Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty — IV: Extensions to Many Dimensions; Generalized Prolate Spheroidal Functions

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In two earlier papers in this series, the extent to which a square-integrable function and its Fourier transform can be simultaneously concentrated in their respective domains was considered in detail. The present paper generalizes much of that work to functions of many variables.*

In treating the case of functions of two variables whose Fourier transforms vanish outside a circle in the two-dimensional frequency plane, we are led to consider the integral equation

$$\gamma\varphi(x) = \int_0^1 J_N(cxy)\sqrt{cxy}\,\varphi(y)dy. \qquad (i)$$

It is shown that the solutions are also the bounded eigenfunctions of the differential equation

$$(1 - x^2)\frac{d^2\varphi}{dx^2} - 2x\frac{d\varphi}{dx} + \left(\chi - c^2x^2 + \frac{\frac{1}{4} - N^2}{x^2}\right)\varphi = 0, \qquad (ii)$$

a generalization of the equation for the prolate spheroidal wave functions. The functions φ (called "generalized prolate spheroidal functions") and the eigenvalues of both (i) and (ii) are studied in detail here, and both analytic and numerical results are presented.

Other results include a general perturbation scheme for differential equations and the reduction to two dimensions of the case of functions of $D > 2$ variables restricted in frequency to the D sphere.

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* See Refs. 1 and 2.

I. INTRODUCTION

In two earlier papers[1,2] in this series, the extent to which a square-integrable function and its Fourier transform can be simultaneously concentrated was considered in detail. In that analysis, the eigenfunctions and eigenvalues of the finite Fourier transform played a key role. These functions, defined for $|x| \leqq 1$ by the integral equation

$$\alpha_j \psi_j(x) = \int_{-1}^{1} e^{icxy} \psi_j(y) dy, \tag{1}$$

can be continued analytically throughout the complex plane. They possess a number of special properties that make them most useful for the study of bandlimited functions. The functions are complete in the class of bandlimited functions; they are orthogonal in $(-1,1)$ and also in $(-\infty, \infty)$; the ψ_j are also the eigenfunctions of the integral equation

$$\begin{aligned} \lambda \psi(x) &= \int_{-1}^{1} \frac{\sin c(x-y)}{\pi(x-y)} \psi(y) dy \\ \lambda &= \frac{c}{2\pi} |\alpha|^2 \end{aligned} \tag{2}$$

whose kernel is positive definite; ψ_o, the eigenfunction of (2) belonging to the largest eigenvalue, is in an appropriate sense most concentrated among bandlimited functions of given energy. These and other properties are discussed in detail in Refs. 1 and 2. Some familiarity with these papers will be assumed in the following.

In the present paper we consider certain aspects of the generalization of this earlier work to functions of many variables. Many of the structural results of Refs. 1 and 2 (as was pointed out there) depend only on the fact that the operator defined by the right of (2) is completely continuous and positive definite. The generalizations to D dimensions are perfectly straightforward: we comment briefly on some of them in Section II, but do not belabor them. Our main concern here is with details of the explicit solution of some of the integral equations that are multidimensional generalizations of (1). An unexpected dividend of this work is that one of these equations is of interest in the theory of masers.

In Section III, we point out some general features of the integral equations to be considered. Section IV treats the case of functions of two variables whose Fourier transforms vanish outside a circle in the two-dimensional frequency plane. The analog of (1) is shown to be the integral equation

$$\gamma\varphi(r) = \int_0^1 J_N(crr')\sqrt{crr'}\varphi(r')dr', \qquad 0 \leqq r \leqq 1. \tag{3}$$

This integral equation also describes the modes in a maser interferometer with confocal spherical mirrors of circular cross section (Ref. 3, p. 488). The eigenfunctions of (3) are shown to be the bounded solutions of

$$(1 - x^2)\frac{d^2\varphi}{dx^2} - 2x\frac{d\varphi}{dx} + \left(\chi - c^2x^2 - \frac{\frac{1}{4} - N^2}{x^2}\right)\varphi = 0 \tag{4}$$

that vanish at $x = 0$.

We call the solutions of (4) generalized prolate spheroidal functions. Section V is devoted to their study: 5.1 treats the case of small c; 5.2 and 5.3 treat various asymptotic cases.*

In Section VI, the results of Section V are used to discuss the eigenvalues of (3). Various asymptotic forms for these quantities are derived.

Section VII treats the case of functions of $D > 2$ variables whose Fourier transforms vanish outside a sphere in the D-dimensional frequency space. It is shown that this more general problem can be reduced completely to the case already treated in Sections IV, V and VI.

