Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty—III: The Dimension of the Space of Essentially Time- and Band-Limited Signals

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The purpose of this paper is to examine the mathematical truth in the engineering intuition that there are approximately 2WT independent signals φ_i *of bandwidth W concentrated in an interval of length T. Roughly speaking, the result is true for the best choice of the* φ_i *(prolate spheroidal wave functions), but not for sampling functions (of the form sin t/t). Some typical conclusions are: Let* $f(t)$*, of total energy 1, be band-limited to bandwidth W, and let*

$$\int_{-T/2}^{T/2} | f^2(t) | \, dt = 1 - \epsilon_T^2 .$$

Then

$$\inf_{\{a_i\}} \int_{-\infty}^{\infty} \left| f(t) - \sum_{0}^{[2WT]+N} a_n \varphi_n \right|^2 dt < C\epsilon_T^2$$

is

(a) *true for all such f with N = 0, C = 12, if the* φ_n *are the prolate spheroidal wave functions;*

(b) *false for some such f for any finite constants N and C if the* φ_n *are sampling functions.*

I. INTRODUCTION AND SUMMARY OF RESULTS

Intuitive considerations based primarily on the sampling theorem have for a long time suggested that the space of signals "essentially" limited in time to the interval $|t| \leqq T/2$ and in frequency to $(-W,W)$ cycles is "essentially" $2WT$-dimensional. It is the object of the present paper to investigate this problem thoroughly. The first step in the process is to see how the above statement may be made precise. The two main difficulties to be overcome in even *formulating* some mathematical problems in this area are contained in the two uses of "essentially" above: What shall we mean by "essentially" limited in time and frequency, and what can we mean by "essentially" $2WT$-dimensional?

Suppose that a function $f(t)$ is actually band-limited. It is then an analytic function of the complex variable t, and cannot vanish in $|t| > T/2$ without vanishing identically. We will therefore think of $f(t)$ as *approximately time-limited* to $|t| \leqq T/2$ if a large fraction of its energy is contained in that interval, that is, if

$$\text{(0.1)} \qquad \frac{\displaystyle\int_{|t| \leqq T/2} |f(t)|^2 \, dt}{\displaystyle\int_{-\infty}^{\infty} |f(t)|^2 \, dt} = 1 - \epsilon_T^2 ,$$

where ϵ_T will, in much of our thinking, be small; ϵ_T shall be used as a measure of the degree to which $f(t)$ fails to be concentrated on the interval $|t| \leqq T/2$. We will denote by $E(\epsilon_T)$ the set of band-limited functions $f(t)$ satisfying (0.1) with the further normalization for convenience that

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = 1.$$

We should point out here that, by previous results,[1] T and ϵ_T are related: as ϵ_T becomes small, T must grow indefinitely.

We have now defined our set of functions; how can we speak precisely about its dimension? $E(\epsilon_T)$ is certainly not finite-dimensional for any $\epsilon_T > 0$, for there is no *finite* set of functions whose linear combinations exactly express each $f(t)$ in $E(\epsilon_T)$. We will, however, say that $E(\epsilon_T)$ is *approximately N-dimensional* if there exist N linearly independent functions $\varphi_0, \cdots \varphi_{N-1}$ whose linear combinations approximate each $f(t)$ in $E(\epsilon_T)$ to within a small fraction of its energy, that is, if

$$\text{(0.2)} \qquad \min_{\{a_i\}} \int_{-\infty}^{\infty} \left| f(t) - \sum_0^{N-1} a_i \varphi_i(t) \right|^2 dt < \delta_N^2 ,$$

where we shall usually think of δ_N as small. Again, δ_N may be used as a measure of the degree to which $E(\epsilon_T)$ is N-dimensional.

In the above definition of the approximate dimension of $E(\epsilon_T)$, we have complete freedom in choosing the "basis" functions $\varphi_0 \cdots \varphi_{N-1}$

with which we will attempt to approximate $f(t)$. There are two different objectives we may have in choosing the φ_i. For real understanding of the dimension of $E(\epsilon_T)$ we must use the φ_i which *best approximate* $E(\epsilon_T)$, in the sense of making the error, represented by the left side of (0.2), as small as it can possibly be over the whole set $E(\epsilon_T)$. Alternatively, for practical purposes, we may wish to use the *simplest* available functions, and see how close we can come with them. Thus there is considerable interest in pursuing two lines of investigation:

(*i*) Let us first try to identify the best functions φ_i to use, that is the functions which achieve

$$\min_{\{\varphi_i\}_0^{N-1}} \max_{f \in E(\epsilon_T)} \min_{\{a_i\}_0^{N-1}} \int_{-\infty}^{\infty} \left| f(t) - \sum_0^{N-1} a_i \varphi_i(t) \right|^2 dt. \tag{0.3}$$

Once we have found these best functions, what is the relation between the number N of such functions, the measure of concentration ϵ_T, and the achievable degree of approximation δ_N?

(*ii*) If we pick for the φ's sampling functions, i.e., functions of the form $[\sin \pi(2Wt - r)]/[\pi(2Wt - r)]$, what is now the relation between N, ϵ_T, and δ_N?

It turns out that the answers to (*i*) and (*ii*) are rather different, that is, the degree of approximation achievable by sampling functions is in a very real sense poorer than the degree achievable by the *best* basis functions. And yet the solutions of the two problems are, as we shall see, remarkably intertwined.

In order to give a detailed picture of our results, it is necessary to summarize some of the previous work on time- and band-limiting which has appeared in Refs. 1 and 2.

The space $\mathfrak{L}^2$ of square-integrable functions on $(-\infty,\infty)$ forms a Hilbert space in which the inner product (f,g) is defined by

$$(f,g) = \int_{-\infty}^{\infty} f(t)\, \overline{g(t)}\, dt;$$

the norm squared of f, $\| f \|^2$, is defined by

$$\| f \|^2 = (f,f),$$

and is just the total energy. Two functions f and g are *orthogonal* if

$$(f,g) = 0.$$

To any closed subspace there corresponds a projection operation P, which assigns to every function its orthogonal projection onto the subspace. Projections are characterized completely[3] by the properties

$$P \text{ is self-adjoint, and} \tag{0.4}$$
$$P^2 = P.$$

We single out for consideration two projection operators on the space of square-integrable functions: time-limiting and band-limiting. *Time-limiting* a function f produces a function Df which is f restricted to $|t| \leqq T/2$:

$$Df \equiv \begin{cases} f & \text{if} \quad |t| \leqq T/2 \\ 0 & \text{if} \quad |t| > T/2 \end{cases}.$$

We shall write $D_T f$ if the specific interval is important to the discussion. *Band-limiting* a function f produces a function Bf whose Fourier transform agrees with the Fourier transform of f for $|\omega| \leqq 2\pi W$, and vanishes for $|\omega| > 2\pi W$. If

$$F(\omega) = \int_{-\infty}^{\infty} f(s)\, e^{-i\omega s}\, ds,$$

$$Bf = \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} F(\omega)\, e^{i\omega t}\, d\omega,$$

or, in terms of f directly,

$$Bf = \frac{1}{\pi} \int_{-\infty}^{\infty} f(s) \frac{\sin 2\pi W(t-s)}{t-s}\, ds.$$

The subspace of functions f in $\mathfrak{L}^2$ which are already time-limited, i.e. for which $Df = f$, will be called $\mathfrak{D}$, and similarly band-limited functions, for which $Bf = f$, the subspace $\mathfrak{B}$. The observation made previously that a band-limited function which vanishes for $|t| > T/2$ must vanish identically may now be phrased as

$$\mathfrak{B} \cap \mathfrak{D} = \{0\}.$$

A major result in Ref. 1 was that there is actually a non-zero minimum angle between the spaces $\mathfrak{B}$ and $\mathfrak{D}$.

A doubly orthogonal system of band-limited functions ψ_n was investigated in Refs. 1 and 2, and a number of properties were derived. The following are important to our development:

Given any $T > 0$ and any $W > 0$, we can find a countably infinite set of real functions $\psi_0(t)$, $\psi_1(t)$, $\psi_2(t)$, $\cdots$, and a set of real positive numbers

$$\lambda_0 > \lambda_1 > \lambda_2 > \cdots ,$$

with the following properties:

(*i*) The $\psi_i(t)$ are band-limited, orthonormal on the real line, and complete in the space of square-integrable band-limited functions of bandwidth W cycles.

$$(\psi_i , \psi_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad i, j = 0,1,2, \ldots .$$

(*ii*) In the interval $-T/2 \leqq t \leqq T/2$, the functions $D\psi_i(t)$ are orthogonal and complete in the space of square-integrable functions vanishing for $|t| > T/2$.

$$(D\psi_i , D\psi_j) = \begin{cases} 0, & i \neq j \\ \lambda_i , & i = j \end{cases} \quad i, j = 0,1,2, \ldots .$$

(*iii*) For all values of t, real or complex,

$$\lambda_i \psi_i = BD\psi_i \left(= \int_{-T/2}^{T/2} \psi_i(s) \frac{\sin 2\pi W(t - s)}{\pi(t - s)} ds \right).$$

We shall write $\lambda_i(T)$ if the specific interval is important to the discussion.

We are now in a position to give an account of our results. We repeat our basic definition:

$E(\epsilon_T)$ is the set of functions $f(t) \in \mathfrak{L}^2$ such that

(1) $f \in \mathfrak{B}$
(2) $\| f \| = 1$
(3) $\| Df \|^2 = 1 - \epsilon_T^2$.

Let us turn to the approximate dimension of $E(\epsilon_T)$. As we pointed out above, the basis $\{\varphi_i\}_0^N$ which we wish to use is the one which minimizes (0.3), that is, which minimizes

$$\max_{f \in E(\epsilon_T)} \min_{\{a_i\}_0^N} \left\| f - \sum_0^N a_i \varphi_i \right\|^2 .$$

It seems reasonable that the best basis, for any given N, should be the $(N + 1)$ linearly independent most concentrated band-limited functions, and these are known, from previous work, to be $\psi_0, \cdots, \psi_N$. Although this seems to be harder to prove than one might expect, it is in fact true, and is the subject of

Theorem 1. For any fixed N, the functions $\psi_0, \cdots, \psi_N$ *achieve the minimum in*

$$\min_{\{\varphi_i\}_0^N} \max_{f \in E(\epsilon_T)} \min_{\{a_i\}_0^N} \left\| f - \sum_0^N a_i \varphi_i \right\|^2 .$$

Thus results on the approximation of $E(\epsilon_T)$ by linear combinations of a finite number of ψ_i are in fact best possible results on the approximate dimension of $E(\epsilon_T)$.

Theorem 3. Let $f(t) \in E(\epsilon_T)$. *Then**

$$\left\| f - \sum_0^{[2WT]} a_n \psi_n \right\|^2 \leqq C_1 \epsilon_T^2 ,$$

where the a_n *are the Fourier coefficients of* f *in its expansion in the* ψ*'s, and* C_1 *is independent of* f, ϵ_T, *and* $2WT$, *and may be taken as 12.*

Theorem 3 shows that $[2WT] + 1$ of the best basis functions for $E(\epsilon_T)$ suffice to approximate a concentrated function to a degree proportional to the "unconcentrated part" ϵ_T^2 of the energy. We shall see that this is no longer the case when we use the simpler sampling functions.

In Theorem 3, as we have said, C_1 may be taken as 12. What does it take to make C_1 very close to 1, that is, to make the approximation almost as good as the concentration? First of all, it is important to see that roughly $2WT$ functions are *not* enough to do this, and this is the subject of

Theorem 5. For any $\epsilon_T^2 < 0.915$, *there exists a function* $f \in E(\epsilon_T)$ *such that*

$$\inf_{a_i} \left\| f - \sum_0^{[2WT]-2} a_i \psi_i \right\|^2 \geqq C_2(\epsilon_T^2 - R(WT)),$$

where $C_2 > 1$ *and* $R(WT) \to 0$ *as* $WT \to \infty$. *Here* C_2 *may be taken as* $1/0.915$ *and* $R(WT)$ *as* $2\sqrt{2}e^{-\pi WT/2}$. *(If* $\epsilon_T^2 > 0.915$, *the right side should be replaced by 1.)*

By further analysis, this result may be strengthened so that it includes approximations by $[2WT] + N$ of the ψ_i functions, where N is any finite integer.

Theorem 8. For any given N and $\epsilon_T^2 < 0.916$, *and for* WT *sufficiently*

* $[x]$ means the largest integer $\leqq x$.

large, there will exist a function $f \in E(\epsilon_T)$ *such that*

$$\inf_{a_i} \left\| f - \sum_{0}^{[2WT]+N} a_i \psi_i \right\|^2 \geqq \frac{1}{0.916} (\epsilon_T^2 - 2\sqrt{2}\, e^{-\pi WT/2}).$$

(If $\epsilon_T^2 > 0.916$*, the right side should be replaced by 1.)*
Since, by Theorem 1, the ψ_i are the best approximating functions in $|t| \leqq T/2$, Theorems 7 and 8 hold, a fortiori, for any approximate basis $\{\varphi_i\}$.

