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Current experience with transfinite interpolation

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Abstract

Gordon methods for transfinite interpolation may be extended so as to allow interpolation of derivative information at arbitrary mesh lines. These results are reviewed and applications and examples are presented. © 1999 Elsevier Science B.V. All rights reserved.

1. Introduction

This paper discusses some current experience with the problem of constructing a suitable surface which interpolates a given rectangular mesh in $\mathbb{R}^3$. The problem is of central importance to the field of geometric modeling and has numerous application. Instead of building surfaces as conglomerates of individual patches, as was and still is common practice, Gordon (1968, 1969a, 1969b), introduced a technique whereby certain methods of univariate polynomial spline interpolation are extended so as to allow a rectangular mesh to be interpolated as a single surface. For a number reasons, which include the inability to assign derivative values at arbitrary mesh points, reliance on a scheme of cardinal interpolation, and lack of local control, Gordon methods seem to have fallen by the wayside. Although the Gordon methods dispense with the difficulty of having to specify large amounts of data for each surface patch, they provide no good methods to specify this same data should it be essential.

The present papers builds on the results of (Walker, 1998) in which the Gordon methods are extended so as to avoid some disadvantage. In particular it is shown in (Walker, 1998) that any two univariate interpolation schemes may be extended to a bivariate mesh interpolation method, according to the method set out by Gordon. In the case of univariate schemes based on cardinal polynomial blending functions of odd degree, the result is known and appears in (Gordon, 1968). The general result is very simply proved, and, although perhaps part of the folklore of CAGD, it has not appear and is important in that

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it opens the door to an investigation of bivariate interpolation schemes based on various univariate, perhaps non-polynomial, methods.

Also in (Walker, 1998), it was shown that with the extended Gordon methods it is possible to construct surfaces which interpolate not only the given mesh but also prescribed derivatives of arbitrary order in directions across mesh lines. Unlike the previous, this result has no reflection in any of the literature surrounding Gordon's work

In this paper, we report on experience applying the above results. In Section 2, definitions and notation are introduced together with a review of the results in (Walker, 1998) for the sake of completeness. In Section 3, two applications are discussed, assigning tangent values across mesh lines and the G^k attachment of one surface along a mesh line to a curve in another.

2. Preliminaries

Definition 1. Given strictly increasing sequences of real numbers $u_0 < u_1 < \cdots < u_m$ and $v_0 < v_1 < \cdots < v_n$, if

$$U = \bigcup_{j=0}^{n} [u_0, u_m] \times \{v_j\} \quad \text{and} \quad V = \bigcup_{i=0}^{n} \{u_i\} \times [v_0, v_n],$$

we refer to the set $U \cup V$ as a *domain mesh*. A *rectangular parametric mesh* is a function $\mathbf{f}: U \cup V \to \mathbb{R}^3$. For such a parametric mesh an *interpolating surface* is considered to be an extension of $\mathbf{f}$ to the rectangle $[u_0, u_m] \times [v_0, v_n]$ – namely, a function

$$\mathbf{q}: [u_0, u_m] \times [v_0, v_n] \to \mathbb{R}^3 \text{ such that } \mathbf{q}|_{X \cup Y} = \mathbf{f}.$$

Definition 2. Given a strictly increasing sequence $t_0 < t_1 < \cdots < t_r$, an associated univariate interpolation scheme is a function $\mathbf{h}: [t_0, t_r] \times (\mathbb{R}^3)^{r+1} \to \mathbb{R}^3$ with the property that

$$\mathbf{h}(t_i, P_0, P_1, \ldots, P_r) = P_i, \quad 0 \leqslant i \leqslant r.$$

Given a parametric mesh $\mathbf{f}: U \cup V \to \mathbb{R}^3$ and univariate interpolation schemes

$$\mathbf{h}_1: [u_0, u_m] \times (\mathbb{R}^3)^m \to \mathbb{R}^3 \quad \text{and} \quad \mathbf{h}_2: [v_0, v_n] \times (\mathbb{R}^3)^n \to \mathbb{R}^3,$$

define extensions $\mathbf{q}_1: [u_0, u_m] \times [v_0, v_n] \to \mathbb{R}^3$, $\mathbf{q}_2: [u_0, u_m] \times [v_0, v_n] \to \mathbb{R}^3$, and $\mathbf{q}_3: [u_0, u_m] \times [v_0, v_n] \to \mathbb{R}^3$ by

$$\mathbf{q}_1(u, v) = \mathbf{h}_1(u, \mathbf{f}(u_0, v), \ldots, \mathbf{f}(u_m, v)),$$
$$\mathbf{q}_2(u, v) = \mathbf{h}_2(v, \mathbf{f}((u, v_0), \ldots, \mathbf{f}(u, v_n)), \quad \text{and}$$
$$\mathbf{q}_3(u, v) = \mathbf{h}_2(v, \mathbf{q}_1(u, v_0), \ldots, \mathbf{q}_1(u, v_n)),$$
$$\mathbf{q}_4(u, v) = \mathbf{h}_1(u, \mathbf{q}_2(u_0, v), \ldots, \mathbf{q}_2(u_m, v)).$$