Finally, in Section VIII we present some numerical detail about some of the eigenfunctions and eigenvalues encountered. Applications of these results will be presented elsewhere.[9, 10]

II. GENERALIZATIONS OF EARLIER WORK

We denote points in Euclidean space of D dimensions, E_D, by vectors, $\mathbf{x} = (x_1, x_2, \cdots, x_D)$. A square-integrable function of D variables, $f(\mathbf{x})$, is said to be R-limited if it can be represented as a Fourier integral

$$f(\mathbf{x}) = (2\pi)^{-D}\int_R \exp\,(i\mathbf{x}\cdot\mathbf{y})F(\mathbf{y})dy \tag{5}$$

over the bounded region R. Here $\mathbf{x}\cdot\mathbf{y} = \sum x_i y_i$ is the usual scalar product and we write dy for $\prod dy_i$. If f is of total energy A, then by Parseval's theorem

$$A = \int_{E_D} |\, f(\mathbf{x})\,|^2 dx = (2\pi)^{-D}\int_R |\, F(\mathbf{y})\,|^2 dy,$$

whereas the energy of f in the bounded region S is

* Some of the results of Sections IV and V have been developed independently by J. C. Huertley,[11] who was led to consider (3) from its laser applications.

$$\int_S |f(\mathbf{z})|^2 dz$$

$$= \int_S dz (2\pi)^{-2D} \int_R dx \exp(i\mathbf{z}\cdot\mathbf{x}) F(\mathbf{x}) \int_R dy \exp(-i\mathbf{z}\cdot\mathbf{y}) \bar{F}(\mathbf{y})$$

$$= (2\pi)^{-D} \int_R dx \int_R dy\, K_S(\mathbf{x} - \mathbf{y}) F(\mathbf{x}) \bar{F}(\mathbf{y})$$

where

$$K_S(\mathbf{x} - \mathbf{y}) = (2\pi)^{-D} \int_S \exp[i\mathbf{z}\cdot(\mathbf{x} - \mathbf{y})] dz \tag{6}$$

and an overbar denotes complex conjugate. The largest fraction of energy that an R-limited function can have in the region S is therefore the maximum value of the fraction

$$\int_R dx \int_R dy\, K_S(\mathbf{x} - \mathbf{y}) F(\mathbf{x}) \bar{F}(\mathbf{y}) \Big/ \int_R |F(\mathbf{y})|^2 dy$$

taken over all functions F square-integrable through R. This maximum is the largest eigenvalue of the integral equation

$$\lambda \psi(\mathbf{x}) = \int_R K_S(\mathbf{x} - \mathbf{y}) \psi(\mathbf{y}) dy, \qquad \mathbf{x} \in R, \tag{7}$$

which is the analog of (2).

The kernel (6) of (7) is positive definite, since

$$\int_R dx \int_R dy\, K_S(\mathbf{x} - \mathbf{y}) f(\mathbf{x}) \bar{f}(\mathbf{y})$$

$$= (2\pi)^{-D} \int_S dz \left| \int_R dx \exp(i\mathbf{z}\cdot\mathbf{x}) f(\mathbf{x}) \right|^2 > 0$$

whenever

$$\int_R |f(\mathbf{x})|^2 dx > 0.$$

By well-known theorems (see Ref. 4, Chap. 6), the eigenvalues of (7) are real and positive and the eigenfunctions, orthogonal on R, are complete in the class of functions square-integrable in R. A complete discussion of the simultaneous concentration of square-integrable functions in E_D and their Fourier transforms can be given in terms of the largest eigenvalue of (7) as in Ref. 2, Theorem 2.

The right member of (7) can be used to extend the domain of definition of ψ. We define

$$\psi(\mathbf{x}) = \frac{1}{\lambda} \int_R K_S(\mathbf{x} - \mathbf{y})\psi(\mathbf{y})dy, \qquad \mathbf{x} \in E_D .$$

Then for two different eigenfunctions of (7)

$$\int_{E_D} \psi_i(\mathbf{x})\bar{\psi}_j(\mathbf{x})dx$$

$$= \frac{1}{\lambda_i\lambda_j} \int_R dx \int_R dy\, \psi_i(\mathbf{x})\bar{\psi}_j(\mathbf{y}) \int_{E_D} dz\, K_S(\mathbf{z} - \mathbf{x})\bar{K}_S(\mathbf{z} - \mathbf{y}).$$

To evaluate the innermost integral here, we observe from (6) that K_S is given as a Fourier transform, so that from Parseval's theorem,

$$\int_{E_D} K_S(\mathbf{z} - \mathbf{x})\bar{K}_S(\mathbf{z} - \mathbf{y})dz$$

$$= (2\pi)^{-D} \int_S du \exp\left[-i\mathbf{u}\cdot(\mathbf{x} - \mathbf{y})\right] = \bar{K}_S(\mathbf{x} - \mathbf{y}).$$

One then finds

$$\int_{E_D} \psi_i(\mathbf{x})\bar{\psi}_j(\mathbf{x})dx = \frac{1}{\lambda_i\lambda_j} \int_R dx\, \psi_i(\mathbf{x}) \int_R dy\, \bar{K}_S(\mathbf{x} - \mathbf{y})\bar{\psi}_j(\mathbf{y})$$

$$= \frac{1}{\lambda_i} \int_R dx\, \psi_i(\mathbf{x})\bar{\psi}_j(\mathbf{x}).$$

The orthogonality of the ψ_i over R thus implies orthogonality over E_D as well.

Other results of the one-dimensional case extend as easily to D dimensions, but we do not dwell further here on this general structure.