What, then, does it take to bring the constant C of Theorem 3 arbitrarily close to 1? We do not know the best possible result, but there is considerable information in the following theorem, due to C. E. Shannon:

Theorem 4 (Shannon): Given any $\eta > 0$*, there exist constants* $C_3 = C_3(\eta)$ *and* $C_4 = C_4(\eta)$ *so that for* $f \in E(\epsilon_T)$,

$$\inf_{a_i} \left\| f - \sum_{0}^{[2WT]+C_3 \log^+ 2WT + C_4} a_i \psi_i \right\|^2 \leqq (1 + \eta)\epsilon_T^2 .*$$

Thus a number of functions boundedly more than $2WT$ cannot suffice for approximating $f \in E(\epsilon_T)$ to within $(1 + \eta)\,\epsilon_T^2$, but a logarithmically growing extra number of terms does.

Let us now turn to approximating $E(\epsilon_T)$ by sampling functions. The first result is that $[2WT] + 1$ sample functions will approximate f in energy roughly to within a constant times ϵ_T, that is, within a constant times the *square root* of the unconcentrated energy. The placement of the sample points depends on $2WT$, but of course not on the specific function.

Theorem 2. Let $f(t) \in E(\epsilon_T)$*. Then, if* $WT - [WT] \leqq \frac{1}{2}$,

$$\text{(a)} \qquad \left\| f - \sum_{|k| \leqq WT} f\left(\frac{k}{2W}\right) \frac{\sin \pi(2Wt - k)}{\pi(2Wt - k)} \right\|^2 \leqq \pi\epsilon_T + \epsilon_T^2 ,$$

and if $WT - [WT] > \frac{1}{2}$,

$$\text{(b)} \left\| f - \sum_{|k+\frac{1}{2}| \leqq WT} f\left(\frac{k + \frac{1}{2}}{2W}\right) \frac{\sin \pi(2Wt - k - \frac{1}{2})}{\pi(2Wt - k - \frac{1}{2})} \right\|^2 \leqq \pi\epsilon_T + \epsilon_T^2 .$$

An estimate valid for all WT *may be obtained by replacing* WT *in (a) by* $WT + 1$.
We note that the coefficients $f(k/2W)$ and $f(k + \frac{1}{2}/2W)$ are well-known to be the Fourier coefficients in the sampling series expansion, and hence the best constants to use.

* $\log^+ x = \max(\log x, 0)$.

This theorem is, in one sense, quite satisfactory because $\pi\epsilon_T + \epsilon_T^2$ does go to 0 as the unconcentrated part of the energy ϵ_T^2 goes to 0. On the other hand, $\pi\epsilon_T + \epsilon_T^2$ approaches 0 more slowly than ϵ_T^2 itself. That this estimate of the degree to which sampling functions approximate $E(\epsilon_T)$ cannot be too much improved is established in

Theorem 10. Let $f(t) \in E(\epsilon_T)$. Then an estimate of the form

$$\left\| f - \sum_{|k| \leqq WT+N} f\left(\frac{k}{2W}\right) \frac{\sin \pi(2Wt - k)}{\pi(2Wt - k)} \right\|^2 \leqq C\epsilon_T^2$$

cannot be valid independently of ϵ_T no matter how large the constants C and N are chosen.

Thus a sampling series approximation using ($2WT$ plus a constant) terms will not approximate every concentrated function to a degree proportional to the unconcentrated energy. As we have seen, this is in direct contrast to the theorem previously quoted for approximation with the best functions ψ_i. We also have the following negative result for approximation by sampling series to within $(1 + \eta)\epsilon_T^2$:

Theorem 11. For every $\beta < 1$, there exists $\delta > 0$, and ϵ_T such that

$$\left\| f - \sum_{|k| \leqq WT+(WT)^\beta} f\left(\frac{k}{2W}\right) \frac{\sin \pi(2Wt - k)}{\pi(2Wt - k)} \right\|^2 > (1 + \delta)\epsilon_T^2$$

for some $f \in E(\epsilon_T)$.

Once again, this is in direct contrast to the situation with the best functions ψ_i as given in Theorem 4.

We supposed near the beginning that $f(t)$ is actually band-limited. Suppose that it is only *almost* band-limited, that is, that

$$\frac{\int_{|\omega| \leqq 2\pi W} |F(\omega)|^2 \, d\omega}{\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega} = 1 - \eta_W^2 . \qquad (0.5)$$

It is interesting that our approximation theorems are stable in the sense that they continue to hold approximately for approximately band-limited functions. A sample is the following

Theorem 12. If $f(t) \in \mathfrak{L}^2$ with $\| f \| = 1$, and satisfies (0.1) and (0.5), then for some constants a_n we have

$$\left\| f - \sum_{0}^{[2WT]} a_n \psi_n \right\|^2 \leqq 12(\epsilon_T + \eta_W)^2 + \eta_W^2 .$$

An analogous result (Theorem 13) holds for a sampling approximation to f.

Before we proceed to the detailed exposition, let us mention one theorem, required for the proof of Theorem 10, which is of interest in its own right.

Theorem 9: When restricted to $t > 0$, the sample functions centered at the negative sample points are dense in $\mathfrak{L}^2(0, \infty)$, but those centered at the positive sample points are not dense in $\mathfrak{L}^2(0, \infty)$, nor even in $\mathfrak{B}$ restricted to $t > 0$. Specifically, given any square-integrable $f(t)$ we may find constants N and $a_n^{(N)}$ which make

$$\int_0^\infty \left| f(t) - \sum_{n=1}^{N} a_n^{(N)} \frac{\sin \pi(2Wt + n)}{\pi(2WT + n)} \right|^2 dt$$

as small as desired, but there exists a band-limited $g(t)$ for which

$$\int_0^\infty \left| g(t) - \sum_{n=1}^{N} b_n \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)} \right|^2 dt$$

cannot be made arbitrarily small regardless of the choice of N or b_n.

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III. DETAILED EXPOSITION

1. Given N functions $\varphi_0, \varphi_1, \cdots, \varphi_{N-1}$ in $\mathfrak{L}^2$, let us denote by S_φ^N the subspace spanned by them. The quantity $\min_{\{a_i\}} \| f - \sum_0^{N-1} a_i\varphi_i \|^2$ of (0.2) now represents the square of the distance $\rho(f, S_\varphi^N)$, measured in $\mathfrak{L}^2$, of f from S_φ^N. The number δ_N in (0.2) may therefore be taken to equal

$$\delta_N = \sup_{f \in E(\epsilon_T)} \rho(f, S_\varphi^N),$$

which, following the terminology of Ref. 4, we will call *the deflection of $E(\epsilon_T)$ from S_φ^N.*

We will first identify, for given T and N, that subspace of dimension N which best approximates $E(\epsilon_T)$, in the sense of minimizing this deflection.

Theorem 1: Let T be given. Then, for every N, the subspace spanned by the (orthonormal) functions $\psi_0, \cdots, \psi_{N-1}$ best approximates $E(\epsilon_T)$, in the sense that the deflection of $E(\epsilon_T)$ from that subspace is smaller than from any other subspace of dimension N.

Proof: We first compute the deflection of $E(\epsilon_T)$ from S_ψ^N. By definition, $f(t)$ is in $E(\epsilon_T)$ if and only if $f \in \mathcal{B}$, with $\| f \|^2 = 1$ and $\| Df \|^2 = 1 - \epsilon_T^2$; thus, expanding f in the complete orthonormal system $\{\psi_i\}_0^\infty$, if and only if

$$f = \sum_0^\infty \alpha_i \psi_i, \quad \text{with} \quad \sum_0^\infty |\alpha_i|^2 = 1 \quad \text{and} \quad \sum_0^\infty \lambda_i |\alpha_i|^2 = 1 - \epsilon_T^2.$$

Now by the orthonormality of the ψ_i,

$$\rho^2(f, S_\psi^N) = \min_{\{a_i\}} \left\| f - \sum_0^{N-1} a_i \psi_i \right\|^2 = \left\| \sum_N^\infty \alpha_i \psi_i \right\|^2 = \sum_N^\infty |\alpha_i|^2.$$

To find the deflection of $E(\epsilon_T)$ from S_ψ^N we therefore compute

$$\sup_{f \in E(\epsilon_T)} \rho(f, S_\psi^N),$$

equivalently $[\sup \sum_N^\infty |\alpha_i|^2]^{\frac{1}{2}}$ subject to the conditions $\sum_0^\infty |\alpha_i|^2 = 1$ and $\sum_0^\infty \lambda_i |\alpha_i|^2 = 1 - \epsilon_T^2 \leqq \lambda_0$. We find

$$\text{(1.1)} \qquad \begin{matrix}\text{deflection of} \\ E(\epsilon_T) \text{ from } S_\psi^N\end{matrix} = \begin{cases} 1, & 0 < 1 - \epsilon_T^2 \leqq \lambda_N \\ \left[\dfrac{\lambda_0 - (1 - \epsilon_T^2)}{\lambda_0 - \lambda_N}\right]^{\frac{1}{2}}, & \lambda_N < 1 - \epsilon_T^2 \leqq \lambda_0 \end{cases}.$$

Next suppose that $\varphi_0, \cdots, \varphi_{N-1}$ are any N given functions in $\mathcal{L}^2$. By the Pythagorean theorem, the distance of $f \in \mathcal{B}$ to any linear combination of the φ_i is no smaller than its distance to the same linear combination of the functions $B\varphi_i$, hence we may assume $\varphi_i \in \mathcal{B}$. As before, let S_φ^N be the subspace spanned by $\varphi_0, \cdots, \varphi_{N-1}$, and denote by P_φ the operation of projecting orthogonally onto S_φ^N; explicitly, $P_\varphi f$ is the element of S_φ^N closest to f. In terms of P_φ, the quantity of interest in (0.2) can therefore be written simply as

$$\text{(1.2)} \qquad \rho^2(f, S_\varphi^N) = \| f - P_\varphi f \|^2 = \| f \|^2 - \| P_\varphi f \|^2;$$

the last equality in (1.2) follows from the orthogonality of $P_\varphi f$ and $(f - P_\varphi f)$.

Now assign to every $f \in \mathcal{B}$ the point in the $x - y$ plane whose x and y coordinates are $\| Df \|^2 / \| f \|^2$ and $[\| f \|^2 - \| P_\varphi f \|^2] / \| f \|^2$ respectively; denote by R_T the set of points so obtained. The significance of this map is that it sends every f in $E(\epsilon_T)$ into the line $x = 1 - \epsilon_T^2$, with y-coordinate equaling $\rho^2(f, S_\varphi^N)$; hence we see that

$$\text{(1.3)} \qquad \text{deflection of } E(\epsilon_T) \text{ from } S_\psi^N = \left[\sup_{x = 1 - \epsilon_T^2} y \right]^{\frac{1}{2}}.$$

By previous results,[1] the x-coordinates of points in R_T satisfy $0 < x \leqq \lambda_0$; $x = \lambda_0$ is achieved only by the functions $k\psi_0(t)$, with k any constant. The y-coordinates of points in R_T satisfy $0 \leqq y \leqq 1$; $y = 1$ is achieved only by functions orthogonal to S_φ^N, equivalently to $\{\varphi_i\}_0^{N-1}$. Therefore, applying the Weyl-Courant lemma (Ref. 3, p. 238), we find

$$\sup_{y=1} x = \sup_{f \perp \{\varphi_i\}_0^{N-1}} \frac{\| Df \|^2}{\| f \|^2} \geqq \lambda_N .$$

Since there exist infinitely-dimensional subspaces of $\mathcal{B}$ over which $\| Df \|^2/\| f \|^2$ is arbitrarily small (for example those spanned by ψ_m, $\psi_{m+1}, \cdots$ for m sufficiently large), while S_φ^N is finite-dimensional, there are functions in those larger subspaces orthogonal to S_φ^N, and consequently $\inf_{y=1} x = 0$.