Theorem 3. *If* $\boldsymbol{\alpha}:[u_0, u_m] \times [v_0, v_n] \to \mathbb{R}^3$ *and* $\boldsymbol{\beta}:[u_0, u_m] \times [v_0; v_n] \to \mathbb{R}^3$ *are functions with the property that* $\boldsymbol{\alpha}(u,v) + \boldsymbol{\beta}(u,v) = 1$, *then the function* $\mathbf{q}:[u_0, u_m] \times [v_0, v_n] \to \mathbb{R}^3$ *defined by*

$$\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2 - (\boldsymbol{\alpha}\mathbf{q}_3 + \boldsymbol{\beta}\mathbf{q}_4)$$

interpolates the mesh $\mathbf{f}: U \cup V \to \mathbb{R}^3$.

In the context of Theorem 3, suppose we are given an integer k, a mesh point (u_r, v), $0 \leqslant r \leqslant m$, a neighborhood W of (u_r, v) together with functions

$$\sigma_i : W \cap \{u_r\} \times [v_0, v_n] \to \mathbb{R}^3, \quad 1 \leqslant i \leqslant k,$$

which define desired derivatives across the mesh line $\{u_r\} \times [v_0, v_n]$. For

$$\boldsymbol{\psi} : [u_0, u_m] \times [v_0, v_n] \to [v_0, v_n] \times (\mathbb{R}^3)^n$$

defined by

$$\boldsymbol{\psi}(u,v) = (u, \mathbf{f}(u_0, v), \ldots, \mathbf{f}(u_m, v)),$$

we then have the following theorem.

Theorem 4. *If the surface* $\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2 - (\boldsymbol{\alpha}\mathbf{q}_3 + \boldsymbol{\beta}\mathbf{q}_4)$ *is such that:*
- $\boldsymbol{\alpha}(u,v) = 1$, *for* $(u,v) \in W$,
- $(\partial^i \mathbf{h}_1(\mathbf{x})/\partial u^i)|_{\mathbf{x}=\boldsymbol{\psi}(u_r,v)} = \boldsymbol{\sigma}_i(v)$, *for* $1 \leqslant i \leqslant k$, $0 \leqslant r \leqslant m$, *and* $(u_r, v) \in W \cap (\{u_r\} \times [v_0, v_n])$,

then

$$\frac{\partial^i \mathbf{q}}{\partial u^i}(u_r, v) = \boldsymbol{\sigma}_i(v), \quad \textit{for } (u_r, v) \in W \cap (\{u_r\} \times [v_0, v_n]).$$

3. Applications and examples

Using preceding results, if $\mathbf{f}: U \cup V \to \mathbb{R}^3$ is a parametric mesh as described in Definition 1, it is possible to construct an interpolating surface $\mathbf{q}$ whose derivatives across a sequence of parallel mesh lines interpolate prescribed values. In the case of a single mesh line $\mathbf{f}(\{u_r\} \times [v_0, v_n])$, $0 \leqslant r \leqslant m$, and some choice of a positive integer k, it is desired to be able to assign derivatives $(\partial^i \mathbf{q}/\partial u^i)(u_r, v)$, $v_0 \leqslant v \leqslant v_n$ and $1 \leqslant i \leqslant k$, according to the values of specified functions $\boldsymbol{\sigma}_i : [v_0, v_n] \to \mathbb{R}^3$, where $\boldsymbol{\sigma}_i(v_j) = (\partial^i \mathbf{f}/\partial u^i)(u_r, v_j)$, for $0 \leqslant j \leqslant n$. The task of constructing the surface $\mathbf{q}$, according to the requirements of Theorem 4, then reduces to that of choosing an appropriate univariate interpolation method. In the examples below, Figs. 1 and 2, the mesh being interpolated consists, from front to back, of a trigonometric curve, an exponential curve, a straight line, and a parabola. From left to right connecting curves are polynomial splines which satisfy $(\partial \mathbf{f}/\partial u)(u_0, v_j) = (1,0,0)$, $0 \leqslant j \leqslant 5$, in the case of Fig. 1 and $(\partial \mathbf{f}/\partial u)(u_0, v_j) = (2,0,3)$, $0 \leqslant j \leqslant 5$, for Fig. 2. Choosing $k = 1$, for Figs. 1 and 2, manipulation of the derivatives $(\partial \mathbf{q}/\partial u)(u_0, v)$ was achieved by setting $\boldsymbol{\alpha} = 1$ and choosing $\boldsymbol{\sigma} : [v_0, v_n] \to \mathbb{R}^3$ so that $\boldsymbol{\sigma}(v) = (1,0,0)$ and $\boldsymbol{\sigma}(v) = (2,0,3)$, respectively. The univariate interpolation method h_1 was chosen to be a

