \in V | \gamma_i > 1\}$ from the boundary vertices $V_b = \{v_i \in V | \gamma_i = 1\}$. The orientation of Γ and G is given by the incidence matrix $D = (d_{ij})_{n \times N}$ with

$$d_{ij} = \begin{cases} 1 & \text{if } \pi_j(l_j) = v_i, \\ -1 & \text{if } \pi_j(0) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

The adjacency matrix $\mathfrak{A}(\Gamma) = (e_{ih})_{n \times n}$ of the graph Γ is defined by

$$e_{ih} = \begin{cases} 1 & \text{if } v_i \text{ and } v_h \text{ are adjacent in } \Gamma, \\ 0 & \text{else.} \end{cases}$$

Note that $\mathfrak{A}(\Gamma)$ is indecomposable since Γ is connected. Moreover, we set

$$s(i,h) = \begin{cases} s & \text{if } k_s \cap V = \{v_i, v_h\}, \\ 1 & \text{otherwise.} \end{cases}$$

For further graph theoretical terminology we refer to [10] and [27]. For a function $u : G \to \mathbb{R}$ we set $u_j := u \circ \pi_j : [0, l_j] \to \mathbb{R}$ and use the abbreviations

$$u_j(v_i) := u_j(\pi_j^{-1}(v_i)), \quad \partial_j u_j(v_i) := \frac{\partial}{\partial x_j} u_j(x_j)\Big|_{\pi_j^{-1}(v_i)} \quad \text{etc.}$$

As special subspaces of $\mathcal{C}(G)$ we introduce for $1 \leq k \in \mathbb{N}$ the Banach spaces $\mathcal{C}^k(G)$ endowed with the norm $|u|_{k,G} = \sum_{j=1}^N |u_j|_{\mathcal{C}^k([0,l_j])}$ by

$$\mathcal{C}^k(G) = \{u \in \mathcal{C}(G) | \forall j \in \{1, ..., N\} : u_j \in \mathcal{C}^k([0, l_j])\},$$

where the Banach space $\mathcal{C}^k([0, l_j])$ is endowed with the usual $\mathcal{C}^k$–norm. For the Hilbert spaces $\mathcal{H}^k := \prod_{j=1}^N H^k[0, l_j]$ endowed with the usual H^k product norm, the Sobolev embedding theorem permits to evaluate the components at vertices for $k \geq 1$. Thus,

$$\mathcal{H}^k(G) := \{(w_j)_{N \times 1} \in \mathcal{H}^k | \exists u \in \mathcal{C}(G) \forall j \in \{1, ..., N\} : w_j = u \circ \pi_j\}$$

is a closed subspace of $\mathcal{H}^k$ for $k \geq 1$.

As the basic geometric transition condition at ramification nodes we impose the following continuity condition

$$\forall v_i \in V_r \,:\, k_j \cap k_s = \{v_i\} \implies u_j(v_i) = u_s(v_i), \qquad (1)$$

that clearly is contained in the condition $u \in C(G)$. Moreover, at each vertex $v_i \in V$ we impose the classical Kirchhoff condition

$$\forall i \in \{1,\ldots,n\} \,:\, \sum_{j=1}^{N} d_{ij}\partial_j u_j(v_i) = 0. \qquad (K)$$

In fact, the Kirchhoff condition (K) generalizes the Neumann boundary condition on an interval to a vertex transition condition in networks. For operators and function spaces on G, let the super- or subscript K indicate the validity of condition (K). Accordingly, the spaces $\mathcal{C}^1_K(G)$, $\mathcal{C}^2_K(G)$ and $\mathcal{H}^2_K(G)$ are well defined.

3 THE LAPLACIAN ON A NETWORK

The canonical Laplacian on a $\mathcal{C}^2$–network G is defined as the operator

$$\Delta_G^K = \left(u \mapsto \left(\partial_j^2 u_j\right)_{N\times 1}\right) \,:\, \mathcal{C}^2_K(G) \longrightarrow \prod_{j=1}^{N} \mathcal{C}([0,l_j])$$

or as an operator $\Delta_G^K \,:\, \mathcal{H}^2_K(G) \longrightarrow \mathcal{H}$. Owing to the results in [4], $\sigma(\Delta_G^K)$ is real and infinitely countable without finite accumulation point, and $\mathcal{H}$ possesses an orthonormal basis of eigensolutions of Δ_G in $\mathcal{C}^2_K(G)$. Moreover, we note that there are no positive eigenvalues:

$$\sigma(\Delta_G^K) \le 0,$$

since due to (1) and (K):

$$\sum_{j=1}^{N} \int_0^{l_j} u_j \partial_j^2 u_j dx_j = -\sum_{j=1}^{N} \int_0^{l_j} (\partial_j u_j)^2 dx_j + \sum_{j=1}^{N} [u_j \partial_j u_j]_0^{l_j}$$

$$= -\sum_{j=1}^{N} \int_0^{l_j} (\partial_j u_j)^2 dx_j + \sum_{i=1}^{n} u(v_i) \underbrace{\sum_{j=1}^{N} d_{ij}\partial_j u_j(v_i)}_{=0} \le 0.$$