III. SYMMETRY CONSIDERATIONS

In what follows, we shall be concerned with the explicit solution of a number of instances of (7). Considerable simplification occurs when the region R is symmetric, i.e., when $\mathbf{x} \in R$ implies $-\mathbf{x} \in R$, and when S is a scaled version of R. We write $S = cR$ where $\mathbf{x} \in cR$ if and only if $\mathbf{x}/c \in R$ with c a positive constant. We restrict our attention henceforth to this case.

Somewhat simpler than (7) is the integral equation

$$\alpha\psi(\mathbf{x}) = \int_R \exp\,(ic\mathbf{x}\cdot\mathbf{y})\psi(\mathbf{y})dy, \qquad \mathbf{x} \in R \tag{8}$$

which is a natural generalization of (1). We shall show in this section that solution of this equation is completely equivalent to solution of (7) when the symmetries just discussed maintain. We shall accordingly hereafter take (8) as our equation of fundamental concern.

From the symmetry of R, it readily follows that if $\psi(\mathbf{x})$ is a solution of (8), so also is $\psi(-\mathbf{x})$, so that both $\psi_e(\mathbf{x}) = \psi(\mathbf{x}) + \psi(-\mathbf{x})$ and $\psi_o(\mathbf{x}) = \psi(\mathbf{x}) - \psi(-\mathbf{x})$ are solutions as well. *The eigenfunctions of* (8) *can be chosen to be either even or odd functions of* x.

The complex conjugate of (8) is

$$\bar{\alpha}\bar{\psi}(\mathbf{x}) = \int_R \exp\,(-ic\mathbf{x}\cdot\mathbf{y})\bar{\psi}(\mathbf{y})dy, \qquad \mathbf{x} \in R. \tag{9}$$

Multiply (8) by $\bar{\psi}(\mathbf{x})$ and integrate over R. Multiply (9) by $\psi(\mathbf{x})$ and integrate over R. Combining these equations, one finds on using the symmetry of R that

$$(\alpha \pm \bar{\alpha}) \int_R \psi(\mathbf{x})\bar{\psi}(\mathbf{x})dx$$
$$= \int_R dx \int_R dy \exp\,(ic\mathbf{x}\cdot\mathbf{y})\bar{\psi}(\mathbf{x})[\psi(\mathbf{y}) \pm \psi(-\mathbf{y})].$$

If then ψ is even, by choosing the negative sign in this equation, one obtains $\alpha - \bar{\alpha} = 0$, whereas if ψ is odd, by choosing the plus sign, one finds $\alpha + \bar{\alpha} = 0$. *The eigenvalues of* (8) *associated with even eigenfunctions are real: the eigenvalues of* (8) *associated with odd eigenfunctions are pure imaginary*. It follows then that (8) is equivalent to the pair of equations

$$\beta_e\psi_e(\mathbf{x}) = \int_R \cos c\mathbf{x}\cdot\mathbf{y}\psi_e(\mathbf{y})dy \tag{10}$$

$$\beta_o\psi_o(\mathbf{x}) = \int_R \sin c\mathbf{x}\cdot\mathbf{y}\psi_o(\mathbf{y})dy \tag{11}$$

in which β_e and β_o are real. These equations have real symmetric kernels and we can fall back on the extensive theory in the literature treating such equations. We observe that the eigenfunctions of (10) must be even and that $\beta_e = 0$ cannot be an eigenvalue of this equation, for by Fourier theory the only even square-integrable function in R for which

$$\int_R \cos c\mathbf{x}\cdot\mathbf{y}\psi(\mathbf{y})dy = 0, \qquad \mathbf{x} \in R$$

is $\psi(\mathbf{y}) \equiv 0$. It follows then from the theorem on page 234 of Ref. 4 that the eigenfunctions of (10) are complete in the class of even functions square-integrable in R. A similar argument shows that the solutions of (11) are complete in the class of odd functions square-integrable in R. The solutions of (10) can be chosen real and orthogonal in R, as can the solutions of (11). Solutions of (10) are automatically orthogonal to solutions of (11) by symmetry.

We have now shown that the solutions of (8) are complete in the class of functions square-integrable in R. The eigenfunctions can be chosen real, orthogonal, and either even (in which case the eigenvalue α is real) or odd (in which case α is pure imaginary). We henceforth assume the ψ so chosen.

By iterating (8), one finds that the ψ also satisfy

$$\lambda\psi(\mathbf{x}) = \int_R K_c(\mathbf{x} - \mathbf{y})\psi(\mathbf{y})dy \tag{12}$$

$$\lambda = \left(\frac{c}{2\pi}\right)^D |\alpha|^2 \tag{13}$$

with

$$K_c(\mathbf{x}) = \left(\frac{c}{2\pi}\right)^D \int_R \exp(ic\mathbf{z}\cdot\mathbf{x})\, dz = (2\pi)^{-D} \int_{cR} e^{i\mathbf{z}\cdot\mathbf{x}}\, dz \tag{14}$$

which is (7) in slightly altered notation and is the D-dimensional analog of (2). Since the solutions ψ of (8) are complete, it follows that they are also a complete set of solutions of (12). As was asserted, to solve (12), it suffices to solve (8).