We show next that R_T is convex, equivalently that if P_1 and P_2 are two points in R_T, the line segment joining them is also contained in R_T. Let l be a line whose equation is $ax + by = c$. By definition of R_T, a function $f \in \mathcal{B}$ will be sent on a point of l if and only if

$$\frac{a \| Df \|^2 + b[\| f \|^2 - \| P_\varphi f \|^2]}{\| f \|^2} = c,$$

equivalently, if and only if $a(Df,Df) - b(P_\varphi f,P_\varphi f) = (c - b)(f,f)$, or, using (0.4) and the fact that $f = Bf$, if and only if

$$([aBDB - bP_\varphi] f,f) = (c - b)\,(f,f). \tag{1.4}$$

An operator is completely continuous[3] if it transforms every bounded sequence (i.e. a sequence of functions $\{f_n\}$ for which $\| f_n \| \leqq k$ with some k) into a sequence which possesses a subsequence converging in $\mathcal{L}^2$ norm. Since B is a projection, $\| Bf_n \| \leqq \| f_n \| \leqq k$. Writing $Bf_n(t)$ in terms of its Fourier transform $F_n(\omega)$ we obtain

$$Bf_n(t) = \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} F_n(\omega)\, e^{i\omega t}\, d\omega,$$

whence $Bf_n(t)$ is an entire function of the complex variable t. Since a function and its Fourier transform have the same $\mathcal{L}^2$ norm, Schwarz's inequality applied to this representation yields

$$| Bf_n(t) | \leqq c_1 e^{2\pi W | \mathrm{Im}\{t\} |}\, \| F_n \| \leqq c_1 k e^{2\pi W | \mathrm{Im}\{t\} |}, \tag{1.5}$$

so that the functions $Bf_n(t)$ are uniformly bounded on any compact set of the t-plane. Consequently (Ref. 5, p. 171), they form a normal family, and the sequence $Bf_n(t)$ possesses a subsequence $Bf_{n_k}(t)$ converging uniformly on any compact set of the t-plane, in particular on the interval $|t| \leqq T/2$ of the real t-axis. Therefore, the functions $DBf_{n_k}(t)$ converge in $\mathfrak{L}^2$ norm as well, whence, since B is bounded, so do the functions $BDBf_{n_k}$. We have established the complete continuity of BDB. Since S_φ^N is finite-dimensional, the projection P_φ is completely continuous. By (0.4), both operators are self-adjoint. Consequently, the operator $A = aBDB - bP_\varphi$, which takes $\mathfrak{B}$ into itself, is also self-adjoint and completely continuous. Therefore[3] it has a set of orthonormal eigenfunctions $\theta_k(t) \in \mathfrak{B}$ with corresponding eigenvalues μ_k, and every function $f \in \mathfrak{B}$ has an expansion of the form

$$f = h_f + \sum_{k=0}^{\infty} \alpha_k \theta_k , \tag{1.6}$$

where $Ah_f = 0$ or, equivalently, h_f is orthogonal to all the θ_k. Using this representation, condition (1.4) becomes

$$\sum |\alpha_k|^2 \mu_k = (c - b)[\| h_f \|^2 + \sum |\alpha_k|^2] \quad \text{or}$$

$$\sum_1^{\infty} |\alpha_k|^2 (c - b - \mu_k) + (c - b) \| h_f \|^2 = 0. \tag{1.7}$$

We now argue that this set of functions is connected. For suppose that $f = h_f + \sum \alpha_k \theta_k$ and $g = h_g + \sum \beta_k \theta_k$ are each of the form (1.6) and satisfy (1.7). For every $0 \leqq u \leqq 1$ define, for $k = 0,1, \cdots$

$$\gamma_k^{(u)} = +\sqrt{u |\alpha_k|^2 + (1 - u) |\beta_k|^2}\, e^{i[u \arg \alpha_k + (1-u) \arg \beta_k]},$$

$$h_u = \frac{uh_f + (1 - u)h_g}{\| uh_f + (1 - u)h_g \|} \sqrt{u \| h_f \|^2 + (1 - u) \| h_g \|^2},$$

and set

$$r_u = h_u + \sum_0^{\infty} \gamma_k^{(u)} \theta_k .$$

We see that $Ah_u = 0$, since h_u is a linear combination of h_f and h_g, so that r_u is of the form (1.6); it is easily seen to satisfy (1.7). But as u varies between 1 and 0, the functions r_u trace a connected path in $\mathfrak{B}$ between f and g. Consequently, those functions in $\mathfrak{B}$ which map into the line l form a connected set in $\mathfrak{B}$. Since the map from $\mathfrak{B}$ onto R_T is continuous, it takes this connected set into a connected set, that is into a single segment of l. Thus, the intersection of R_T with any line l is a single segment, whence R_T is convex.

Combined with the information already derived about the points in

R_T, the convexity of R_T implies that

$$\text{(1.8)}\qquad \begin{aligned} &\sup_{x=1-\epsilon_T^2} y = 1, && 0 < 1 - \epsilon_T^2 \leqq \lambda_N \\ &\sup_{x=1-\epsilon_T^2} y \geqq \frac{\lambda_0 - 1 + \epsilon_T^2}{\lambda_0 - \lambda_N}, && \lambda_N \leqq 1 - \epsilon_T^2 \leqq \lambda_0 . \end{aligned}$$

Combined with (1.3) and (1.1), (1.8) implies that deflection of $E(\epsilon_T)$ from $S_\varphi^N \geqq$ deflection of $E(\epsilon_T)$ from S_ψ^N. Theorem 1 is established.

We conclude from Theorem 1 that the quantity δ_N of (0.2), measuring the degree to which $E(\epsilon_T)$ is N-dimensional, may be taken to be equal to (1.1). Since, for $\lambda_N < 1 - \epsilon_T^2$,

$$\frac{\lambda_0 - (1 - \epsilon_T^2)}{\lambda_0 - \lambda_N} < \frac{\epsilon_T^2}{1 - \lambda_N},$$

and, for $\lambda_N \geqq 1 - \epsilon_T^2$,

$$1 \leqq \frac{\epsilon_T^2}{1 - \lambda_N}.$$

we find

$$\text{(1.9)}\qquad \delta_N < \frac{\epsilon_T}{\sqrt{\lambda_0 - \lambda_N}}.$$

Thus to establish an inequality of the form $\delta_k \leqq C\epsilon_T$ with C independent of T, it is sufficient to show that $\lambda_k(T)$ is bounded uniformly away from 1 independently of T. This will be done for $k = [2WT] + 1$ in Lemma 2, and for $k = [2WT] - N$, provided T is sufficiently large, in Theorem 8.1.

2. *Lemma 1. Let $f(s)$ be differentiable on $(-\infty, \infty)$. Then for any integers m and n, $m \leqq n$, and any $0 \leqq {}^{\alpha}_{\beta} \leqq 1$,*

$$f(m) + \cdots + f(n) = \int_{m-\alpha}^{n+\beta} f(s)\, ds + (\tfrac{1}{2} - \beta)\, f(n + \beta) + (\tfrac{1}{2} - \alpha)\, f(m - \alpha) + \int_{m-\alpha}^{n+\beta} (s - [s] - \tfrac{1}{2})\, f'(s)\, ds.$$

Proof: The standard form of the Euler Summation Formula (Ref. 6, p. 539) gives

$$f(m) + f(m+1) + \cdots + f(n) = \int_m^n f(s)\, ds + \tfrac{1}{2}f(n) + \tfrac{1}{2}f(m) + \int_m^n (s - [s] - \tfrac{1}{2})f'(s)\, ds.$$

Our result then follows if

$$0 = \int_{m-\alpha}^{m} f(s)\, ds + (\tfrac{1}{2} - \alpha) f(m - \alpha) - \tfrac{1}{2} f(m) + \int_{m-\alpha}^{m} (s - [s] - \tfrac{1}{2}) f'(s)\, ds,$$

and if

$$0 = \int_{n}^{n+\beta} f(s)\, ds + (\tfrac{1}{2} - \beta)\, f(n + \beta) - \tfrac{1}{2} f(n) + \int_{n}^{n+\beta} (s - [s] - \tfrac{1}{2})\, f'(s)\, ds.$$

Both follow immediately by partial integration on the last integrals, where $[s] = m - 1$ and n respectively. Lemma 1 is established.

We are now in a position to prove

Theorem 2. Let $g(t) \in E(\epsilon_T)$. Then if $WT - [WT] \leqq \frac{1}{2}$,

$$\text{(a)} \qquad \left\| g - \sum_{|k| \leqq WT} g\left(\frac{k}{2W}\right) \frac{\sin \pi(2Wt - k)}{\pi(2Wt - k)} \right\|^2 \leqq \pi\epsilon_T + \epsilon_T^2,$$

and if $WT - [WT] > \frac{1}{2}$,

$$\text{(b)} \qquad \left\| g - \sum_{|k+\frac{1}{2}| \leqq WT} g\left(\frac{k + \frac{1}{2}}{2W}\right) \frac{\sin \pi(2Wt - k - \frac{1}{2})}{\pi(2Wt - k - \frac{1}{2})} \right\|^2 \leqq \pi\epsilon_T + \epsilon_T^2.$$

An estimate valid for all WT may be obtained by replacing WT in (a) by $WT + 1$.

Proof: Without loss of generality, we assume $W = \frac{1}{2}$ for convenience. We apply Lemma 1 with $\beta = 0$ and f replaced by $| g |^2$. Then if $\alpha \leqq 1$,

$$| g^2(m) | + \cdots + | g^2(n) | = \int_{m-\alpha}^{n} | g^2(s) |\, ds + \tfrac{1}{2} | g^2(n) | + (\tfrac{1}{2} - \alpha) | g^2(m - \alpha) | + \int_{m-\alpha}^{n} (s - [s] - \tfrac{1}{2})\, 2\mathrm{Re}(g\bar{g}')\, ds.$$

It follows that if $\frac{1}{2} \leqq \alpha < 1$,

$$| g^2(m) | + \cdots + | g^2(n - 1) | \leqq \int_{m-\alpha}^{n} | g^2(s) |\, ds + \int_{m-\alpha}^{n} (s - [s] - \tfrac{1}{2})\, 2\mathrm{Re}(g\bar{g}')\, ds. \tag{2.1}$$

If $0 \leqq \alpha < \frac{1}{2}$, we set $f(s) = | g^2(s + \frac{1}{2}) |$ and $\alpha' = \alpha + \frac{1}{2}$. We have

$$| g^2(m + \tfrac{1}{2}) | + \cdots + | g^2(n + \tfrac{1}{2}) | = \int_{m-\alpha}^{n} | g^2(s + \tfrac{1}{2}) | \, ds + \tfrac{1}{2} | g^2(n + \tfrac{1}{2}) | + (\tfrac{1}{2} - \alpha') | g^2(m + \tfrac{1}{2} - \alpha') | + \int_{m-\alpha'}^{n} (s - [s] - \tfrac{1}{2}) \, 2\mathrm{Re}(g(s + \tfrac{1}{2})\bar{g}'(s + \tfrac{1}{2})) \, ds,$$

or

$$| g^2(m + \tfrac{1}{2}) | + \cdots + | g^2(n + \tfrac{1}{2}) | = \int_{m-\alpha}^{n+\frac{1}{2}} | g^2(u) | \, du + \tfrac{1}{2} | g^2(n + \tfrac{1}{2}) | - \alpha | g^2(m - \alpha) | + \int_{m-\alpha}^{n+\frac{1}{2}} (u - 1 - [u - \tfrac{1}{2}]) \, 2\mathrm{Re}(g(u)\bar{g}'(u)) \, du,$$

and

$$| g^2(m + \tfrac{1}{2}) | + \cdots + | g^2(n - \tfrac{1}{2}) | \leqq \int_{m-\alpha}^{n+\frac{1}{2}} | g^2(u) | \, du + \int_{m-\alpha}^{n+\frac{1}{2}} (u - 1 - [u - \tfrac{1}{2}]) \, 2\mathrm{Re}(g(u)\bar{g}'(u)) \, du. \tag{2.2}$$

If $\frac{1}{2} \leqq \alpha < 1$, we may apply (2.1) to $| g^2(t) |$ and $| g^2(-t) |$, and add the results. We obtain

$$\sum_{m \leqq |k| < n} | g^2(k) | \leqq \int_{m-\alpha \leqq |s| \leqq n} | g^2(s) | \, ds + \int_{m-\alpha \leqq |s| \leqq n} (s - [s] - \tfrac{1}{2}) \, 2\mathrm{Re}(g\bar{g}') \, ds.$$

Now $| 2(s - [s] - \frac{1}{2}) | \leqq 1$; hence

$$\left| \int (s - [s] - \tfrac{1}{2}) \, 2\mathrm{Re}(g\bar{g}') \, ds \right| \leqq \int | g | \, | g' | \, ds \leqq \sqrt{\int | g |^2 \, ds} \sqrt{\int | g' |^2 \, ds},$$

by the Schwarz inequality. But

$$\int_{m-\alpha \leqq |s| \leqq n} |g'|^2 \, ds < \int_{-\infty}^{\infty} |g'|^2 \, ds$$

$$\leqq \pi^2 \int_{-\infty}^{\infty} |g|^2 \, ds = \pi^2.$$

The last inequality holds because

$$g(t) = \int_{-\pi}^{\pi} G(x) \, e^{ixt} \, dx$$

$$g'(t) = \int_{-\pi}^{\pi} ixG(x) \, e^{ixt} \, dx$$

$$\int_{-\infty}^{\infty} |g'(t)|^2 \, dt = 2\pi \int_{-\pi}^{\pi} x^2 |G(x)|^2 \, dx$$

$$\leqq 2\pi \cdot \pi^2 \int_{-\pi}^{\pi} |G(x)|^2 \, dx$$

$$= \pi^2 \int_{-\infty}^{\infty} |g(t)|^2 \, dt.$$

Hence

$$\sum_{m \leqq |k| \leqq n} |g^2(k)| \leqq \int_{m-\alpha \leqq |s| \leqq n} |g^2(s)| \, ds + \pi \left[\int_{m-\alpha \leqq |s| \leqq n} |g^2(s)| \, ds \right]^{\frac{1}{2}}.$$

Now let $n \to \infty$; the preceding equation becomes

$$\sum_{m \leqq |k|} |g^2(k)| \leqq \epsilon_{2(m-\alpha)}^2 + \pi\epsilon_{2(m-\alpha)}.$$

Furthermore,

$$g(t) = \sum_{-\infty}^{\infty} g(k) \frac{\sin \pi(t-k)}{\pi(t-k)},$$

and the functions $\sin \pi(t-k)/\pi(t-k)$ are orthonormal. Hence,

$$\sum_{m \leqq |k|} g^2(k) = \left\| g(t) - \sum_{|k| < m} g(k) \frac{\sin \pi(t-k)}{\pi(t-k)} \right\|^2.$$

If we now set $m = [T/2] + 1$ and $m - \alpha = T/2$, then $\alpha \geqq \frac{1}{2}$ if

$$T/2 - [T/2] \leqq \tfrac{1}{2},$$

and we obtain (a).

Exactly the same argument, based on (2.2) rather than (2.1), gives the result (b) for the case $T/2 - [T/2] > \frac{1}{2}$.