Of course, one can also consider the Laplacian Δ_G under other transition conditions, especially under replacing the Kirchhoff condition by the homogeneous Dirichlet condition at some vertices. But, as it stands, in view of the results in [3]–[5], the present case is the essential model case that can also help in treating other ones.

Let us precise the inverse spectral problem in question. By definition, two networks G_1 and G_2 are *isometric* ($G_1 \cong G_2$) if there is an homeomorphism $H : G_1 \longrightarrow G_2$ such that for each edge $k \subset G_1$, $H\big|_k$ is an isometric diffeomorphism onto some edge of G_2. In particular, H is length preserving. Moreover, the underlying abstract graphs Γ_1 and Γ_2 are called *isomorphic* as graphs ($\Gamma_1 \simeq \Gamma_2$) if there is a bijection $V(\Gamma_1) \longrightarrow V(\Gamma_2)$ that preserves the adjacency relation between vertices. If G_1 and G_2 are isometric networks, then the underlying abstract graphs Γ_1 and Γ_2 are isomorphic as graphs.

PROBLEM *Suppose that for two networks G_1 and G_2 in $\mathbb{R}^m$ the spectra of $\Delta_{G_1}^K$ and $\Delta_{G_2}^K$ coincide counting multiplicities. Does this imply that G_1 and G_2 are isometric?*

$$\sigma\left(\Delta_{G_1}^K\right) \overset{\mathrm{m}}{=} \sigma\left(\Delta_{G_2}^K\right) \quad \overset{?}{\Longrightarrow} \quad G_1 \cong G_2 \tag{2}$$

For the underlying abstract graphs Γ_1 and Γ_2, we are led to the reduced problem

$$\sigma\left(\Delta_{\Delta_1}^K\right) \overset{\mathrm{m}}{=} \sigma\left(\Delta_{\Delta_2}^K\right) \quad \overset{?}{\Longrightarrow} \quad \Gamma_1 \simeq \Gamma_2 \tag{3}$$

In Section 4, we shall show that the implication (3) is wrong, which in turn shows the same for (2). We can restrict ourselves to the case where all edge lengths are equal:

$$\forall j \in \{1, ..., N\} : \; l_j = 1 \tag{4}$$

In fact, under condition (4), the definition of a $\mathcal{C}^2$-network implies that $G_1 \cong G_2 \iff \Gamma_1 \simeq \Gamma_2$. The eigenvalue problem for Δ_G^K reads

$$0 \neq u \in \mathcal{C}_K^2(G) \quad \text{and} \quad \partial_j^2 u_j = -\lambda u_j \quad \text{for} \quad 1 \leq j \leq N. \tag{5}$$

Following the transformations in [3], we formulate Problem (5) as an equivalent matrix differential boundary eigenvalue problem incorporating the adjacency structure of the network. For that purpose we recall that the Hadamard matrix product is defined as

$$(a_{ik})_{n \times n} \star (b_{ik})_{n \times n} = (a_{ik} b_{ik})_{n \times n} \, .$$

For a function $u : G \to \mathbb{R}$ we set for $x \in [0,1]$

$$u_{ih}(x) = e_{ih} u_{s(i,h)} \left(\frac{1 + d_{is(i,h)}}{2} - x d_{is(i,h)} \right)$$

and

$$\varphi = (u(v_i))_{n \times 1}, \qquad \mathbf{U} = (u_{ih})_{n \times n}, \qquad \mathbf{e} = \mathbf{e}_n = (1)_{n \times 1}.$$

Thus,

$$\mathbf{U}(0) = \underbrace{\begin{pmatrix} \varphi_1 & \varphi_1 & \cdots & \varphi_1 & \varphi_1 \\ \varphi_2 & \varphi_2 & \cdots & \varphi_2 & \varphi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_n & \varphi_n & \cdots & \varphi_n & \varphi_n \end{pmatrix}}_{\varphi \mathbf{e}^*} \star \mathfrak{A},$$

and (5) becomes equivalent to the following differential boundary eigenvalue problem (6) for the matrix $\mathbf{U}$:

$$(5) \iff (6) \quad \begin{cases} u_{ih} \in C^2([0,1]) & \text{for } 1 \leq i,h \leq n & (6.1) \\ e_{ih} = 0 \Rightarrow u_{ih} = 0 & \text{for } 1 \leq i,h \leq n & (6.2) \\ \mathbf{U}'' = -\lambda \mathbf{U} & \text{in } [0,1] & (6.3) \\ \mathbf{U}(0) = \varphi \mathbf{e}^* \star \mathfrak{A} & (\text{continuity in } V_r) & (6.4) \\ \mathbf{U}^*(x) = \mathbf{U}(1-x) & \text{for } x \in [0,1] & (6.5) \\ \mathbf{U}'(0)\mathbf{e} = 0 & (\mathrm{K}) & (6.6) \end{cases}$$