The eigenfunctions of (8) can be extended by demanding that equation to hold for all $\mathbf{x} \in E_D$. It is then easy to show that the extended ψ are orthogonal in E_D and that they are complete in the class of cR-limited functions.

IV. THE CASE $D = 2$, R A CIRCLE

We now treat in detail the equation

$$\alpha\psi(x_1, x_2) = \int_R e^{ic(x_1y_1+x_2y_2)}\psi(y_1, y_2)dy_1dy_2 \tag{15}$$

where R is the unit circle $y_1^2 + y_2^2 \leqq 1$. Change to polar coordinates gives

$$\begin{aligned}\alpha\psi(r,\theta) &= \int_0^1 dr'\, r' \int_0^{2\pi} d\theta'\, e^{icrr'\cos(\theta-\theta')}\psi(r',\theta') \\ &= \sum_{-\infty}^{\infty} i^m e^{im\theta} \int_0^1 dr'\, r' J_m(crr') \int_0^{2\pi} d\theta'\, e^{-im\theta'}\, \psi(r',\theta')\end{aligned} \tag{16}$$

on making the usual Bessel function expansion. Here $\psi(r,\theta)$ is exhibited as a Fourier series in θ. A simple argument then gives for the eigenfunctions of (15) and their corresponding eigenvalues

$$\begin{aligned} \psi_{o,n}(r,\theta) &= R_{o,n}(r), & \alpha_{o,n} &= 2\pi\beta_{o,n} \\ \psi_{N,n}(r,\theta) &= R_{N,n}(r) \begin{matrix}\cos N\theta, \\ \sin N\theta,\end{matrix} & \alpha_{N,n} &= 2\pi i^N \beta_{N,n} \\ & N = 1, 2, \dots, & n &= 0, 1, 2, \dots \end{aligned} \tag{17}$$

where

$$\beta_{N,n} R_{N,n}(r) = \int_0^1 J_N(crr') R_{N,n}(r') r' \, dr', \qquad 0 \leqq r \leqq 1, \qquad n, N = 0, 1, 2, \dots . \tag{18}$$

All the eigenvalues of (15), except possibly the $\alpha_{o,n}$ have at least a two-fold degeneracy inherited from the symmetry of the circle.

Our task now is to study the integral equation

$$\beta R(r) = \int_0^1 J_N(crr') R(r') r' \, dr', \qquad 0 \leqq r \leqq 1.$$

It is convenient to make the substitutions

$$\gamma = \sqrt{c}\beta, \qquad \varphi(r) = \sqrt{r} R(r) \tag{19}$$

to obtain the symmetric equation

$$\gamma \varphi(r) = \int_0^1 J_N(crr') \sqrt{crr'} \varphi(r') \, dr', \qquad 0 \leqq r \leqq 1. \tag{20}$$

Note that $\varphi(0) = 0$. We shall show that the eigenfunctions $\varphi_{N,n}(r)$ of (20) can be obtained as the solution of a Sturm-Liouville differential equation.

Let

$$K_N(x) = J_N(x) \sqrt{x} \tag{21}$$

and let the operator M be defined by

$$[M\psi](x) = \int_0^1 K_N(cxy) \psi(y) dy.$$

Denote by L_x the differential operator

$$L_x = \frac{d}{dx}(1 - x^2)\frac{d}{dx} + \left(\frac{\frac{1}{4} - N^2}{x^2} - c^2 x^2 \right).$$

Consider now

$$
\begin{aligned}
[ML\psi](x) &= \int_0^1 K_N(cxy)\left[\frac{d}{dy}(1-y^2)\frac{d}{dy} + \left(\frac{\frac{1}{4}-N^2}{y^2} - c^2y^2\right)\right]\psi(y)dy \\
&= [K_N(cxy)(1-y^2)\psi'(y) - cx(1-y^2) \\
&\quad \cdot K_N'(cxy)\psi(y)]_{y=0}^1 + \int_0^1 \psi(y)\left[c^2x^2(1-y^2)K_N''(cxy)\right. \\
&\quad \left. - 2cxyK'(cxy) + \left(\frac{\frac{1}{4}-N^2}{y^2} - c^2y^2\right)K_N(cxy)\right]dy
\end{aligned}
\tag{22}
$$

where the right member is obtained by integration by parts. Here primes denote differentiation of the function in question with respect to its argument. The integrated expression vanishes if $\psi(0) = 0$, since from (21), $K_N(0) = 0$. Also from (21) and the differential equation satisfied by Bessel functions, one has the identity

$$
K_N''(cxy) = -\left(1 + \frac{\frac{1}{4}-N^2}{c^2x^2y^2}\right)K_N(cxy). \tag{23}
$$