If in (a), we use for $[T/2]$ the integer $m + 1$, then

$$\left\| g(t) - \sum_{|k| \leqq m+1} g(k) \frac{\sin \pi(t-k)}{\pi(t-k)} \right\|^2 \leqq \epsilon_{2(m+1)}^2 + \pi\epsilon_{2(m+1)},$$

and, since ϵ_α is monotone decreasing in α, the last statement of Theorem 2 follows.

Corollary 2.1. Let $g(t) \in E(\epsilon_T)$, and let $W = \frac{1}{2}$ for simplicity. If, in addition, $g(k) = 0$, $|k| \leqq T/2$, when $T/2 - [T/2] \leqq \frac{1}{2}$, or if $g(k + \frac{1}{2}) = 0$, $|k + \frac{1}{2}| \leqq T/2$, when $T/2 - [T/2] > \frac{1}{2}$, then

$$\| Dg \|^2 \leqq \pi\epsilon_T .$$

Proof: This follows immediately from substitution into Theorem 2(a) and (b) of the additional conditions on $g(t)$.

Notice that the number of points at which g is required to vanish is $[T] + 1$, except if $T/2 - [T/2] = \frac{1}{2}$, when it is one less.

Lemma 2. With the normalization of $W = \frac{1}{2}$, for any $T > 0$

$$\lambda_{[T]+1}(T) \leqq 0.915.$$

Proof: Let us consider a function of the form

$$f = \sum_{n=0}^{[T]+1} a_n \psi_n(t). \tag{2.3}$$

The series contains $[T] + 2$ coefficients to be determined; it is therefore possible to make f vanish at the (at most) $[T] + 1$ integer or half-integer points α_k of Corollary 2.1 without having f vanish identically. More precisely, we wish

$$\sum_{m=0}^{[T]+1} a_n \psi_n(\alpha_k) = 0, \qquad k = 0,1, \cdots, [T].$$

The rank of the matrix $\{\psi_n(\alpha_k)\}$, $n = 0,1, \cdots [T] + 1$, $k = 0, \cdots, [T]$, is at most $[T]$, and hence there exists a solution vector $\{a_n\}$ not all of whose elements vanish. We may then pick the a_n so that $\sum |a_n|^2 = 1$. We have thus found a function of the form (2.3) and of total energy one, which vanishes at the $[T] + 1$ points of the Corollary 2.1.

We know for this function that

$$\int_{|t| \leqq T/2} |f|^2 \, dt = \sum_0^{[T]+1} |a_n|^2 \lambda_n ,$$

$$\int_{|t| > T/2} |f|^2 \, dt = 1 - \sum_0^{[T]+1} |a_n|^2 \lambda_n$$

$$= \sum_0^{[T]+1} (1 - \lambda_n) |a_n|^2.$$

Since the λ_n are decreasing in n, we have, remembering $\sum |a_n|^2 = 1$,

$$\begin{aligned}\lambda_{[T]+1} &\leqq \sum_0^{[T]+1} |a_n|^2 \lambda_n \\ &\leqq \pi \sqrt{\sum_0^{[T]+1} (1 - \lambda_n) |a_n|^2} \text{ by Corollary 2.1,} \\ &\leqq \pi\sqrt{1 - \lambda_{[T]+1}}.\end{aligned}$$

Therefore $\lambda_{[T]+1}$ is bounded from 1, and is, in fact, no larger than the root of the equation

$$x = \pi\sqrt{1 - x},$$

which is

$$\frac{-\pi^2 + \sqrt{\pi^4 + 4\pi^2}}{2} = 0.915.$$

Lemma 2 is established.

3. *Theorem 3. Let $f(t) \in E(\epsilon_T)$. Then*

$$\left\| f - \sum_0^{[2WT]+1} a_n \psi_n \right\|^2 \leqq 12\, \epsilon_T^2,$$

where the a_n are the Fourier coefficients of f in its expansion in the functions ψ_n.

Proof: The quantity defined in the theorem represents the square of the distance from $f \in E(\epsilon_T)$ to the subspace $S_\psi^{[2WT]+1}$ spanned by the functions ψ_n, with $0 \leqq n \leqq [2WT]$. Thus, by definition, it does not exceed $\delta_{[2WT]+1}^2$, the square of the deflection of $E(\epsilon_T)$ from $S_\psi^{[2WT]+1}$. Combining (1.9) and Lemma 2 now yields

$$\left\| f - \sum_0^{[2WT]} a_n \psi_n \right\|^2 \leqq \frac{\epsilon_T^2}{1 - \lambda_{[2WT]+1}} \leqq \frac{\epsilon_T^2}{1 - 0.916} \leqq 12\epsilon_T^2.$$

Theorem 3 is established.

4. *Theorem 4 (Shannon). Given any $\eta > 0$, there exist constants $C_3 = C_3(\eta)$ and $C_4 = C_4(\eta)$ so that for $f \in E(\epsilon_T)$,*

$$\inf_{a_i} \left\| f - \sum_0^{[2WT]+C_3\log^+ 2WT+C_4} a_i \psi_i \right\|^2 \leqq (1 + \eta)\epsilon_T^2.$$

Proof: Using properties (*ii*) and (*iii*) of the eigenfunctions ψ_i, and known results (Ref. 3, p. 242), we obtain

$$\rho_1(t - s) = \frac{\sin 2\pi W(t - s)}{\pi(t - s)} = \sum_0^\infty \psi_i(s)\psi_i(t).$$

Therefore

$$\rho_1(0) = \sum_0^\infty \psi_i^2(t), \quad \text{and}$$

$$(4.1) \qquad \int_{-T/2}^{T/2} \rho_1(0)\, dt = 2WT = \sum_0^\infty \int_{-T/2}^{T/2} \psi_i^2(t)\, dt = \sum_0^\infty \lambda_i .$$

We now proceed to estimate $\sum_0^\infty \lambda_i^2$. The functions ψ_i satisfy the integral equation

$$\lambda_i \psi_i(t) = \int_{-T/2}^{T/2} \psi_i(s) \rho_1(t - s)\, ds.$$

Then

$$\lambda_i \int_{-T/2}^{T/2} \psi_i^2(t)\, dt = \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \rho_1(t - s) \psi_i(s) \psi_i(t)\, ds\, dt,$$

and if we sum on i, we obtain

$$\sum_{i=0}^\infty \lambda_i^2 = \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \rho_1^2(t - s)\, ds\, dt.$$

We now set $s' = 2s/T$, $t' = 2t/T$, $c = \pi WT$, and $\rho(u) = \sin cu/(\pi u)$. Then

$$\sum_{i=0}^\infty \lambda_i^2 = \int_{-1}^{1} \int_{-1}^{1} \rho^2(t' - s')\, ds'\, dt'$$

$$= \int_{-1}^{1} ds' \int_{-1-s'}^{1-s'} \rho^2(u)\, du.$$

Integration by parts, and the substitution $cu = x$, give

$$\sum_{i=0}^\infty \lambda_i^2 = \frac{4c}{\pi^2} \int_0^{2c} \frac{\sin^2 x}{x^2}\, dx - \frac{2}{\pi^2} \int_0^{2c} \frac{\sin^2 x}{x}\, dx.$$

Asymptotically for large c, this is easily seen to equal

$$\frac{2c}{\pi} - \frac{1}{\pi^2} \log c + 0(1),$$

but we desire an actual lower bound. For $c \geqq \pi/8$,

$$\sum_{i=0}^\infty \lambda_i^2 = \frac{2c}{\pi} - \frac{4c}{\pi^2} \int_{2c}^\infty \frac{\sin^2 x}{x^2}\, dx - \frac{2}{\pi^2} \int_0^{3\pi/4} \frac{\sin^2 x}{x}\, dx - \frac{2}{\pi^2} \int_{3\pi/4}^{2c} \frac{\sin^2 x}{x}\, dx.$$

Therefore,

$$\sum_{i=0}^{\infty} \lambda_i^2 \geqq \frac{2c}{\pi} - \frac{4c}{\pi^2} \int_{2c}^{\infty} \frac{dx}{x^2} - \frac{2}{\pi^2} \int_0^{3\pi/4} x\,dx - \frac{2}{\pi^2} \int_{3\pi/4}^{2c} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) \frac{dx}{x}$$

$$\geqq \frac{2c}{\pi} - \frac{2}{\pi^2} - \frac{9}{16} - \frac{1}{\pi^2} \log \frac{2c}{3\pi/4},$$

since

$$\int_{3\pi/4}^{2c} \frac{\cos 2x\,dx}{x} > 0 \quad \text{if} \quad c \geqq \pi/8$$

Thus

$$\sum_0^{\infty} \lambda_i^2 \geqq \frac{2c}{\pi} - \frac{1}{\pi^2} \log^+ c - 1 \tag{4.2}$$

for all c, since the inequality is trivially true for $c < \pi/8$.

Let us now introduce the following combinatorial problem. We consider infinite sequences of non-negative numbers μ_j such that

(a) $1 \geqq \mu_0 \geqq \mu_1 \geqq \cdots$,

(b) $\sum_0^{\infty} \mu_j = A$, a given positive constant,

(c) for a given integer $m \geqq A$, μ_m has a prescribed value,

and we seek to maximize $\sum_0^{\infty} \mu_j^2$ over all such sequences. Clearly the optimum $\{\mu_j\}$ will have $\mu_j = 0$ if $j > m$.

We claim that, with the possible exception of one μ_j, all the others in the optimum solution equal either 1 or μ_m. For suppose they do not, i.e. suppose $\{\mu_n\}$ takes on two values α and β such that $\mu_m < \alpha < \beta < 1$. If we now vary α and β between the limits μ_m and 1, keeping $\alpha + \beta$ a constant, and maximize $\alpha^2 + \beta^2$, we find an end-point maximum. In detail, if $\alpha + \beta = s$, then $\alpha^2 + \beta^2 = 2\,[(\alpha - s/2)^2 + s^2/4]$, which is maximized at an end-point value of α. Thus, the maximizing sequence $\{\mu_n\}_0^m$ can contain only one value which is neither 1 nor μ_m. This odd value is due to "breakage" in obtaining the exact total A. Let the maximizing sequence have k "$1's$", $(m - k)$ "$\mu_m's$" and one value α, $\mu_m < \alpha < 1$. Then

$$k + (m - k)\,\mu_m + \alpha = A,$$

so that

$$k = \frac{A - \alpha - m\mu_m}{1 - \mu_m}.$$

Then

$$\sum_0^m \mu_j^2 = \frac{A - \alpha - m\mu_m}{1 - \mu_m} + \frac{m - A + \alpha}{1 - \mu_m}\mu_m^2 + \alpha^2 \tag{4.3}$$
$$= (A - \alpha)(1 + \mu_m) - m\mu_m + \alpha^2.$$

This is the maximum achievable value of $\sum_0^\infty \mu_j^2$ under the conditions (a), (b) and (c) above. But with $A = 2c/\pi$, and λ_m given, $m \geqq 2c/\pi$, the sequence of eigenvalues λ_j satisfies the above conditions. It therefore competes for the maximum, and hence

$$\frac{2c}{\pi} - \frac{1}{\pi^2}\log^+ c - 1 \leqq \sum_0^\infty \lambda_i^2 \leqq \frac{2c}{\pi}(1 + \lambda_m) - m\lambda_m .$$

Thus, for any $m \geqq 2c/\pi$,

$$\lambda_m \leqq \frac{\dfrac{\log^+ c}{\pi^2} + 1}{m - \dfrac{2c}{\pi}}.$$

For any given $\eta > 0$, if

$$m \geqq \frac{2c}{\pi} + \frac{12}{\eta}\left(\frac{\log^+ c}{\pi^2} + 1\right) \tag{4.4}$$

it follows that $\lambda_m \leqq \eta/12$. Then, by the reasoning of Theorem 3,

$$\inf_{a_i} \left\| f - \sum_0^m a_n\psi_n \right\|^2 \leqq \frac{\epsilon_T^2}{1 - \eta/12}.$$

If $\eta \leqq 11$, this implies

$$\inf_{a_i} \left\| f - \sum_0^m a_n\psi_n \right\|^2 \leqq (1 + \eta)\epsilon_T^2;$$

larger values of η are covered by Theorem 3. Theorem 4 is proved.

Note: If only small values of η are of interest, the "12" in (4.4) is of course unnecessarily large.