We set

$$\Phi := \mathbf{U}(0) = \varphi \mathbf{e}^* \star \mathfrak{A}, \qquad \Psi := \mathbf{U}'(0),$$

and recall the following elementary rules for a $n \times n$–matrix M:

$$(M \star \mathbf{e}\varphi^*)\,\mathbf{e} = M\varphi \qquad\qquad (M \star \varphi\mathbf{e}^*)\,\mathbf{e} = \mathrm{Diag}(M\mathbf{e})\varphi \qquad (7)$$

For $\lambda = 0$, any solution $\mathbf{U}$ of (6) satisfies $\mathbf{U}(x) = \Phi + x\,(\Phi^* - \Phi)$ and $(\Phi^* - \Phi)\,\mathbf{e} = 0$, which implies $\mathrm{Diag}_i\left(\gamma_i^{-1}\right)\mathfrak{A}(\Gamma)\,\varphi = \varphi$. Since $\mathfrak{A}$ is indecomposable, the Perron–Frobenius Theorem [20] yields that 0 is a simple eigenvalue of Δ_G^K with eigenfunctions $\mathbf{U} = \text{const.}\mathfrak{A}$

For $\lambda > 0$, a fundamental solution of (6.3) is given by

$$\mathbf{U}(x) = \cos(x\sqrt{\lambda})\Phi + \frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}}\Psi \tag{8}$$

In the case $\sin\sqrt{\lambda} \neq 0$, (6.5) and (6.6) yield

$$\mathbf{U}(1) = \Phi^* = \Phi\cos\sqrt{\lambda} + \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\Psi,$$

$$\Psi = \frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}}\left(\mathbf{e}\varphi^* - \cos\sqrt{\lambda}\,\varphi\mathbf{e}^*\right) \star \mathfrak{A},$$

and using (7)

$$(\mathfrak{A} \star \mathbf{e}\varphi^*)\,\mathbf{e} - \cos\sqrt{\lambda}\;(\mathfrak{A} \star \varphi\mathbf{e}^*)\,\mathbf{e} = 0.$$

Thus we are led to the following characteristic equation

$$\mathfrak{A}(\Gamma)\varphi = \cos\sqrt{\lambda}\,\mathrm{Diag}_i\,(\gamma_i)\,\varphi. \tag{9}$$

This is part of the following theorem that we recall from Section 5 of [3] in the special case (4) here.

THEOREM 1 *Under condition (4) and using the above definitions, $\lambda \in \sigma(-\Delta_G^K)$ iff either $\varphi = 0$, $\Psi \neq 0$ and $\sin\sqrt{\lambda} = 0$ or φ is an eigenvector belonging to the eigenvalue $\cos\sqrt{\lambda}$ of the row–stochastic matrix*

$$\mathbf{Z} = \mathrm{Diag}_i\left(\gamma_i^{-1}\right)\mathfrak{A}(\Gamma).$$

Moreover, the multiplicities $m(\lambda)$ are

$$m(0) = 1,$$

$$m(\lambda) = \dim\ker\left(\mathbf{Z} - \cos\sqrt{\lambda}\,\mathbf{I}_n\right),\ \textit{if}\ \sin\sqrt{\lambda} \neq 0,$$

$$m\left(\pi^2 4k^2\right) = N - n + 2,$$

$$m\left(\pi^2(2k+1)^2\right) = N - n + 2,\ \textit{if}\ \Gamma\ \textit{is bipartite},$$

$$m\left(\pi^2(2k+1)^2\right) = N - n,\ \textit{if}\ \Gamma\ \textit{is not bipartite}.$$