Substitute this expression in (22) to yield

$$
\begin{aligned}
[ML\psi](x) &= \int_0^1 \psi(y)[-2cxyK'(cxy) \\
&\quad + (\tfrac{1}{4} - N^2 + c^2x^2y^2 - c^2x^2 - c^2y^2)K(cxy)]dy,
\end{aligned}
\tag{24}
$$

$$
\psi(0) = 0.
$$

On the other hand, by direct calculation and use of (23), one has

$$
\begin{aligned}
[LM\psi](x) &= L_x \int_0^1 K_N(cxy)\psi(y)dy \\
&= \int_0^1 \psi(y)\left[(1-x^2)c^2y^2K_N''(cxy) - 2xcyK_N'(cxy)\right. \\
&\quad \left. + \left(\frac{\frac{1}{4}-N^2}{x^2} - c^2x^2\right)K_N(cxy)\right]dy \\
&= \int_0^1 \psi(y)\left[-2xcyK_N'(cxy) + \left\{-(1-x^2)c^2y^2\right.\right. \\
&\quad \left.\left. \cdot\left(1 + \frac{\frac{1}{4}-N^2}{c^2x^2y^2}\right) + \frac{\frac{1}{4}-N^2}{x^2} - c^2x^2\right\}K_N(cxy)\right]dy
\end{aligned}
$$

$$= \int_0^1 \psi(y)[-2cxyK_N'(cxy) + (\tfrac{1}{4} - N^2 + c^2x^2y^2 - c^2x^2 - c^2y^2)K_N(cxy)]dy$$

$$= [ML\psi](x)$$

on comparison with (24).

Let C be the class of functions square-integrable in (0,1) and twice differentiable there that vanish at the origin. Operating on functions in C, the operators M and L commute. It follows that solutions of

$$L_x\varphi(x) = -\chi\varphi(x)$$

in C are also solutions of (20). Consequently, we next turn our attention to the differential equation.

$$(1 - x^2)\frac{d^2\varphi}{dx^2} - 2x\frac{d\varphi}{dx} + \left(\frac{\tfrac{1}{4} - N^2}{x^2} - c^2x^2 + \chi\right)\varphi = 0. \qquad (25)$$

V. GENERALIZED PROLATE SPHEROIDAL FUNCTIONS

When $N = \tfrac{1}{2}$ in (25), this equation reduces to the equation for prolate spheroidal functions of order zero. We shall refer to bounded solutions of (25) for arbitrary values of N as *generalized prolate spheroidal functions*. These functions are similar in many respects to prolate spheroidal functions, as the development that follows shows. Bounded solutions of (25) exist only for discrete values of χ, say $\chi_{N,n}$, $n = 0, 1, 2, \ldots$ which we label so that $\chi_{N,o} \leqq \chi_{N,1} \leqq \chi_{N,2} \leqq \ldots$. We denote the corresponding eigenfunctions by $\varphi_{N,n}(x)$.

5.1 *Expansions in Powers of c*

Consider first the case when $c = 0$. Substitution of the series

$$\varphi = \sum_0^\infty a_j x^{\alpha+2j}$$

into (25) shows that we must have $\alpha = \tfrac{1}{2} \pm N$. If $N \neq 0$, the negative sign leads to solutions having a singularity at $x = 0$. If $N = 0$, a second solution can be found, but it has a logarithmic singularity at $x = 0$. We must have therefore

$$\alpha = \tfrac{1}{2} + N.$$

The coefficients are given by the recurrence

$$a_{j+1} = a_j \frac{(\alpha + 2j)(\alpha + 2j + 1) - \chi}{(\alpha + 2j + 2)(\alpha + 2j + 1) + \tfrac{1}{4} - N^2}.$$

For large j, $a_{j+1}/a_j \to 1$, so unless the series terminates, this solution becomes unbounded as $x \to 1$. Choosing χ to terminate the series at $x^{\alpha+2n}$, we have

$$\chi = \chi_{N,n}(0) = (N + 2n + \tfrac{1}{2})(N + 2n + \tfrac{3}{2}) \tag{26}$$

for the eigenvalues of (25) when $c = 0$. The series solution now becomes (when a_o is set equal to unity)*

$$\begin{aligned} \varphi &= T_{N,n}(x) = x^{N+\frac{1}{2}} R_{N,n}(x) \\ R_{N,n}(x) &= F(-n, n + N + 1; N + 1; x^2) \end{aligned} \tag{27}$$

where

$$F(a,b;c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots$$

is the usual Gaussian hypergeometric function. The polynomial $R_{N,n}(x)$ is readily expressed in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ (Ref. 5, Chap. IV). Adopting the notation of Szegö, we have

$$R_{N,n}(x) = \binom{n+N}{n}^{-1} P_n^{(N,0)}(1 - 2x^2). \tag{28}$$

From (27), (28) and the known properties of the Jacobi polynomials, one finds

$$T_{N,n}(1) = (-1)^n \binom{n+N}{n}^{-1} \tag{29}$$

$$\int_0^1 T_{N,n}(x) T_{N,n'}(x)\,dx = \frac{\delta_{nn'}}{2(2n + N + 1)\binom{n+N}{n}} \tag{30}$$

$$\begin{aligned} 2(n + N + 1)^2(2n + N)R_{N,n+1} &= (2n + N + 1)\,[(2n + N)(2n + N + 2)(1 - 2x^2) \\ &\quad + N^2]R_{N,n} - 2n^2(2n + N + 2)R_{N,n-1} \end{aligned}$$

$$\begin{aligned} (2n + N)x(1 - x^2)\frac{d}{dx}R_{N,n}(x) &= n[(2n + N)(1 - 2x^2) - N]R_{N,n}(x) - 2n^2R_{N,n-1}(x) \end{aligned}$$

$$x^2T_{N,n}(x) = \gamma_{N,n}{}^1 T_{N,n+1}(x) + \gamma_{N,n}{}^0 T_{N,n}(x) + \gamma_{N,n}{}^{-1} T_{N,n-1}(x) \tag{31}$$