Lemma 3: With the normalization of $W = \frac{1}{2}$, we have for any $T > 1$,

$$\lambda_{[T]-1}(T) \geqq 0.085.$$

Proof: We begin, again, with Lemma 1. If we consider first the case $T/2 - [T/2] \geqq \frac{1}{2}$, we let $f(s) = | g^2(s) |$, and

(a) $m = 1, \quad \alpha = \dfrac{1}{2}, \quad n = \left[\dfrac{T}{2}\right], \quad \beta = \dfrac{T}{2} - \left[\dfrac{T}{2}\right];$

(b) $m = -\left[\dfrac{T}{2}\right] - 1, \quad \alpha = -\dfrac{T}{2} - \left[\dfrac{T}{2}\right], \quad n = 0, \quad \beta = \dfrac{1}{2}.$

We obtain

$$| g^2(1) | + \cdots | g^2([T/2]) | \leqq \int_{\frac{1}{2}}^{T/2} | g^2(s) | \, ds + \int_{\frac{1}{2}}^{T/2} (s - [s] - \tfrac{1}{2}) \, 2\mathrm{Re}(g\bar{g}') \, ds$$

and

$$| g^2(-[T/2]) | + \cdots + | g^2(0) | \leqq \int_{-T/2}^{\frac{1}{2}} | g^2(s) | \, ds + \int_{-T/2}^{\frac{1}{2}} (s - [s] - \tfrac{1}{2}) \, 2\mathrm{Re}(g\bar{g}') \, ds.$$

Adding and applying the Schwarz inequality, we find, as in Theorem 2,

$$\sum_{|n| \leqq [T/2]} | g^2(n) | \leqq \| Dg \|^2 + \pi \| Dg \| \| g \|. \tag{4.5}$$

The Weyl-Courant lemma (Ref. 3, p. 238) asserts that

$$\lambda_n = \inf_{A_n} \sup_{\varphi \perp A_n} \frac{\| D\varphi \|^2}{\| \varphi \|^2},$$

where A_n ranges over all n-dimensional subspaces of $\mathcal{B}$. If B_{n+1} is an $(n + 1)$-dimensional subspace of $\mathcal{B}$, the orthogonal complement of every A_n must have at least one vector in common with B_{n+1}. Thus

$$\sup_{\varphi \perp A_n} \frac{\| D\varphi \|^2}{\| \varphi \|^2} \geqq \inf_{\varphi \in B_{n+1}} \frac{\| D\varphi \|^2}{\| \varphi \|^2},$$

and since the right-hand side of the inequality is independent of A_n, the Weyl-Courant lemma implies

$$\lambda_n \geqq \inf_{\varphi \in B_{n+1}} \frac{\| D\varphi \|^2}{\| \varphi \|^2}. \tag{4.6}$$

Now let $B_{[T]}$ be the subspace of $\mathcal{B}$ spanned by the $[T]$ (orthonormal) functions $[\sin \pi(t - k)]/\pi(t - k)$, $| k | \leqq [T/2]$. For $g \in B_{[T]}$ we have

$$\| g \|^2 = \sum_{|n| \leqq [T/2]} | g(n) |^2,$$

since $g(n) = 0$ when $| n | > [T/2]$, so that (4.5) yields

$$1 \leqq \frac{\| Dg \|^2}{\| g \|^2} + \pi \frac{\| Dg \|}{\| g \|}.$$

Letting g vary in $B_{[T]}$, and using (4.6), we now find

$$1 \leqq \inf_{g \in B[T]} \frac{\| Dg \|^2}{\| g \|^2} + \pi \frac{\| Dg \|}{\| g \|} = \inf_{g \in B[T]} \frac{\| Dg \|^2}{\| g \|^2} + \pi \inf_{g \in B[T]} \frac{\| Dg \|}{\| g \|}$$
$$\leqq \lambda_{[T]-1} + \pi \sqrt{\lambda_{[T]-1}},$$

whence

$$\lambda_{[T]-1} \geqq 0.085.$$

Similarly, if $T/2 - [T/2] < \frac{1}{2}$,

$$\sum_{|n+\frac{1}{2}| \leqq [T/2]} | g(n + \tfrac{1}{2}) |^2 \leqq \| Dg \|^2 + \pi \| Dg \| \| g \|,$$

and letting $B_{[T]}$ be the subspace of $\mathcal{B}$ spanned by the $[T]$ functions $[\sin \pi(t - k - \frac{1}{2})]/[\pi(t - k - \frac{1}{2})]$ with $| k + \frac{1}{2} | \leqq [T/2]$ we may apply the identical argument to find $\lambda_{[T]-1} \geqq 0.085$, as before. Lemma 3 is established.

Lemma 4: For any $WT > 0$,

$$\lambda_0 > 1 - 2\sqrt{2}e^{-\pi WT/2}.$$

Proof: For convenience, let $\Omega = 2\pi W$, and normalize so that $T = 2$. Consider the function $f(t)$ whose Fourier Transform $F(x)$ is given by

$$F(x) = \begin{cases} \dfrac{1}{(\Omega\pi)^{\frac{1}{4}}} e^{-x^2/2\Omega} & \text{if } | x | \leqq \Omega \\ 0 & \text{if } | x | > \Omega. \end{cases}$$

Then

$$\int_{-\infty}^{\infty} | f(t) |^2 \, dt = 2\pi \int_{-\Omega}^{\Omega} F^2(x) \, dx = 4\sqrt{\pi} \int_0^{\sqrt{\Omega}} e^{-u^2} \, du.$$

On the other hand,

$$f(t) = \int_{-\Omega}^{\Omega} F(x) \cos xt \, dx$$

$$= \frac{2}{(\pi\Omega)^{\frac{1}{4}}} \left[\sqrt{\frac{\pi\Omega}{2}} e^{-t^2\Omega/2} - \int_{\Omega}^{\infty} e^{-x^2/2\Omega} \cos xt \, dx \right] \tag{4.7}$$

$$\geqq \frac{2}{(\pi\Omega)^{\frac{1}{4}}} \left[\sqrt{\frac{\pi\Omega}{2}} e^{-t^2\Omega/2} - \int_{\Omega}^{\infty} e^{-x^2/2\Omega} \, dx \right].$$

It is easy to check that the expression in brackets is non-negative for $t = 1$, and hence for $|t| \leqq 1$. In fact,

$$\int_{\Omega}^{\infty} e^{-x^2/2\Omega}\, dx = \sqrt{\Omega} \int_{\sqrt{\Omega}}^{\infty} e^{-u^2/2}\, du,$$

and

$$G(\Omega) = \sqrt{\frac{\pi}{2}}\, e^{-\Omega/2} - \int_{\sqrt{\Omega}}^{\infty} e^{-u^2/2}\, du$$

is non-negative since it equals 0 at both $\Omega = 0$ and $\Omega = \infty$, and

$$G'(\Omega) = e^{-\Omega/2}\left(\frac{1}{2\sqrt{\Omega}} - \frac{\sqrt{\pi}}{2\sqrt{2}}\right)$$

is positive for $\Omega < 2/\pi$ and negative for $\Omega > 2/\pi$. Thus

$$\int_{-1}^{1} f^2(t)\, dt > 4\sqrt{\pi} \int_0^{\sqrt{\Omega}} e^{-u^2}\, du - 8\sqrt{2} \int_{\sqrt{\Omega}}^{\infty} e^{-u^2/2}\, du \int_0^{\sqrt{\Omega}} e^{-u^2/2}\, du.$$

Hence, from (4.7),

$$\frac{\int_{-1}^{1} f^2(t)\, dt}{\int_{-\infty}^{\infty} f^2(t)\, dt} > 1 - 2\sqrt{\frac{2}{\pi}} \int_{\sqrt{\Omega}}^{\infty} e^{-u^2/2}\, du \left[\frac{\int_0^{\sqrt{\Omega}} e^{-u^2/2}\, du}{\int_0^{\sqrt{\Omega}} e^{-u^2}\, du}\right].$$

But

$$\int_{\sqrt{\Omega}}^{\infty} e^{-u^2/2}\, du < \sqrt{\frac{\pi}{2}}\, e^{-\Omega/2} \text{ because } G(\Omega) \geqq 0,$$

and the expression in parentheses is bounded by $\sqrt{2}$. Thus

$$\frac{\int_{-1}^{1} f^2(t)\, dt}{\int_{-\infty}^{\infty} f^2(t)\, dt} > 1 - 2\sqrt{2}\, e^{-\Omega/2}.$$

But $f \in \mathcal{B}$ by definition, and hence competes in the maximum problem[2] which defines λ_0. Hence

$$\lambda_0 = \max_{\mathcal{B}} \frac{\int_{-1}^{1} |f(t)|^2\, dt}{\int_{-\infty}^{\infty} |f(t)|^2\, dt} > 1 - 2\sqrt{2}\, e^{-\Omega/2}.$$

Lemma 4 is established.

5. *Theorem 5. For any* $\epsilon_T^2 < 0.915$, *there exists a function* $f \in E(\epsilon_T)$ *such that*

$$\inf_{a_i} \left\| f - \sum_0^{[2WT]-2} a_i \psi_i \right\|^2 \geqq \frac{1}{0.915} (\epsilon_T^2 - 2\sqrt{2}\, e^{-\pi WT/2}).$$

(If $\epsilon_T^2 \geqq 0.915$, *the right-hand side of the inequality should be replaced by 1.)*

Proof: Theorem 5 asserts the existence of a lower bound for the deflection of $E(\epsilon_T)$ from the subspace $S_\psi^{[2WT]-1}$, spanned by the functions ψ_k, with $0 \leqq k \leqq [2WT] - 2$. This deflection has already been calculated in (1.1) and is easily seen to be assumed by a function in $E(\epsilon_T)$. Thus there exists $f \in E(\epsilon_T)$ such that

$$\inf_{a_i} \left\| f - \sum_0^{[2WT]-2} a_i \psi_i \right\|^2 = \min\left[1, \frac{\lambda_0 - (1 - \epsilon_T^2)}{\lambda_0 - \lambda_{[2WT]-1}}\right].$$

By Lemma 3, $\lambda_{[2WT]-1} \geqq 0.085$; this ensures that when $\epsilon_T^2 \geqq 0.915$ the smaller of the terms is 1. For other values of ϵ_T^2, since

$$\lambda_0 > 1 - 2\sqrt{2}\, e^{-\pi WT/2}$$

by Lemma 4, and $\lambda_0 < 1$, we find

$$\min\left(1, \frac{\lambda_0 - 1 + \epsilon_T^2}{\lambda_0 - \lambda_{[2WT]-1}}\right) \geqq \min\left[1, \frac{1}{0.915} (\epsilon_T^2 - 2\sqrt{2}\, e^{-\pi WT/2})\right],$$

and now the second of the bracketed terms is the smaller. Theorem 5 is established.

6. In $\mathfrak{L}^2(-\infty, \infty)$ let D' denote the operation of projecting onto $[0, \infty]$, that is

$$D'f(t) = \begin{cases} f(t) & t \geqq 0 \\ 0 & t < 0. \end{cases}$$

Arguing as with DBD in the proof of Theorem 1 we see that $D'BD'$, which takes $\mathfrak{L}^2(0, \infty)$ into itself, is self-adjoint, positive, and bounded by 1 (though no longer completely continuous). It therefore has a spectrum[3] contained in the unit interval; we will show that its spectrum consists of all $0 \leqq \lambda \leqq 1$.

Theorem 6. The $\mathfrak{L}^2$ *spectrum of the operator*

$$D'BD'f = \frac{1}{\pi} \int_0^\infty \frac{\sin (x - y)}{x - y} f(y)\, dy, \qquad x \geqq 0,$$

consists of all $0 \leqq \lambda \leqq 1$.

Proof: Theorem 6 follows immediately as a special case of much more general results of H. Widom[7] and M. Rosenblum (unpublished), which determine the spectra in $\mathfrak{L}^2$ of Wiener-Hopf equations with kernels whose Fourier transforms are bounded. We include a separate proof only because it is constructive.

By definition, λ is in the spectrum of an operator A if and only if for every $\epsilon > 0$ there exists φ_ϵ such that

$$\frac{\| A\varphi_\epsilon - \lambda\varphi_\epsilon \|}{\| \varphi_\epsilon \|} < \epsilon. \tag{6.1}$$

We will prove the theorem by constructing functions which satisfy (6.1) for any given $0 < \lambda < 1$. The spectrum being a closed set, it must then include all $0 \leqq \lambda \leqq 1$, but by the introductory remarks it is also contained in the closed unit interval, hence it consists of precisely the points $0 \leqq \lambda \leqq 1$.

Lemma 5: Let $\mu > 0$ be given. Then corresponding to any $\delta > 0$ there exists a function $H_\delta(z)$ satisfying

a. $H_\delta(z)$ *is analytic in* $|z| < 1$, *continuous in* $|z| \leqq 1$,

b. $H_\delta(0) = 0$,

c. $$\frac{\int_0^\pi | \mu H_\delta(e^{i\theta}) + H_\delta(e^{-i\theta}) |^2 \sin\theta \, d\theta}{\int_0^\pi | H_\delta(e^{i\theta}) |^2 \sin\theta \, d\theta} < \delta.$$

Proof: Suppose $0 < \alpha < \pi/2$. Denote in the z-plane by $P_1, P_2, P_3, P_4, P_5, P_6$ the points $1, e^{i\alpha}, -e^{-i\alpha}, -1, 0$, and i respectively, and let γ_1, γ_2 represent respectively the arcs P_1P_2, P_3P_4 of the unit circle. Let $w = P_0(z)$ be a conformal map of the upper half of the unit disc onto the region in the w-plane defined by $1 < |w| < q < \infty$, $\text{Im}\{w\} > 0$, which takes the points P_1, P_2, P_3, P_4, onto $w = 1$, $w = q$, $w = -q$, $w = -1$ respectively. The required map exists as soon as q is chosen appropriately (for example, so as to make the extremal length of the family of curves joining γ_1 to γ_2 in the upper semicircle equal to the extremal length of the family of curves joining the two segments of the real axis in the image domain), and it defines q uniquely. Now by reflection, $P_0(z)$ is extendable across the diameter of the unit circle to a map of $|z| < 1$ onto the domain $1/q < |w| < q$, $\text{Im}\{w\} > 0$, and satisfies

$$P_0(e^{i\theta})/q^2 = P_0(e^{-i\theta}) \qquad \alpha < \theta < \pi - \alpha$$

$$| P_0(e^{i\theta}) | = q \qquad \alpha < \theta < \pi - \alpha$$

$$1/q < | P_0(e^{i\theta}) |, | P_0(e^{-i\theta}) | < q \qquad e^{i\theta} \in \gamma_1, \gamma_2 .$$