Note that the multiplicities depend only on $n, N, \gamma_1, \ldots, \gamma_n$ and the adjacency matrix $\mathfrak{A}$. Thus, the spectrum of the Laplacian Δ_G^K is entirely determined by combinatorial quantities of the underlying graph Γ. As an immediate consequence of Theorem 1, we note the following result contained in the results [3] by von Below 1985 and independently shown by Ali Mehmeti [1] 1986 and by Nicaise [21] 1987. Let $b(z)$ denote the number of eigenvalues of $-\Delta_G^K$ in $[0, z]$. Using the above multiplicity formulae, we find $b((2\pi k)^2) = 2Nk + 1$ and

$$\lim_{x \to \infty} \frac{b(x)}{\sqrt{x}} = \frac{N}{\pi}.$$

This yields the

COROLLARY 1 *Let $\{\lambda_k \,|\, k \in \mathbb{N}\}$ denote the monotonically increasing sequence of eigenvalues of $-\Delta_G^K$ under condition (4). Then*

$$\lim_{k \to \infty} \frac{\lambda_k}{k^2} = \frac{\pi^2}{N^2}.$$

In view of the results of the next section it will be of interest to determine common invariants of isospectral networks. Here we mention only the following

COROLLARY 2 *Under condition (4), isospectral networks have the same number of vertices and the same number of edges. Moreover, networks that are isospectral with a bipartite network are also bipartite.*

Proof: By the Lemma in Section 5 of [3], the eigenvalues of $\mathbf{Z}$ are real, $\mathbb{R}^n$ possesses a basis formed by eigenvectors of $\mathbf{Z}$, and the underlying graph Γ is bipartite iff $-1 \in \sigma(\mathbf{Z})$. By the Perron–Frobenius Theorem [20], all eigenvalues of $\mathbf{Z}$ of modulus 1 are simple. Let G_1 and G_2 be two C^2–networks in $\mathbb{R}^m$ and indicate all corresponding entities by 1 and 2 respectively. Now suppose that $\sigma\left(\Delta_{G_1}^K\right)$ and $\sigma\left(\Delta_{G_2}^K\right)$ coincide counting multiplicities. Then apply Theorem 1 in order to conclude $\sigma(\mathbf{Z_1}) \overset{\mathrm{m}}{=} \sigma(\mathbf{Z_2})$. This shows the last assertion. Furthermore, the multiplicities formulae in Theorem 1 imply

$$n_1 = \sum_{\mu \in \sigma(\mathbf{Z_1})} \dim \ker(\mathbf{Z_1} - \mu\, \mathbf{I}_n) = \sum_{\mu \in \sigma(\mathbf{Z_2})} \dim \ker(\mathbf{Z_2} - \mu\, \mathbf{I}_n) = n_2$$

and, finally, $N_1 = N_2$. $\square$

Thus, under condition (4), for subclasses of networks that are characterized only by the number of vertices n or only by the number of edges N, the implications (2) and (3) are true within the subclass. For instance, isospectral paths are isometric, isospectral circuits are isometric, or isospectral star shaped networks, i.e. $|V_r| = 1$, are isometric.

4 COUNTEREXAMPLES

Recall that a graph Γ is called γ–regular if all vertex valencies γ_i are equal to γ. In this case the characteristic equation (9) reads

$$\mathfrak{A}(\Gamma)\varphi = \gamma \cos\sqrt{\lambda}\, \varphi. \tag{10}$$

Combining (10) with the Lemma in Section 5 of [3], we obtain the

COROLLARY 3 *Suppose that* Γ *is a regular. Then the following conditions are equivalent:*

(a) Γ *is bipartite.*

(b) $-1 \in \sigma(\mathbf{Z})$

(c) $\sigma(\mathbf{Z}) \stackrel{\mathrm{m}}{=} -\sigma(\mathbf{Z})$

(d) $\cos\sqrt{\sigma(-\Delta_G^K)} \stackrel{\mathrm{m}}{=} -\cos\sqrt{\sigma(-\Delta_G^K)}$

COROLLARY 4 *Suppose that* Γ *is a regular. If* Γ *is bipartite or a circuit of odd length, then*

$$\gamma \cos\sqrt{\sigma(-\Delta_G^K)} = \sigma\left(\mathfrak{A}(\Gamma)\right).$$

If Γ *is neither bipartite nor a circuit of odd length, then*

$$\gamma \cos\sqrt{\sigma(-\Delta_G^K)} = \sigma\left(\mathfrak{A}(\Gamma)\right) \cup \{-\gamma\}.$$

Proof: The regular nonbipartite connected graphs fulfilling $N = n$ are exactly the circuits of odd length. Now apply Theorem 1, (10) and Corollary 3. □