* It has been called to our attention that our $T_{N,n}(x)$ are closely related to the Zernike polynomials. These latter arise in the diffraction theory of aberrations.[12]

$$\gamma_{N,n}^{\ 1} = -\frac{(n + N + 1)^2}{(2n + N + 1)(2n + N + 2)}$$
$$\gamma_{N,n}^{\ 0} = \frac{1}{2}\left(1 + \frac{N^2}{(2n + N)(2n + N + 2)}\right) \tag{32}$$
$$\gamma_{N,n}^{\ -1} = -\frac{n^2}{(2n + N)(2n + N + 1)}$$
$$| T_{N,n}(x) | \leqq 1 \quad \text{for} \quad 0 \leqq x \leqq 1.$$

The function $T_{N,n}(x)$ has n zeros in (0,1]. We define $T_{N,n}(x) = 0$ if $n < 0$.

Returning now to (25) for arbitrary values of c, we attempt a power series solution in c^2 by writing

$$\varphi(x) = \varphi_{N,n}(x) = T_{N,n}(x) + \sum_{j=1}^{\infty} c^{2j} Q_j(N,n,x) \tag{33}$$

$$\chi = \chi_{N,n}(c) = \chi_{N,n}(0) + \sum_{j=1}^{\infty} c^{2j} a_j(N,n), \tag{34}$$

where the Q's and a's are independent of c. When this latter quantity is zero, this solution reduces to that already found. As is shown in Appendix A, the Q's and a's can be determined recursively in an elementary manner. We have

$$Q_j(N,n,x) = \sum_{k=-j}^{j} A_k^{\ j}(N,n) T_{N,n+k}(x) \tag{35}$$

with

$$a_j(N,n) = \sum_{k=-1}^{1} A_{-k}^{\ j-1}(N,n) \gamma_{N,n-k}^{\ k}, \qquad j = 1, 2, \ldots \tag{36}$$

$$[\chi_{N,n+m}(0) - \chi_{N,n}(0)] A_m^{\ j}(N,n)$$
$$= \sum_{k=1}^{j} a_k(N,n) A_m^{\ j-k}(N,n) - \sum_{k=-1}^{1} A_{-k+m}^{\ j-1}(N,n) \gamma_{N,n-k+m}^{\ k}, \tag{37}$$
$$m = -j, -j + 1, \ldots, j; \quad j = 1, 2, \ldots .$$

Here $A_k^{\ j}(N,n)$ is defined to be zero if $| k | > j$, or $k < -n$ or $k = 0$ and $j \neq 0$. In addition we have $A_m^{\ 0}(N,n) = 0$, $m \neq 0$, $A_o^{\ 0}(N,n) = 1$, $a_o(N,n) = 0$. For use in (36) and (37), the γ's of (32) must be defined so that for $n < 0$, $\gamma_{N,n}^{\ 1} = \gamma_{N,n}^{\ 0} = \gamma_{N,n+1}^{\ -1} = 0$.

To terms of order c^2 the eigenfunctions and eigenvalues of (25) are explicitly

$$\chi_{N,n}(c) = \left(2n + N + \frac{1}{2}\right)\left(2n + N + \frac{3}{2}\right) + \frac{1}{2}\left(1 + \frac{N^2}{(2n + N)(2n + N + 2)}\right)c^2 + O(c^4) \tag{38}$$

$$\varphi_{N,n}(x) = T_{N,n}(x) + \left(\frac{n^2 T_{N,n-1}(x)}{4(2n + N)^2(2n + N + 1)} - \frac{(n + N + 1)^2 T_{N,n+1}(x)}{4(2n + N + 1)(2n + N + 2)^2}\right)c^2 + O(c^4). \tag{39}$$

In view of (35), the series (33) can be formally regrouped to give

$$\varphi_{N,n}(x) = \sum_0^\infty d_j^{N,n}(c) T_{N,j}(x). \tag{40}$$

Substitution in (25) yields the three-term recurrence

$$c^2\gamma_{N,j-1}^{1} d_{j-1}^{N,n} + [c^2\gamma_{N,j}^{0} + (2j + N + \tfrac{1}{2})(2j + N + \tfrac{3}{2}) - \chi]\, d_j^{N,n} + c^2\gamma_{N,j+1}^{-1} d_{j+1}^{N,n} = 0. \tag{41}$$

This recurrence can be used to determine the $d_j^{N,n}$ and the eigenvalues in a manner quite parallel to that used in the study of prolate spheroidal wave functions. The method of Bouwkamp[6] can be adopted and used advantageously for the computation of the $d_j^{N,n}$ and the eigenvalues for values of c too large to permit effective use of (33) and (34). The d's can, of course, be expressed in terms of the A's of (35). One has

$$d_{n+j}^{N,n}(c) = \sum_{l=|j|}^{\infty} A_j^{l}(N,n)c^{2l}, \qquad j = -n, -n + 1, \ldots . \tag{42}$$