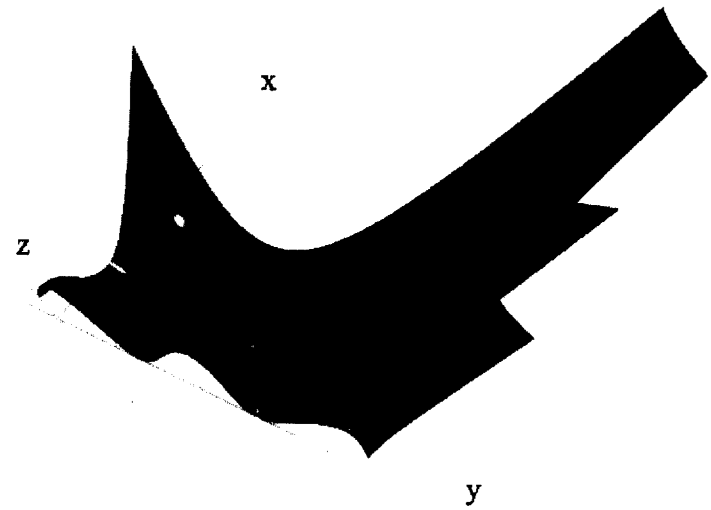

Fig. 1. The red curves front to back are: a trigonometric curve, an exponential curve, a straight line, and a parabola. From left to right they are polynomial splines. The blue surface interpolates the red mesh in such a way that partial derivatives in the x direction along the front edge all have value $(1, 0, 0)$.

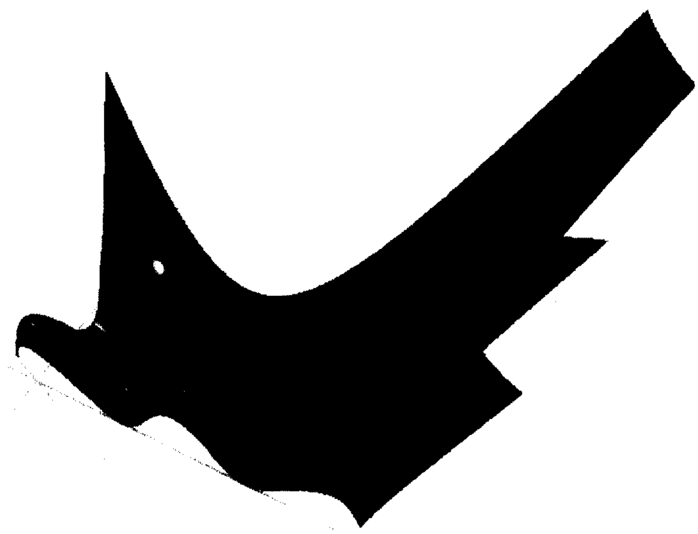

Fig. 2. The same as Fig. 1 with the exception that partial derivatives in the x direction across the leading edge now have value $(2, 0, 3)$.

form of Catmull–Rom interpolation consisting of a B-spline blending local interpolating polynomial functions each possessing derivatives equal to $\sigma(v)$ at the points (u_0, v). The interpolation method h_2 was likewise chosen to be Catmull–Rom interpolation—but with no restrictions on derivatives of local interpolating functions.

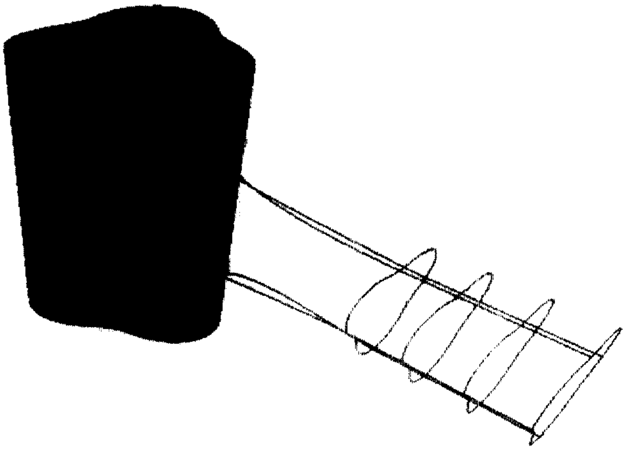

Fig. 3. A cylinder with radius scalloped with trigonometric variation together with a mesh of curves to guide the attachment of another such surface.