In the regular case, the multiplicities are $m(0) = 1$ and for $\lambda > 0$

$$m(\lambda) = \dim\ker\left(\mathfrak{A}(\Gamma) - \gamma\cos\sqrt{\lambda}\,\mathbf{I}_n\right), \text{ if } \sin\sqrt{\lambda} \neq 0,$$

$$m(\lambda) = \begin{cases} N - n + 2, \\ N - n, \end{cases}, \text{ if } \sin\sqrt{\lambda} = 0,$$

and do only depend on n, N and the adjacency matrix $\mathfrak{A}$. This can be applied in order to find two isospectral networks G_1 and G_2 with underlying topological graphs Γ_1 and Γ_2, respectively, such that the corresponding problems (5) have the same eigenvalues counted according to their multiplicities, while Γ_1 and Γ_2 are not isomorphic as abstract graphs. We only have to find graphs Γ_1 and Γ_2 that are regular, nonisomorphic, and isospectral as graphs, i.e. $\sigma(\mathfrak{A}(\Gamma_1)) = \sigma(\mathfrak{A}(\Gamma_2))$ with identical multiplicities. A large variety of such examples can be obtained with the aid of a result of A. J. Hoffman, see e.g. Theorem 6.1 in [12], which states that for any natural number M there is an integer n_0 such that for any $n \geq n_0$ there exist M nonisomorphic isospectral regular connected graphs with n vertices. Obviously, all these must have the same valency. As a concrete example, we display the following two 4–regular graphs Γ_1 and Γ_2 with $n = 12$ and $N = 24$

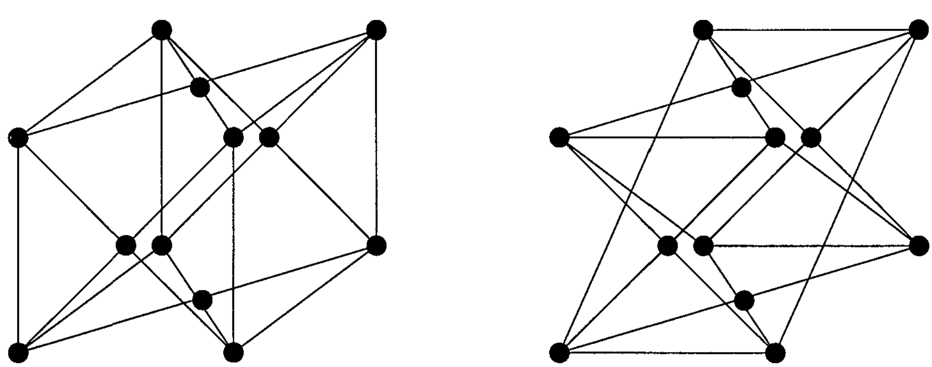


Fig.1 Two isospectral networks that are not isometric.

as depicted in Fig.1. The corresponding networks G_1 and G_2 can be easily realized under condition (4). We readily compute

$$\sigma(\mathfrak{A}(\Gamma_1)) = \sigma(\mathfrak{A}(\Gamma_2)) = (4, 2, 2, 2, 0, 0, 0, -2, -2, -2, -2, -2).$$

But, both graphs are not isomorphic, since the left one Γ_1 is planar while the right one Γ_2 is not realizable in $\mathbb{R}^2$, see Fig.2, where in each graph the two half-moon like vertices have to be identified. On the

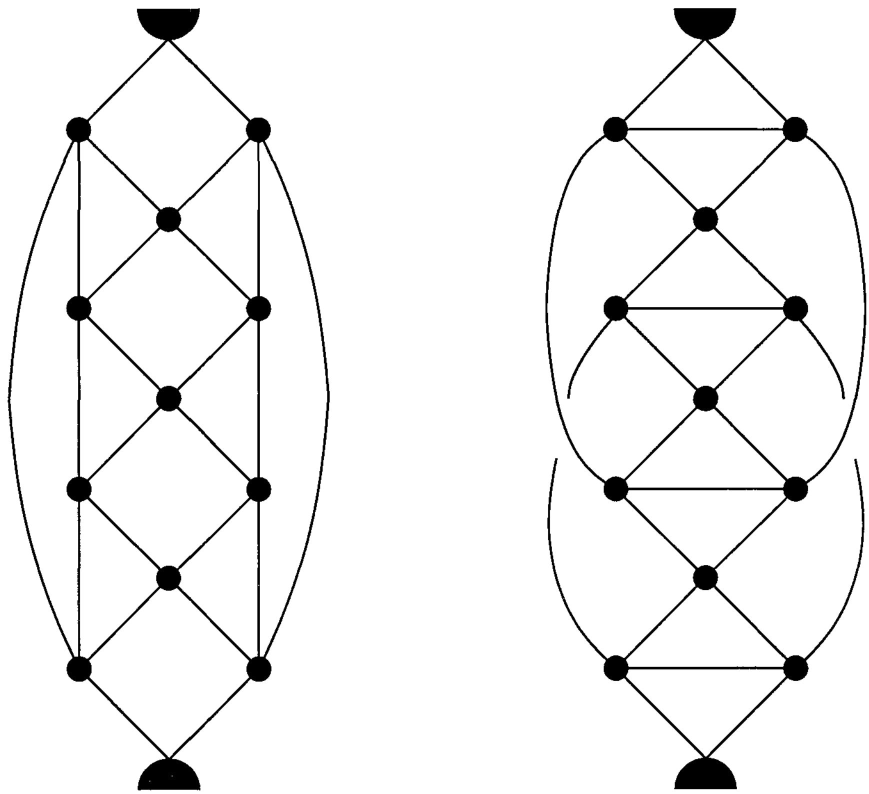


Fig.2 The nonisomorphic underlying graphs.