The series solutions (40) or (33), (35) for the generalized prolate spheroidal function are, of course, valid only for $0 \leqq x \leqq 1$. To obtain a series valid for $x > 1$, we use (20) and the fact (established in Appendix B) that

$$\int_0^1 J_N(cxy)\sqrt{cxy}\, T_{N,n}(y)\,dy = \binom{N + n}{n}^{-1} \frac{J_{N+2n+1}(cx)}{\sqrt{cx}}. \tag{43}$$

The solution (40) then extends for all x by the series

$$\varphi_{N,n}(x) = \frac{1}{\gamma_{N,n}} \sum_{j=0}^{\infty} d_j^{N,n} \frac{J_{N+2j+1}(cx)}{\binom{N + j}{j}\sqrt{cx}} \tag{44}$$

which is obtained by inserting (40) in the right of (20) and integrating term by term.

The eigenvalue $\gamma_{N,n}$ can be expressed in terms of the $d_j^{N,n}$. Divide both sides of the equation

$$\gamma_{N,n}\varphi_{N,n}(x) = \int_0^1 J_N(cxx')\sqrt{cxx'}\,\varphi_{N,n}(x')dx' \tag{45}$$

by $x^{N+\frac{1}{2}}$ and take limits as $x \to 0$. From (27) and (40), we see that the left member of (45) becomes

$$\gamma_{N,n}\sum_j d_j^{N,n}R_{N,n}(0) = \gamma_{N,n}\sum_j d_j^{N,n}.$$

Since $J_N(x)\sqrt{x} \sim (x/2)^N\sqrt{x}/\Gamma(N+1)$, the right of (45) becomes

$$\frac{1}{\Gamma(N+1)2^N}\int_0^1 (cx')^{N+\frac{1}{2}}\varphi_{N,n}(x')dx'$$

$$= \frac{c^{N+\frac{1}{2}}}{\Gamma(N+1)2^N}\sum_{j=0} d_j^{N,n}\int_0^1 x'^{N+\frac{1}{2}}T_{N,j}(x')dx'$$

$$= \frac{c^{N+\frac{1}{2}}}{\Gamma(N+1)2^N}\sum_{j=0} d_j^{N,n}\int_0^1 T_{N,o}(x')T_{N,j}(x')dx$$

$$= \frac{c^{N+\frac{1}{2}}d_o^{N,n}}{\Gamma(N+2)2^{N+1}}$$

where we have used successively (40), (27) and (30). The combined result is

$$\gamma_{N,n} = \frac{c^{N+\frac{1}{2}}d_o^{N,n}}{2^{N+1}\Gamma(N+2)\sum_{j=0} d_j^{N,n}}. \tag{46}$$

The integral equation (45) is also useful for obtaining the asymptotic behavior of $\varphi_{N,n}(x)$ as $x \to \infty$. We have

$$\gamma_{N,n}\varphi_{N,n}(x) = \frac{1}{cx}\int_0^{cx} du\, u^{N+1}J_N(u)\,\frac{\varphi_{N,n}(u/cx)}{u^{N+\frac{1}{2}}} \tag{47}$$

on letting $cxx' = u$. Now $(u^{N+1}J_{N+1})' = (u^{N+1}J_N)$, so that (47) can be integrated by parts to yield

$$\gamma_{N,n}\varphi_{N,n}(x) = \frac{1}{cx}\left[u^{N+1}J_{N+1}(u)\,\frac{\varphi_{N,n}(u/cx)}{u^{N+\frac{1}{2}}}\Bigg|_0^{cx}\right.$$

$$\left. - \int_0^{cx} du\, u^{N+1}J_{N+1}(u)\,\frac{d}{du}\,\frac{\varphi_{N,n}(u/cx)}{u^{N+\frac{1}{2}}}\right]$$

$$= \frac{1}{cx}\sqrt{cx}\,\varphi_{N,n}(1)J_{N+1}(cx) - R.$$

For large x, this becomes

$$\gamma_{N,n}\varphi_{N,n}(x) = \varphi_{N,n}(1)\sqrt{\frac{2}{\pi}}\,\frac{\cos\,[cx - (N+1)(\pi/2) - (\pi/4)]}{cx} + O\left(\frac{1}{x^2}\right). \tag{48}$$

This of course is consistent with (44). If now we define $\varphi_{N,n}{}^*(x)$ to be a generalized prolate spheroidal function normalized so that for large x

$$\varphi_{N,n}{}^*(x) \sim \frac{\cos\,[cx - (N+1)(\pi/2) - (\pi/4)]}{cx}, \tag{49}$$

(48) gives us

$$\gamma_{N,n} = \sqrt{\frac{2}{\pi}}\,\varphi_{N,n}{}^*(1) \tag{50}$$

a relation that will be useful to us later.