Choose r so that $\mu = q^{-2r}$, and let $P(z) \equiv [qP_0(z)]^r$. Since $P_0(z)$ is bounded away from zero and infinity in $|z| \leqq 1$, the function $P(z)$ is also analytic in $|z| < 1$, continuous in $|z| \leqq 1$ (though no longer necessarily univalent) and satisfies

$$\mu P(e^{i\theta}) = P(e^{-i\theta}), \qquad \alpha < \theta < \pi - \alpha$$

$$|P(e^{i\theta})| = 1/\mu, \qquad \alpha < \theta < \pi - \alpha$$

$$m_\mu = \min(1/\mu,1) < |P(e^{i\theta})|, |P(e^{-i\theta})| < \max(1/\mu,1) = M_\mu,$$

$$e^{i\theta} \in \gamma_1, \gamma_2 .$$

Next let $w = Q(z)$ map the region defined by $|z| < 1$, $\mathrm{Im}\{z\} > 0$, $\mathrm{Re}\{z\} > 0$ onto itself, taking the points P_5, P_1, P_2 onto $w = 0$, $w = 1$, and $w = i$ respectively. $Q(z)$ may be constructed from elementary maps and is given explicitly by

$$Q^2(z) \equiv \frac{\sqrt{\cos^2\alpha + \left(\frac{1+z^2}{1-z^2}\right)^2 \sin^2\alpha} - 1}{\sqrt{\cos^2\alpha + \left(\frac{1+z^2}{1-z^2}\right)^2 \sin^2\alpha} + 1}.$$

It may be extended by reflection to yield a map of $|z| < 1$ onto the domain in the w-plane formed by cutting the unit circle along the imaginary axis from $i[\sin\alpha/(1 + \cos\alpha)]$ to i and from $-i[\sin\alpha/(1 + \cos\alpha)]$ to $-i$. It satisfies

$$Q(0) = 0,$$

$$Q(e^{i\theta}) = -Q(e^{-i\theta}), \qquad \alpha < \theta < \pi - \alpha$$

$$|Q(e^{i\theta})| = |Q(e^{-i\theta})| = 1, \qquad e^{i\theta} \in \gamma_1, \gamma_2 .$$

Now form $H(z) \equiv P(z)Q(z)$. We see that $H(z)$ satisfies conditions (a) and (b) of Lemma 5. Furthermore, by definition of H,

$$\mu H(e^{i\theta}) + H(e^{-i\theta}) = 0, \qquad \alpha < \theta < \pi - \alpha$$

$$|H(e^{i\theta})| = \frac{1}{\mu}|Q(e^{i\theta})|, \qquad \alpha < \theta < \pi - \alpha$$

$$m_\mu < |H(e^{i\theta})|, |H(e^{-i\theta})| < M_\mu, \qquad e^{i\theta} \in \gamma_1, \gamma_2 .$$

Thus

$$\begin{aligned}(6.2)\qquad &\int_0^\pi |\mu H(e^{i\theta}) + H(e^{-i\theta})|^2 \sin\theta\, d\theta \\ &= \int_0^\alpha + \int_{\pi-\alpha}^\alpha |\mu H(e^{i\theta}) + H(e^{-i\theta})|^2 \sin\theta\, d\theta \\ &\leqq 2(\mu+1)^2 M_\mu^2 \int_0^\alpha \sin\theta\, d\theta = 2(\mu+1)^2 M_\mu^2 (1 - \cos\alpha).\end{aligned}$$

$$\begin{aligned}(6.3)\qquad &\int_0^\pi |H(e^{i\theta})|^2 \sin\theta\, d\theta > \int_\alpha^{\pi-\alpha} |H(e^{i\theta})|^2 \sin\theta\, d\theta \\ &= \frac{1}{\mu^2}\int_\alpha^{\pi-\alpha} |Q(e^{i\theta})|^2 \sin\theta\, d\theta = \frac{2}{\mu^2}\int_\alpha^{\pi/2} |Q(e^{i\theta})|^2 \sin\theta\, d\theta.\end{aligned}$$

Using the expression for $Q(z)$ we find, for $\alpha < \theta < \pi - \alpha$,

$$|Q(e^{i\theta})|^2 = \left| \frac{\sqrt{\cos^2\alpha - \dfrac{\cos^2\theta \sin^2\alpha}{\sin^2\theta}} - 1}{\sqrt{\cos^2\alpha - \dfrac{\cos^2\theta\sin^2\alpha}{\sin^2\theta}} + 1} \right| = \left| \frac{\sqrt{1 - \dfrac{\sin^2\alpha}{\sin^2\theta}} - 1}{\sqrt{1 - \dfrac{\sin^2\alpha}{\sin^2\theta}} + 1} \right|$$

$$= \frac{\sin^2\theta}{\sin^2\alpha} \left| 1 - \sqrt{1 - \frac{\sin^2\alpha}{\sin^2\theta}} \right|^2 \geqq \frac{1}{4}\frac{\sin^2\alpha}{\sin^2\theta}.$$

Introducing this into (6.3) yields

$$\int_0^\pi |H(e^{i\theta})|^2 \sin\theta\, d\theta > \frac{1}{2\mu^2} \sin^2\alpha \log\left(\frac{1+\cos\alpha}{\sin\alpha}\right),$$

whence, by (6.2),

$$\frac{\displaystyle\int_0^\pi |\mu H(e^{i\theta}) + H(e^{-i\theta})|^2 \sin\theta\, d\theta}{\displaystyle\int_0^\pi |H(e^{i\theta})|^2 \sin\theta\, d\theta} < \frac{K_\mu}{\log\csc\alpha},$$

where K_μ depends only on μ. Thus, if α is chosen sufficiently small, $H(z)$ satisfies the remaining condition (c). Lemma 5 is established.

We now pass to the construction of the functions φ_ϵ of (6.1). Given $0 < \lambda < 1$ and $\epsilon > 0$, set $0 < \mu = (1 - \lambda)/\lambda$, choose δ so small that that $\sqrt{\delta}/(1 - \sqrt{\delta}) < \epsilon$, and let $H_\delta(z) = H(z)$ be the function of Lemma 5 corresponding to δ.

Introduce the map $u + iv = w = \frac{1}{2}(z + 1/z)$, taking $|z| < 1$ onto the w-plane slit along the real axis from $w = -1$ to $w = 1$, and in that region define $F(w) \equiv H(z)$. The function $F(w)$ is then analytic except on the slit. If $F_1(w)$ denotes $F(w)$ in the upper half-plane, $F_1(w)$ is continuous in the closed half plane $v \geqq 0$. If $v > 0$,

$$\int_{-\infty}^{\infty} |F_1(u + iv)|^2\, du = \frac{1}{2} \int_{\Gamma_v} |H(z)|^2 \left| 1 - \frac{1}{z^2} \right| |dz|, \tag{6.4}$$

where Γ_v is the curve in the upper half of the unit circle defined in polar coordinates by $(r - 1/r) \sin \theta = 2v$. Since $H(0) = 0$, the function $[H(z)/z]\sqrt{1 - z^2}$ is analytic in $|z| < 1$, continuous in $|z| \leqq 1$, and by the maximum principle

$$\left| \frac{H(z)}{z} \sqrt{1 - z^2} \right| \leqq \sup_{|z|=1} \left| \frac{H(z)}{z} \sqrt{1 - z^2} \right| \leqq \sup_{|z|=1} |H(z)|.$$

By property (a) of Lemma 5, $H(z)$ is bounded in $|z| \leqq 1$, hence so is the integrand on the right-hand side of (6.4). Since the curves Γ_v have lengths bounded independently of v, it follows that

$$\int_{-\infty}^{\infty} |F_1(u + iv)|^2\, du < c, \qquad v > 0.$$

Consequently, by a theorem of Paley-Wiener,[8] $F_1(w)$ coincides in $v \geqq 0$ with the Fourier transform of a function $\psi_1(t) \in \mathfrak{L}^2$ which vanishes for $t \geqq 0$. Letting $F_2(w)$ denote $F(w)$ in the lower half plane, the identical argument establishes that $F_2(w)$ coincides in $v \leqq 0$ with the Fourier transform of a function $\psi_2(t) \in \mathfrak{L}^2$ which vanishes for $t \leqq 0$. Let $\varphi_1(t) \equiv (1/\lambda)\, \psi_1(t)$, $\varphi_2(t) = -(1/\lambda)\psi_2(t)$, and $\varphi(t) = \varphi_1(t) + \varphi_2(t)$. Then with $\chi(u)$ the characteristic function of the interval $-1 \leqq u \leqq 1$, using the norm-preserving property of the Fourier transform, we have

$$\begin{aligned} \| BD'\varphi - \lambda\varphi \|^2 &= \| (B - \lambda)\varphi_2 - \lambda\varphi_1 \|^2 \\ &= \int_{-\infty}^{\infty} \left| [\chi(u) - \lambda] \frac{F_2(u)}{\lambda} + F_1(u) \right|^2 du \\ &= \int_{|u|>1} |F_1(u) - F_2(u)|^2\, du \\ &\quad + \int_{-1}^{1} |\mu F_2(u) + F_1(u)|^2\, du \\ &= \int_0^{\pi} |\mu H(e^{i\theta}) + H(e^{-i\theta})|^2 \sin\theta\, d\theta, \end{aligned} \tag{6.5}$$

while

$$\| \varphi \|^2 \geqq \| \varphi_2 \|^2 = \frac{1}{\lambda^2} \| \psi_2 \|^2$$

$$(6.6) \qquad = \frac{1}{\lambda^2} \int_{-\infty}^{\infty} | F_2(u) |^2 \, du > \frac{1}{\lambda^2} \int_{-1}^{1} | F_2(u) |^2 \, du$$

$$= \frac{1}{\lambda^2} \int_0^{\pi} | H(e^{i\theta}) |^2 \sin \theta \, d\theta.$$

Thus, combining (6.5) and (6.6),

$$(6.7) \qquad \frac{\| BD'\varphi - \lambda\varphi \|^2}{\| \varphi \|^2} < \lambda^2 \frac{\int_0^{\pi} | \mu H(e^{i\theta}) + H(e^{-i\theta}) |^2 \sin \theta \, d\theta}{\int_0^{\pi} | H(e^{i\theta}) |^2 \sin \theta \, d\theta} < \lambda^2 \delta.$$

From (6.7),

$$(6.8) \qquad \| D'\varphi \| \geqq \| BD'\varphi \| \geqq \lambda(1 - \sqrt{\delta}) \| \varphi \|,$$

and

$$(6.9) \qquad \| D'BD'\varphi - \lambda \, D'\varphi \| = \| D'(BD'\varphi - \lambda\varphi) \| \leqq \lambda\sqrt{\delta} \, \| \varphi \|.$$

Setting $\varphi_\epsilon = D'\varphi$ and combining (6.8) and (6.9) we obtain

$$\| D'BD'\varphi_\epsilon - \lambda\varphi_\epsilon \| / \| \varphi_\epsilon \| \leqq \sqrt{\delta}/(1 - \sqrt{\delta}) < \epsilon,$$

which is the required inequality (6.1). Theorem 6 is established.

7. *Theorem 7. Given any subinterval* $0 < \alpha \leqq x \leqq \beta < 1$ *of the unit interval, there exists* T_0 *such that for all* $T > T_0$, *the operator* BD_TB *has an eigenvalue contained in* $[\alpha,\beta]$.

Proof: Let $\lambda = \frac{1}{2}(\alpha + \beta)$ and choose ϵ so small that $3\epsilon/(\lambda - 3\epsilon) < (\beta - \alpha)/2$. Since $0 < \lambda < 1$, by Theorem 6 there exists a function $\varphi \in \mathfrak{L}^2$ such that

$$\frac{\| D'BD'\varphi - \lambda\varphi \|}{\| \varphi \|} < \epsilon.$$

Since φ and $D'BD'\varphi$ are fixed functions in $\mathfrak{L}^2$, there exists T_0 such that for each $T > T_0$

$$\| (D' - D_T)\varphi \|^2 = \int_T^{\infty} | \varphi(t) |^2 \, dt < \epsilon^2 \| \varphi \|^2$$

and

$$\| (D' - D_T)BD'\varphi \|^2 = \int_T^\infty | BD'\varphi(t) |^2 \, dt < \epsilon^2 \| \varphi \|^2.$$

Using the inequality $\| D_T B(D' - D_T)\varphi \| \leqq \| (D' - D_T)\varphi \|$ we then find

$$\frac{\| D_T B D_T \varphi - \lambda\varphi \|}{\| \varphi \|}$$

$$= \frac{\| D'BD'\varphi - \lambda\varphi - (D' - D_T)BD'\varphi - D_T B(D' - D_T)\varphi \|}{\| \varphi \|}$$

$$\leqq \frac{\| D'BD'\varphi - \lambda\varphi \|}{\| \varphi \|} + \frac{\| (D' - D_T)BD'\varphi \|}{\| \varphi \|} + \frac{\| D_T B(D' - D_T)\varphi \|}{\| \varphi \|} \tag{7.1}$$

$$\leqq 3\epsilon.$$

Now from (7.1) we see

$$\| D_T \varphi \| \geqq \| D_T B D_T \varphi \| \geqq (\lambda - 3\epsilon) \| \varphi \| \tag{7.2}$$

and

$$\| D_T B D_T \varphi - \lambda D_T \varphi \| = \| D_T(D_T B D_T \varphi - \lambda\varphi) \| \leqq 3\epsilon \| \varphi \| \tag{7.3}$$

so that, combining (7.2) and (7.3),

$$\| D_T B D_T \varphi - \lambda D_T \varphi \| / \| D_T \varphi \| \leqq 3\epsilon/(\lambda - 3\epsilon). \tag{7.4}$$

Now by property *ii* of the functions ψ_i, we may expand $D_T \varphi$ in a series $D_T \varphi = \sum a_n \varphi_n$, where $\varphi_n \equiv D_T \psi_n / \sqrt{\lambda_n(T)}$. Inserting this into (7.4), and using the fact (*iii*) that the $\psi_n(t)$ (which depend also on T) are eigenfunctions of $BD_T B$, we find

$$\left(\frac{3\epsilon}{\lambda - 3\epsilon}\right)^2 \geqq \frac{\| D_T B D_T \varphi - \lambda D_T \varphi \|^2}{\| D_T \varphi \|^2} = \frac{\| \sum a_n(\lambda_n(T) - \lambda)\varphi_n \|^2}{\| \sum a_n \varphi_n \|^2}$$

$$= \frac{\sum | a_n |^2 \, | \lambda_n(T) - \lambda |^2}{\sum | a_n |^2}$$

$$\geqq \inf_n | \lambda_n(T) - \lambda |^2.$$

We conclude that for every $T > T_0$ there exists an eigenvalue $\lambda_n(T)$

of the operator BD_TB with $|\lambda_n(T) - \lambda| \leqq 3\epsilon/(\lambda - 3\epsilon) < (\beta - \alpha)/2$, or equivalently, since the $\lambda_n(T)$ are all real, that $\alpha < \lambda_n(T) < \beta$. Theorem 7 is established.