other hand, by Theorem 1, (10) and (11) we conclude that the spectra $\sigma\left(\Delta^K_{G_1}\right)$ and $\sigma\left(\Delta^K_{G_2}\right)$ coincide counting multiplicities. Thus, we can resume:

THEOREM 2 *In general, the shape of a C^2–network G cannot be reconstructed from the spectrum of its Laplacian Δ^K_G. There exist pairs of regular isospectral networks that are not isometric.*

The same phenomenon can occur when the underlying abstract graphs are isomorphic, but different edge lengths are admitted. Roth [24] has already given an example of a nonisometric pair of networks with the same multiple underlying graph and with the same eigenvalues of the Laplacian. For the simple graphs considered here, we consider the case of a weighted Laplacian on a network where the diffusion rates on all edges are given by the squares of the edge lengths. The corresponding eigenvalue problem including a consistent Kirchhoff condition reads

$$(12) \qquad \begin{cases} u \in C^2(G), \\ l_j^2 \partial_j^2 u_j = -\lambda u_j & \text{in } [0,1] \text{ for } 1 \leq j \leq N, \\ \sum_{j=1}^{N} d_{ij} l_j^2 \partial_j u_j(v_i) = 0 & \text{for all } v_i \in V. \end{cases}$$

Note that the consistency of the Kirchhoff condition is indispensable in order to ensure the symmetry of the Laplacian, see [3] Section 7 and [4], except for a very special class of networks containing one inconsistent ramification node, see [5]. Using the transformations of Section 3 with the modifications

$$u_{ih}(x) = e_{ih} u_{s(i,h)} \left(l_{s(i,h)} \left[\frac{1 + d_{is(i,h)}}{2} - x d_{is(i,h)} \right] \right)$$

and

$$\mathbf{L} = \left(e_{ih} l_{s(i,h)} \right)_{n \times n},$$

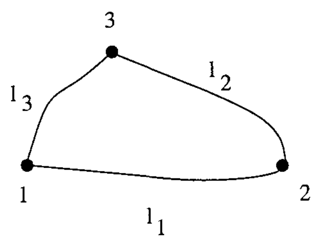
the equivalent matrix differential boundary eigenvalue problem incorporating the adjacency structure of the network becomes:

$$(13) \qquad \begin{cases} u_{ih} \in C^2([0,1]) & \text{for } 1 \leq i,h \leq n \\ e_{ih} = 0 \Rightarrow u_{ih} = 0 & \text{for } 1 \leq i,h \leq n \\ \mathbf{U}'' = -\lambda \mathbf{U} & \text{in } [0,1] \\ \mathbf{U}(0) = \varphi \mathbf{e}^* \star \mathfrak{A} & \\ \mathbf{U}^*(x) = \mathbf{U}(1-x) & \text{for } x \in [0,1] \\ [\mathbf{L} \star \mathbf{U}'(0)]\, \mathbf{e} = 0 & \end{cases}$$

By the results of Section 5 of [3], all the assertions of Theorem 1 remain valid for the eigenvalues of (12) and (13) with the row–stochastic matrix

$$\mathbf{Z} = \mathrm{Diag}\,(\mathbf{Le})^{-1}\,\mathbf{L}.$$

Thus, in order to find the desired example, we only have to ensure that for some nonisometric networks the corresponding matrices $\mathbf{Z}$ all have the same eigenvalues with multiplicities counted and the same zero pattern. Take for instance $\Gamma = \mathcal{K}_3$ to be the circuit of length 3 and G as indicated in Fig.3 with $(l_1+l_2)(l_1+l_3)(l_2+l_3) = 8l_1l_2l_3$. Then the set of eigenvalues of (12) with multiplicities counted coincides with the one in the special case $l_1 = l_2 = l_3 = 1$ given by (14), (15) and (17) in Section 5 for $n = 3$, since the common characteristic polynomial for all matrices $\mathbf{Z}$ reads $-\lambda^3 + \frac{3}{4}\lambda + \frac{1}{4}$.



$$(l_1 + l_2)(l_1 + l_3)(l_2 + l_3) = 8l_1l_2l_3$$

Fig.3 A family of nonisometric isospectral networks with isomorphic underlying graph.