3.1. Attaching surfaces

The techniques above may be used to attach two parametric surfaces with G^k continuity in such a way that one is attached along a mesh line $\mathbf{f}(\{u_r\} \times [v_0, v_n])$, $0 \leqslant r \leqslant m$, to a curve in another. Following Gregory (1988), let D be a compact region of the plane homeomorphic to a closed disk, $\mathbf{p}: D \to \mathbb{R}^3$ be a C^k continuous parametric surface, and $\boldsymbol{\xi}: [a, b] \to D$ be a C^k continuous curve in the domain of $\mathbf{p}$. Given a domain mesh $U \cup V$, as in Definition 1, let $\lambda: [v_0, v_n] \to [a, b]$ be a C^k continuous change of variable function with $\lambda'(s) \neq 0$, for $s \in [v_0, v_n]$, and define $\mathbf{E}_1: [v_0, v_n] \to D$ by $\mathbf{E}_1 = \boldsymbol{\xi} \circ \lambda$. For $0 \leqslant r \leqslant m$, let $\mathbf{E}_2: [v_0, v_n] \to U \cup V$ be defined by $\mathbf{E}_2(s) = (u_r, s)$. For W a suitable neighborhood of $\mathbf{E}_2([v_0, v_n])$, let $\boldsymbol{\varphi}: W \to \mathbb{R}^2$ be a C^k diffeomorphism such that for $s \in [v_0, v_n]$, $\mathbf{E}_1(s) = \boldsymbol{\varphi}(\mathbf{E}_2(s))$. According to (Gregory, 1988) Lemma 5.2, a surface $\mathbf{q}: [u_0, u_m] \times [v_0, v_n] \to \mathbb{R}^3$ is connected with G^k continuity to the surface $\mathbf{p}: D \to \mathbb{R}^3$ so that the mesh line $\mathbf{q}(\mathbf{E}_2(s))$ is joined to the curve $\mathbf{p}(\boldsymbol{\varphi}(\mathbf{E}_2(s)))$, provided that for derivatives in the u-direction,

$$\left.\frac{\partial^i \mathbf{q}(\mathbf{x})}{\partial u^i}\right|_{\mathbf{x}=\mathbf{E}_2(s)} = \left.\frac{\partial^i \mathbf{p}(\boldsymbol{\varphi}(\mathbf{x}))}{\partial u^i}\right|_{\mathbf{x}=\mathbf{E}_2(s)}, \quad 0 \leqslant i \leqslant k.$$

In order to guarantee that this condition is satisfied, let $\mathbf{q}$ be as constructed in Theorem 4 and choose the univariate interpolation method $\mathbf{h}_1$ so that

$$\left.\frac{\partial^i \mathbf{h}_1(\mathbf{x})}{\partial u^i}\right|_{\mathbf{x}=\boldsymbol{\varphi}(\mathbf{E}_2(s))} = \left.\frac{\partial^i \mathbf{p}(\boldsymbol{\varphi}(\mathbf{x}))}{\partial u^i}\right|_{\mathbf{x}=\boldsymbol{\varphi}(\mathbf{E}_2(s))}, \quad 0 \leqslant i \leqslant k.$$

In Figs. 3 and 4, we see how two cylinders, each scalloped according to a trigonometric variation of the radius, may be attached. In Fig. 5 a cylinder is attached to the example occurring in Fig. 2. In each of these examples, the univariate interpolation scheme $\mathbf{h}_1$ was chosen to consist of a Hermite segment in the region of attachment and Catmull–Rom interpolatory segments elsewhere; the interpolation scheme $\mathbf{h}_2$ was chosen to consist only of Catmull–Rom segments.

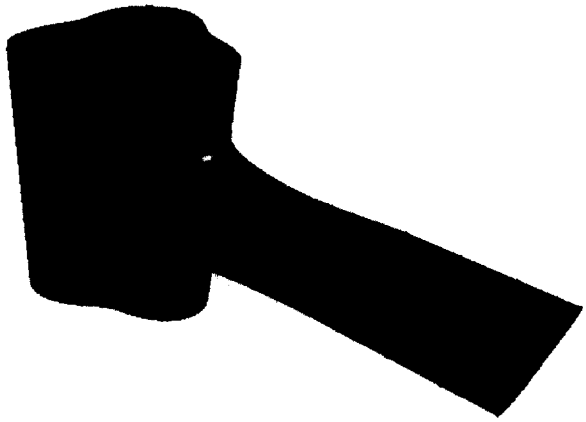

Fig. 4. A second scalloped cylinder attached with G^1 continuity to the first in such a way that it interpolates the curves in Fig. 3.

Fig. 5. A cylinder attached with G^1 continuity to the surface of Fig. 2.

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