Corollary 7.1 The number of eigenvalues of the operator BD_TB contained in any subinterval J of the unit interval cannot remain bounded as $T \to \infty$.

Proof: Given any integer N, subdivide J into N disjoint intervals J_n. By Theorem 7, for all T sufficiently large each J_n will contain an eigenvalue of BD_TB, hence J will contain at least N such eigenvalues. Since N was arbitrary, Corollary 7.1 is established.

8. *Theorem 8. Let any integer N and $\epsilon_T^2 < 0.916$ be given. Then as soon as WT is sufficiently large, there will exist a function $f \in E(\epsilon_T)$ such that*

$$\inf_{a_i} \left\| f - \sum_0^{[2WT]+N} a_i \psi_i \right\|^2 \geqq \frac{1}{0.916} \left(\epsilon_T^2 - 2\sqrt{2}\, e^{-\pi WT/2}\right).$$

(If $\epsilon_T^2 \geqq 0.916$, the right-hand side of the inequality should be replaced by 1.)

Proof: By Lemma 3, we have

$$\lambda_{[2WT]-1}\,(2WT) \geqq 0.085.$$

By Corollary 7.1, there exists a constant k_0, depending only on N, such that for all $WT > k_0$ the interval $0.084 \leqq x \leqq 0.085$ will contain at least $N + 2$ eigenvalues of BD_TB. Hence

$$\lambda_{[2WT]+N+1}\,(2WT) \geqq 0.084 \quad \text{for} \quad WT > k_0 .$$

Now the proof of Theorem 5, applied without change to $\lambda_{[2WT]+N+1}$, establishes Theorem 8.

Theorem 8.1 Let ϵ_T and any integer N be given. Then as soon as T is sufficiently large

$$\inf_{a_i} \left\| f - \sum_0^{[2WT]-N} a_i \psi_i \right\|^2 \leqq 12\, \epsilon_T^2 ,$$

for all $f \in E(\epsilon_T)$.

Proof: According to Lemma 2,

$$\lambda_{[2WT]+1}\,(2WT) \leqq 0.915.$$

By Corollary 7.1, there exists a constant k_1, depending only on N, such that for all $WT > k_1$ the interval $0.915 \leqq x \leqq 0.916$ will contain at least $N + 1$ eigenvalues of BD_TB. Hence,

$$\lambda_{[2WT]-N}\,(2WT) \leqq 0.916 \quad \text{for} \quad WT > k_1 .$$

Applying now the proof of Theorem 3 to $\lambda_{[2WT]-N}(2WT)$ establishes Theorem 8.1.

9. *Theorem 9. A. The restrictions to $t > 0$ of the functions*

$$[\sin \pi(2Wt - n)]/(2Wt - n),$$

for $n \leqq -1$, are dense in $\mathfrak{L}^2(0,\infty)$.

B. Their restrictions to $t < 0$ are not dense in $\mathfrak{L}^2(-\infty,0)$, nor even in $\mathfrak{B}$ restricted to $t < 0$.

Proof: Without loss of generality we may take $W = \frac{1}{2}$, to simplify notation. We begin with part A. Let

$$c(t) = \begin{cases} 1, & t \geqq 0 \\ 0, & t < 0. \end{cases}$$

The functions

$$\varphi_n(t) \equiv c(t) \frac{\sin \pi\,(t - n)}{t - n}$$

all lie in $\mathfrak{L}^2(0,\infty)$, so that their being dense in $\mathfrak{L}^2(0,\infty)$ is equivalent to the statement that $h(t) \equiv 0$ is the only function in $\mathfrak{L}^2(0,\infty)$ which is orthogonal to $\varphi_n(t)$, $n \leqq -1$ (Ref. 3, p. 72). We will prove A in this form.

Accordingly, suppose that $(h(t),\varphi_n(t)) = 0$, $n \leqq -1$. Using the Parseval theorem, and letting $\chi(u)$ be the characteristic function of the interval $|u| \leqq \pi$, we find

$$\begin{aligned} 0 = (h(t),\varphi_n(t)) &= \left[c(t)h(t), \frac{\sin \pi(t-n)}{t-n}\right] \\ &= (H(u),\chi(u)e^{inu}) \\ &= (\chi(u)H(u),e^{inu}), \qquad n \leqq -1, \end{aligned} \tag{9.1}$$

where $H(u)$ is the inverse Fourier transform of $c(t)h(t)$. The function $\chi(u)H(u)$ is in $\mathfrak{L}^2(-\pi,\pi)$ and may therefore be expanded there in a Fourier series $\chi(u)H(u) = \sum_{k=-\infty}^{\infty} a_k e^{iku}$. By (9.1) the coefficients a_k vanish for $k \leqq -1$, so that

$$\chi(u)H(u) = \sum_{k=0}^{\infty} a_k e^{iku}; \tag{9.2}$$

also

$$\sum_{k=0}^{\infty} |a_k|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(u)|^2\,du < \infty. \tag{9.3}$$

The function $H(u)$ may be continued analytically into the upper half of the $w = u + iv$ plane by its defining formula

$$H(w) = \int_0^\infty h(x)\, e^{iwx}\, dx ,$$

from which, by Parseval's theorem,

$$\int_{-\infty}^{\infty} | H(u + iv) |^2 \, du < A < \infty , \qquad v \geqq 0 . \tag{9.4}$$

Set $G(w) \equiv \sum_{k=0}^{\infty} a_k e^{ikw}$; the function $G(w)$ is then also analytic in the upper half-plane $v > 0$, and is periodic there, with period 2π. We will now show that (9.2) implies $H(w) \equiv G(w)$ for $v > 0$, consequently that $H(w)$ is also periodic in $v > 0$ with period 2π. It then follows from (9.4) that $H(w) \equiv 0$, hence that $h(x) \equiv 0$, which was to be proved. We model our argument on one given by A. Beurling (unpublished).

Applying the Schwarz inequality to the defining expressions for $H(w)$ and $G(w)$ we find

$$| H(u + iv) |, \quad | G(u + iv) | \leqq k/\sqrt{v}, \quad 0 < v < 2. \tag{9.5}$$

Next set $F(w) \equiv H(w) - G(w)$ in $v > 0$.

Let $0 < \epsilon < \frac{1}{8}$, and in the w-plane denote by $P_1, P_\epsilon, P_2, Q_2, Q_\epsilon, Q_1$ the points $\pi, \pi + i\epsilon, \pi + i, -\pi + i, -\pi + i\epsilon, -\pi$ respectively. Let Γ, Γ_ϵ be the arcs made up of the line segments $P_1P_2 + P_2Q_2 + Q_2Q_1$ and $P_1P_\epsilon + P_\epsilon P_\epsilon + Q_\epsilon Q_1$ respectively. Let R_1, R_2 be the rectangles $| u | < \pi/2$, $\frac{1}{4} < v < \frac{1}{2}$ and $| u | < \pi/2$, $-\frac{1}{2} < v < -\frac{1}{4}$ respectively, and R a region which contains R_1 and R_2 and whose closure does not intersect Γ.

Form the function

$$J(w) = \int_\Gamma \frac{F(\zeta)\, d\zeta}{\zeta - w} .$$

By (9.5), $F(\zeta)$ is integrable on Γ, so that $J(w)$ is an analytic function of w for w off Γ, in particular for $w \in R$. Now we rewrite

$$J(w) = \int_{\Gamma - \Gamma_\epsilon} \frac{F(\zeta)\, d\zeta}{\zeta - w} + \int_{\Gamma_\epsilon} \frac{F(\zeta)\, d\zeta}{\zeta - w} \tag{9.6}$$

and estimate the second integral of (9.6). If $w \in R_1 \cup R_2$ and $\zeta \in \Gamma_\epsilon$ we see that $1/| \zeta - w | < B < \infty$. Consequently

$$\begin{aligned} \left| \int_{\Gamma_\epsilon} \frac{F(\zeta)\, d\zeta}{\zeta - w} \right. &= \int_{P_1 P_\epsilon + Q_\epsilon Q_1} \frac{F(\zeta)\, d\zeta}{\zeta - w} + \left. \int_{-\pi}^{\pi} \frac{F(u + i\epsilon)}{u + i\epsilon - w}\, du \right| \\ &\leqq B \int_{P_1 P_\epsilon + Q_\epsilon Q_1} | F(\zeta)\, d\zeta | + B \int_{-\pi}^{\pi} | F(u + i\epsilon) |\, du . \end{aligned} \tag{9.7}$$

By virtue of (9.5),

$$\lim_{\epsilon \to 0} \int_{P_1 P_\epsilon + Q_\epsilon Q_1} | F(\zeta) \, d\zeta | = 0. \tag{9.8}$$

Applying the Schwarz inequality to the remaining integral of (9.7), and using the definition of F and the triangle inequality in $\mathfrak{L}^2$ we find

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} | F(u + i\epsilon) | \, du &\leqq \left\{ \int_{-\pi}^{\pi} | F(u + i\epsilon) |^2 \, du \right\}^{\frac{1}{2}} \\ &\leqq \left\{ \int_{-\pi}^{\pi} | H(u + i\epsilon) - H(u) |^2 \, du \right\}^{\frac{1}{2}} \\ &\quad + \left\{ \int_{-\pi}^{\pi} | H(u) - G(u) |^2 \, du \right\}^{\frac{1}{2}} \\ &\quad + \left\{ \int_{-\pi}^{\pi} | G(u) - G(u + i\epsilon) |^2 \, du \right\}^{\frac{1}{2}}. \end{aligned} \tag{9.9}$$

By definition of $H(w)$,

$$H(u + i\epsilon) - H(u) = \int_0^\infty h(t)[e^{-\epsilon t} - 1]e^{iut} \, dt,$$

whence by Parseval's theorem

$$\begin{aligned} \int_{-\pi}^{\pi} | H(u + i\epsilon) - H(u) |^2 \, du &\leqq \int_{-\infty}^{\infty} | H(u + i\epsilon) - H(u) |^2 \, du \\ &= 2\pi \int_0^\infty | h(t) |^2 \, | e^{-\epsilon t} - 1 |^2 \, dt. \end{aligned} \tag{9.10}$$

For each t, $\lim_{\epsilon \to 0} | h(t) |^2 | 1 - e^{-\epsilon t} |^2 = 0$, and $| h(t) |^2 | 1 - e^{-\epsilon t} |^2 \leqq 4 | h(t) |^2$, which by assumption is an integrable function. Consequently by the theorem on dominated convergence (Ref. 3, p. 37) applied to the last integral of (9.10),

$$\lim_{\epsilon \to 0^+} \int_{-\pi}^{\pi} | H(u + i\epsilon) - H(u) |^2 \, du = 0. \tag{9.11}$$

Similarly from the definition of $G(w)$

$$G(u) - G(u + i\epsilon) = \sum_{k=0}^{\infty} a_k (1 - e^{-\epsilon k}) e^{iuk},$$

whence

$$\int_{-\pi}^{\pi} | G(u) - G(u + i\epsilon) |^2 \, du = 2\pi \sum_{k=0}^{\infty} | a_k |^2 \, |1 - e^{-\epsilon k} |^2,$$

so that using (9.3) and arguing as above

$$\lim_{\epsilon\to 0^+} \int_{-\pi}^{\pi} | G(u) - G(u + i\epsilon) |^2 \, du = 0. \tag{9.12}$$

Combining (9.12), (9.11), (9.2), (9.9), (9.8) and (9.7) we find that uniformly for $w \in R_1 \cup R_2$,

$$\lim_{\epsilon\to 0^+} \left| \int_{\Gamma_\epsilon} \frac{F(\zeta)\, d\zeta}{\zeta - w} \right| = 0. \tag{9.13}$$

Since $\Gamma - \Gamma_\epsilon$ forms the boundary of the rectangle $| u | \leqq \pi$, $\epsilon \leqq v \leqq 1$, in whose interior F is an analytic function, the first integral on the right-hand side of (9.6) is equal to $F(w)$ for $w \in R_1$ and to 0 for $w \in R_2$. From (9.13) it follows that $J(w)$, which is independent of ϵ, must itself coincide with $F(w)$ for $w \in R_1$ and with 0 for $w \in R_2$. But if $J(w) = 0$ in R_2, it must be identically 0 in its whole domain of analyticity, in particular in R, hence also in R_1. We conclude that $F(w) \equiv 0$ in R_1, hence in its whole domain of analyticity $v > 0$. Thus $H(w) \equiv G(w)$ in $v > 0$, whence, as we have already argued, part A of Theorem 9 follows.