5 NETWORK IMMANENT EIGENVALUES

To what extent can one hear a network, i.e. which frequencies are immanent to the network system and cannot be derived from the spectrum of the Laplacian on a single edge under 0–Dirichlet or Neumann boundary conditions? Let us discuss this problem for the complete graph on n vertices $\mathcal{K}_n$. Its adjacency matrix $\mathfrak{A} = \mathbf{e}_n\mathbf{e}_n^* - \mathbf{I}_n$ has the simple eigenvalue $n-1$ and the $(n-1)$-fold eigenvalue -1 leading to the following eigenvalue sequences of (5) by Theorem 1:

$$\lambda = 0,\ m(0) = 1,\ \varphi = \mathbf{e}_n \tag{14}$$

$$\lambda = 4\pi^2k^2,\ k \neq 0,\ m(\lambda) = 2 + \frac{1}{2}n(n-3),\ \varphi = \mathbf{e}_n \tag{15}$$

$$\lambda = \pi^2(2k+1)^2,\ k \neq 0,\ m(\lambda) = \frac{1}{2}n(n-3),\ \varphi = 0, \tag{16}$$

$$\cos\sqrt{\lambda} = -(n-1)^{-1}, \quad m(\lambda) = n-1,\ \varphi \in \ker \mathbf{e}\mathbf{e}^* \tag{17}$$

The results apply for instance to the wave equation on a tetrahedron network $\mathcal{K}_4$. Using the relation $\mathbf{U}'(1) = -\Psi^\star$, all eigenvalues (frequencies) and corresponding eigensolutions are given in terms of the transformations of Section 3 by (8) and Theorem 1 as follows:

$$\lambda = 0,\ m(0) = 1,\ \mathbf{U} = r\left(\mathbf{e}_4\mathbf{e}_4^* - \mathbf{I}_4\right),\ r \in \mathbb{R} \tag{18}$$

$$\begin{aligned} &\lambda = 4\pi^2k^2,\ 1 \leq k \in \mathbb{N},\ m(\lambda) = 4, \\ &\quad \mathbf{U}(x) = r\cos(x2\pi k)\left(\mathbf{e}_4\mathbf{e}_4^* - \mathbf{I}_4\right) + \frac{\sin(x2\pi k)}{2\pi k}\Psi \star \mathfrak{A}, \\ &\quad r \in \mathbb{R}, \Psi^* = -\Psi, \Psi\mathbf{e}_4 = 0 \end{aligned} \tag{19}$$

$$\begin{aligned} &\lambda = \pi^2(2k+1)^2,\ k \in \mathbb{N},\ m(\lambda) = 2,\ \varphi = 0, \\ &\quad \mathbf{U}(x) = \frac{\sin(x\pi(2k+1))}{\pi(2k+1)}\Psi \star \mathfrak{A},\ \Psi^* = \Psi, \Psi\mathbf{e}_4 = 0 \end{aligned} \tag{20}$$

$$\begin{aligned} &\lambda = \left((-1)^k \arccos\left(\frac{-1}{3}\right) + \frac{2k+1-(-1)^k}{2}\pi\right)^2,\ k \in \mathbb{N}, \\ &\quad \text{i.e. } \cos\sqrt{\lambda} = -\frac{1}{3},\ m(\lambda) = 3,\ \mathbf{e}_4^*\varphi = 0, \\ &\quad \mathbf{U}(x) = \cos(x\sqrt{\lambda})\varphi\mathbf{e}_4^* \star \mathfrak{A} + \frac{\sin(x\sqrt{\lambda})}{\sin\sqrt{\lambda}}\left(\mathbf{e}_4\varphi^* - \cos(\sqrt{\lambda})\varphi\mathbf{e}_4^*\right) \star \mathfrak{A} \end{aligned} \tag{21}$$

On the one hand, solutions $\mathbf{U}$ with $\Psi = 0$ in (19) correspond just to single edge solutions with Neumann boundary conditions glued together such that they form a continuous function on G. On the other hand, any $\mathbf{U} \neq 0$ with $\varphi = 0$ leads to an eigensolution on some edge with 0–Dirichlet conditions. Conversely, $u_0(x) := \sin(\pi k x)$ on $[0,1]$ can be extended to an eigensolution of $\mathcal{K}_4$ of the type (19) for $0 < k \equiv 0 \bmod 2$ and of the type (20) for $k \equiv 1 \bmod 2$ as displayed in Fig.4 and Fig.5. The bold edges indicate the circuits along which $u_{s(i,h)} = u_0$ on the edges in the circuit orientation, while on the remaining ones $u_{s(i,h)} \equiv 0$. In this way, a basis of the eigenspace in the case (19) is given by the matrix $\mathfrak{A} = \mathbf{e}_4\mathbf{e}_4^* - \mathbf{I}_4$, corresponding to the Neumann boundary condition on each edge $\Psi = 0$ and $\Phi = \Phi^*$, and the following matrices that correspond to the 0–Dirichlet condition on each edge $\Phi = 0$ and $\Psi^* = -\Psi$, see Fig.4.