We now pass to a proof of part B. We remark first that the restrictions of $\mathfrak{B}$ to $t < 0$ include the functions $[\sin \pi(2Wt - n)]/(2Wt - n)$, $n \geqq 1$, restricted to $t < 0$. Replacing t by $-t$, we see that, by part A, these are already dense in $\mathfrak{L}^2(-\infty,0)$. Consequently to prove part B it is enough to establish its first assertion.

We argue by contradiction. Accordingly, suppose that the restrictions to $t < 0$ of the functions $[\sin \pi(2Wt - n)]/(2Wt - n)$, for $n \leqq -1$, are dense in $\mathfrak{L}^2(-\infty,0)$. Then defining the function $g(t) \epsilon \mathfrak{L}^2(-\infty,0)$ by

$$g(t) = \begin{cases} 1, & -1 \leqq t \leqq 0 \\ 0, & t < -1, \end{cases}$$

we could find a sequence of functions $f_n(t)$, each some linear combination of the $[\sin \pi(2Wt - n)]/(2Wt - n)$, $n \leqq -1$, such that $\{f_n(t)\}$ approaches $g(t)$ in $\mathfrak{L}^2(-\infty,0)$, i.e. such that

$$\int_{-\infty}^{0} | g(t) - f_n(t) |^2 \, dt = \epsilon_n \to 0. \tag{9.14}$$

The triangle inequality in $\mathfrak{L}^2(-\infty,0)$ applied to (9.14) yields

$$\int_{-\infty}^{0} | f_n(t) |^2 \, dt \leqq \left\{ \left[\int_{-\infty}^{0} | g(t) |^2 \, dt \right]^{\frac{1}{2}} + \sqrt{\epsilon_n} \right\}^2 = (1 + \sqrt{\epsilon_n})^2. \tag{9.15}$$

Now the functions $f_n(t)$ are all band-limited and $f_n(k) = 0$ for $k \geqq 0$. Thus by Ref. 9 there exists a constant C_1 such that

$$(9.16) \qquad \int_0^\infty |f_n(t)|^2\,dt \leqq C_1 \int_{-\infty}^0 |f_n(t)|^2\,dt.$$

From (9.15) and (9.16) it follows that, as elements of $\mathfrak{L}^2(-\infty,\infty)$, the functions $f_n(t)$ have uniformly bounded norms as soon as $\epsilon_n < 1$. Applying (1.5), we conclude that the $f_n(t)$ are a uniformly bounded family of analytic functions in the strip $|\operatorname{Im}\{t\}| < 1$ of the complex t-plane, thus a normal family there (Ref. 5, p. 171). We may therefore extract from the sequence $\{f_n(t)\}$ a subsequence $f_{n_k}(t)$ converging (pointwise) in the whole strip, uniformly on any compact subset of the strip, to an analytic function $f(t)$; from (9.14),

$$f(t) = g(t), \qquad t < 0.$$

But $g(t)$ vanishes on an interval without vanishing identically, and so cannot coincide with an analytic function. We have reached a contradiction, and part B follows. Theorem 9 is established.

10. *Theorem 10: Let $f(t) \in E(\epsilon_T)$. Then an estimate of the form*

$$\min_{\{a_k\}} \left\| f - \sum_{|k| \leqq WT+N} a_k \frac{\sin \pi(2Wt - k)}{\pi(2Wt - k)} \right\|^2 \leqq C\epsilon_T^2$$

cannot be valid independently of ϵ_T, no matter how large the constants C and N are chosen.

Proof: Without loss of generality we may take $W = \frac{1}{2}$, to simplify notation.

Any function $f \in \mathfrak{B}$ has the (sampling series) expansion $f(t) = \sum_{-\infty}^{\infty} f(k)[\sin \pi(t-k)]/[\pi(t-k)]$. Since the functions

$$(10.1) \qquad \varphi_k(t) = \frac{\sin \pi(t-k)}{\pi(t-k)} \text{ are orthonormal,}$$
$$\min_{\{a_k\}} \left\| f - \sum_{|k| \leqq (T/2)+N} a_k \varphi_k \right\|^2 = \sum_{|k| > (T/2)+N} |f(k)|^2.$$

Now consider the function $[\sin \pi(t-N-1)]/[\pi(t-N-1)]$ which is in $\mathfrak{B}$. By Theorem 9, we may approximate its restriction to $t > 0$ arbitrarily closely in $\mathfrak{L}^2(0,\infty)$ by finite linear combinations of the functions $[\sin \pi(t-n)]/[\pi(t-n)]$, $n \leqq -1$. That is, given $\eta > 0$, there exists constants $a_{-1}, \dots, a_{-m}$ (depending on η) such that

$$(10.2) \qquad \int_0^\infty \left| \frac{\sin \pi(t-N-1)}{\pi(t-N-1)} - \sum_{k=-1}^{-m} a_k \frac{\sin \pi(t-k)}{\pi(t-k)} \right|^2 dt < \eta.$$

Let

$$(10.3) \qquad \varphi_\eta(t) \equiv \frac{\sin \pi(t-N-1)}{\pi(t-N-1)} - \sum_{k=-1}^{-m} a_k \frac{\sin \pi(t-k)}{\pi(t-k)};$$

the function $\varphi_\eta \in \mathcal{B}$, and $\| \varphi_\eta \|^2 = 1 + \sum_{k=-1}^{-m} | a_k |^2 \geqq 1$. Since in particular φ_η is in $\mathcal{L}^2(-\infty, \infty)$ we may choose an integer $T/2$ so large that

$$\int_{-\infty}^{-T} | \varphi_\eta(t) |^2 \, dt < \eta. \tag{10.4}$$

Now set

$$f(t) \equiv \frac{\varphi_\eta(t - T/2)}{\| \varphi_\eta \|};$$

We see that $f \in \mathcal{B}$ and $\| f \| = 1$. Furthermore, by (10.2) and (10.4),

$$\int_{|t|>T/2} | f(t) |^2 \, dt = \frac{\int_{-\infty}^{-T} | \varphi_\eta(t) |^2 \, dt + \int_0^\infty | \varphi_\eta(t) |^2 \, dt}{\| \varphi_\eta \|^2} < \frac{2\eta}{\| \varphi_\eta \|^2},$$

so that $f \in E(\epsilon_T)$, with $\epsilon_T = (\sqrt{2\eta}/\| \varphi_\eta \|)$; we observe that ϵ_T can be made arbitrarily small by choosing η small, since $\| \varphi_\eta \| \geqq 1$. By definition

$$\sum_{|k|>(T/2)+N} | f(k) |^2$$

$$= \frac{1}{\| \varphi_\eta \|^2} \left[\sum_{k>N} | \varphi_\eta(k) |^2 + \sum_{k<-T-N} | \varphi_\eta(k) |^2 \right] \geqq \frac{1}{\| \varphi_\eta \|^2},$$

whence by (10.1)

$$\min_{\{a_k\}} \left\| f - \sum_{|k| \leqq (T/2)+N} a_k \varphi_k \right\|^2 \geqq \epsilon_T^2 \frac{1}{2\eta}.$$

Since η may be arbitrarily small, Theorem 10 follows.

11. *Theorem 11. For any $\beta < 1$, there exists $\delta > 0$ and ϵ_T such that*

$$\left\| f - \sum_{|k| \leqq WT+(WT)^\beta} f\left(\frac{k}{2W}\right) \frac{\sin \pi(2Wt - k)}{\pi(2Wt - k)} \right\|^2 > (1 + \delta)\epsilon_T^2,$$

for some $f \in E(\epsilon_T)$.

Proof: We again take $W = \frac{1}{2}$ without loss of generality. We follow a line of reasoning used in Ref. 9.

$$\text{Let } g(t) = \sin \pi t \sum_1^\infty \frac{1}{n^{\frac{1}{2}+\epsilon}} \frac{1}{t + n}.$$

Then

$$\frac{1}{2\epsilon} = \int_1^\infty \frac{dt}{t^{1+2\epsilon}} < \sum_1^\infty \frac{1}{n^{1+2\epsilon}} = \pi^2 \int_{-\infty}^\infty g^2(t) \, dt < \int_{\frac{1}{2}}^\infty \frac{dt}{t^{1+2\epsilon}} = \frac{2^{2\epsilon}}{2\epsilon}.$$

Now if $P, N > 0$, with $P > N + 1$, then

Proof: The decomposition

$$f = Bf + (f - Bf) \tag{12.1}$$

expresses f as the sum of its components in $\mathcal{B}$ and orthogonal to $\mathcal{B}$ respectively. The Pythagorean theorem then yields $1 = \| f \|^2 = \| Bf \|^2 + \| f - Bf \|^2$, whence $\| f - Bf \| = \eta_W$. Similarly, $\| f - Df \| = \epsilon_T$.

Let $g = Bf/\sqrt{1 - \eta_W^2}$, so that $g \in \mathcal{B}$ and $\| g \| = 1$. We will apply Theorem 3 to g; to do so, we must estimate its degree of concentration. We first expand

$$\begin{aligned} \| Df - DBf \|^2 &= (Df - DBf, Df - DBf) \\ &= \| Df \|^2 + \| DBf \|^2 - 2\mathrm{Re}(Df, DBf). \end{aligned} \tag{12.2}$$

Moreover, since $\| Df - Bf \| \leqq \| f - Df \| + \| f - Bf \| = \epsilon_T + \eta_W$, we find

$$(\epsilon_T + \eta_W)^2 \geqq \| Df - Bf \|^2 = \| Df \|^2 + \| Bf \|^2 - 2\mathrm{Re}(Df, Bf). \tag{12.3}$$

Since D is a projection, $(Df, DBf) = (Df, Bf)$; hence subtracting (12.3) from (12.2),

$$\| DBf \|^2 - \| Bf \|^2 \geqq \| Df - DBf \|^2 - (\epsilon_T + \eta_W)^2,$$

or

$$\begin{aligned} \| Dg \|^2 = \frac{\| DBf \|^2}{1 - \eta_W^2} &\geqq \frac{\| Bf \|^2}{1 - \eta_W^2} - \frac{(\epsilon_T + \eta_W)^2}{1 - \eta_W^2} \\ &= 1 - \frac{(\epsilon_T + \eta_W)^2}{1 - \eta_W^2}. \end{aligned} \tag{12.4}$$

Consequently, by Theorem 3, there exist constants b_k such that

$$\| g - \sum_0^{[2WT]} b_k \psi_k \|^2 \leqq 12 \frac{(\epsilon_T + \eta_W)^2}{1 - \eta_W^2}. \tag{12.5}$$

Now from (12.1)

$$\frac{f}{\sqrt{1 - \eta_W^2}} - \sum_0^{[2WT]} b_k \psi_k = \left(g - \sum_0^{[2WT]} a_k \psi_k \right) + \left(\frac{f - Bf}{\sqrt{1 - \eta_W^2}} \right),$$

and the bracketed terms remain orthogonal. Thus, with

$$a_k = \sqrt{1 - \eta_W^2}\, b_k ,$$

$$\| f - \sum_0^{[2WT]} a_k \psi_k \|^2 \leqq 12(\epsilon_T + \eta_W)^2 + \eta_W^2 .$$

Theorem 12 is established.

We should point out that by letting $g = Df/\sqrt{1 - \epsilon_2^T}$ and working with the functions $D\psi_k$, the roles of ϵ and η may be interchanged, to yield the inequality

$$\| f - \sum_0^{[2WT]} c_k D\psi_k \|^2 \leqq 12(\epsilon_T + \eta_W)^2 + \epsilon_T^2 .$$

13. *Theorem 13: If* $f(t) \in \mathfrak{L}^2$ *with* $\| f \| = 1$, $\| Df \|^2 = 1 - \epsilon_T^2$,

$$\| Bf \|^2 = 1 - \eta_W^2 ,$$

then for some constants $c_k = c_k(f)$,

$$\| f - \sum_{|k| \leqq WT+1} c_k \frac{\sin \pi(2Wt - k)}{\pi(2Wt - k)} \|^2 \leqq (\epsilon_T + \eta_W)^2 + \eta_W^2 + \pi(\epsilon_T + \eta_W)\sqrt{1 - \eta_W^2}.$$

Proof: We proceed as in Theorem 12, up to (12.4) but now apply Theorem 2 instead of Theorem 3. Thus, for some constants b_k,

$$\| g - \sum_{|k| \leqq WT+1} b_k \frac{\sin \pi(2Wt - k)}{\pi(2Wt - k)} \|^2 \leqq \pi \frac{\epsilon_T + \eta_W}{\sqrt{1 - \eta_W^2}} + \frac{(\epsilon_T + \eta_W)^2}{1 - \eta_W^2}. \tag{13.1}$$

Replacing (12.5) by (13.1) and applying without change the rest of the proof of Theorem 12 establishes Theorem 13.

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