$$\begin{pmatrix} 0 & +1 & -1 & 0 \\ -1 & 0 & +1 & 0 \\ +1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & +1 & 0 & -1 \\ -1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & +1 \\ +1 & 0 & -1 & 0 \end{pmatrix}$$

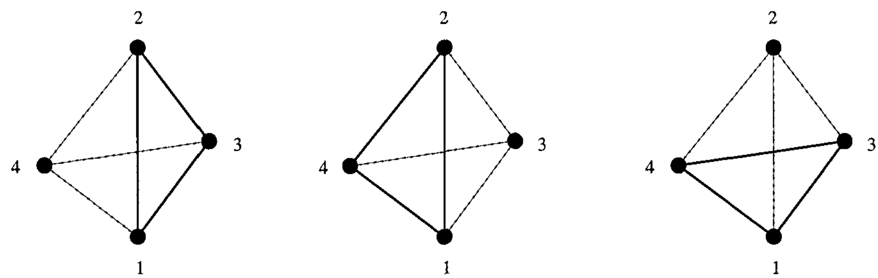


Fig. 4

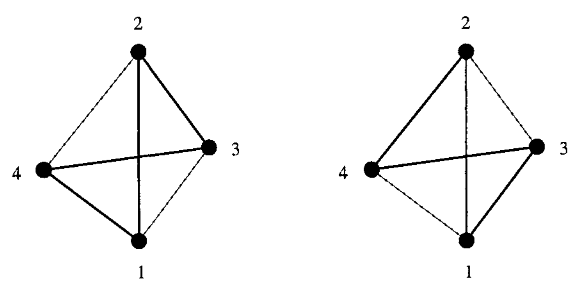


Fig. 5

A basis of the eigenspace in the case (20) is given by the matrices

$$\begin{pmatrix} 0 & +1 & 0 & -1 \\ +1 & 0 & -1 & 0 \\ 0 & -1 & 0 & +1 \\ -1 & 0 & +1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & +1 & -1 & 0 \\ +1 & 0 & 0 & -1 \\ -1 & 0 & 0 & +1 \\ 0 & -1 & +1 & 0 \end{pmatrix}$$

that correspond again to the 0–Dirichlet condition on each edge $\Phi = 0$ and $\Psi^* = \Psi$, see Fig.5. Thus, the eigenvalues (21) are the only ones emanating from the network system and neither from a single clamped branch, nor a single branch with Neumann boundary condition. As for the acoustic interpretation, we observe that $\left(\arccos\left(\frac{-1}{3}\right)\right)^2$ is not compatible with the basic tone of a single edge, while for the triangle network $\mathcal{K}_3$ the corresponding network eigenvalue satisfying $\cos\sqrt{\lambda} = -\frac{1}{2}$ is compatible to the basic tone of a single edge: $\lambda = \frac{4}{9}\pi^2$. That is why tetrahedra sound so bad while triangles sound much better ...

Mutatis mutandis, the same distinction between single edge eigenvalues and network immanent ones holds for general networks. The eigenvalues of (5) belonging to the eigenvalues μ of the matrix $\mathbf{Z} = \mathrm{Diag}_i\left(\gamma_i^{-1}\right)\mathfrak{A}(\Gamma)$ with $|\mu| < 1$ are always immanent to the whole network system and cannot emanate from a single branch. In the bipartite case, there is an additional non zero node vector that belongs to the simple matrix eigenvalue $\mu = \cos\sqrt{\lambda} = -1$ and that can only occur in the whole system. Note that $0 \neq \Phi = -\Phi^*$ is impossible in nonbipartite networks. Note further that a single edge spectrum with zero boundary conditions can only be embedded if and only if Γ contains circuits. On a tree these embeddings are impossible. This also becomes clear by the multiplicity formulae in Theorem 1 that read for a tree $m\left(\pi^2 k^2\right) = 1$ with $\varphi \in \mathbb{R}\mathbf{e}_n$ and $m(\lambda) = \dim\ker\left(\mathfrak{A}(\Gamma) - \cos\sqrt{\lambda}\,\mathbf{I}_n\right)$ for $\sin\sqrt{\lambda} \neq 0$ with $\varphi \neq 0$ as in (9).

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