5.2 *Asymptotics for Fixed n and Large c*

The behavior of generalized prolate spheroidal functions for large c can be determined by methods quite parallel to those used in Ref. 7 in discussing the prolate spheroidal functions. Five different asymptotic forms for $\varphi_{N,n}(x)$ are found, depending on the x range under consideration. These are properly joined to furnish a solution for all x. For most of these regions, we content ourselves here with writing only the leading term of the asymptotic development.

In (25) we make the substitution $t = x\sqrt{c}$. There results

$$\mathbf{L}\varphi - (1/c)\mathbf{M}\varphi + (\chi/c)\varphi = 0 \tag{51}$$

where the operators are given by

$$\begin{aligned}\mathbf{L} &= \frac{d^2}{dt^2} + \frac{\frac{1}{4} - N^2}{t^2} - t^2 \\ \mathbf{M} &= t^2\frac{d^2}{dt^2} + 2t\frac{d}{dt}.\end{aligned} \tag{52}$$

Now the equation

$$\mathbf{L}U + \lambda U = 0$$

has solutions

$$U = U_{N,n}(t) = e^{-t^2/2} t^{N+\frac{1}{2}} L_n^{(N)}(t^2)$$
$$\lambda_n = 4n + 2N + 2, \qquad n = 0, 1, 2, \ldots \tag{53}$$

(see Ref. 5, p. 99) where $L_n^{(\alpha)}(x)$ is the Laguerre polynomial of degree n in Szegö's notation. The function $U_{N,n}(t)$ has n zeros in $(0, \infty)$. This suggests attempting solution of (51) for large c by the series

$$\varphi_{N,n}^{1} = U_{N,n}(t) + \sum_{j=1}^{\infty} (1/c^j) S_j(N,n,t) \tag{54}$$

$$\chi_{N,n}(c)/c = 4n + 2N + 2 + \sum_{j=1}^{\infty} (1/c^j) b_j(N,n). \tag{55}$$

We now note that

$$\mathbf{M} U_{N,n}(t) = \mu_{N,n}^{1} U_{N,n+2} + \mu_{N,n}^{0} U_{N,n} + \mu_{N,n}^{-1} U_{N,n-2} \tag{56}$$

where

$$\mu_{N,n}^{1} = (n + 1)(n + 2)$$
$$\mu_{N,n}^{0} = -[(2n + 1)(n + N + \tfrac{1}{2}) + \tfrac{3}{4}] \tag{57}$$
$$\mu_{N,n}^{-1} = (n + N)(n + N - 1),$$

a fact which can be readily derived from (52), (53) and the properties of Laguerre polynomials. The perturbation scheme of Appendix A applies therefore, and we find at once that

$$S_j(N,n,t) = \sum_{k=-j}^{j} B_k^{j}(N,n) U_{N,n+2k}(t) \tag{58}$$

where the B's and b's are given by the recurrence

$$b_j(N,n) = \sum_{k=-1}^{1} B_{-k}^{j-1}(N,n) \mu_{N,n-2k}^{k}, \qquad j = 1, 2, \ldots$$

$$8mB_m^{j}(N,n) = \sum_{k=1}^{j} b_k(N,n) B_m^{j-k}(N,n) - \sum_{k=-1}^{1} B_{-k+m}^{j-1}(N,n) \mu_{N,n+2}^{k}(m - k) \tag{59}$$

$$m = -j, -j + 1, \ldots, j; \quad j = 1, 2, \ldots$$

with the convention $B_k^{j}(N,n) \equiv 0$ if $|k| > j$, or $k < -n$ or $k = 0$ and $j \neq 0$. We take $B_m^{0}(N,n) = 0$, $m \neq 0$, $B_0^{0}(N,n) = 1$, $b_0(N,n) = 0$.

In this manner we obtain explicitly

$$\chi_{N,n}(c) = (4n + 2N + 2)c - [(2n + 1)(n + N + \tfrac{1}{2}) + \tfrac{3}{4}] - \frac{(N + 2n + 1)[2n^2 + 2n(N + 1) + N + 2]}{4c} + O\left(\frac{1}{c^2}\right) \quad (60)$$

which gives the behavior of $\chi_{N,n}$ for large c.

We write the solution just found as

$$\varphi_{N,n}^{1}(x) = U_{N,n}(t) + \sum_{j=1}^{\infty} \frac{1}{c^j} \sum_{k=-j}^{j} B_k^{\,j}(N,n) U_{N,n+2k}(t), \qquad t = x\sqrt{c}. \quad (61)$$

The right side of (61) is ordered in powers of c^{-1} when expressed in terms of the variable t. However, if $t = x\sqrt{c}$ is substituted, the terms are no longer so ordered since $U_{N,m+2}(x\sqrt{c})/U_{N,m}(x\sqrt{c}) = O(c)$. The range of x values for which the first few terms of (61) furnish information about $\varphi_{N,n}$ vanishes as c gets large. We shall use (61) only for $0 \leqq x \leqq 1/c^{\frac{1}{4}}$.

To obtain an asymptotic form for $\varphi_{N,n}(x)$ for $c^{-\frac{1}{4}} \leqq x \leqq 1 - (1/c)$ it is convenient to write $\varphi_{N,n}(x) = x^{N+\frac{1}{2}}\psi_{N,n}(x)$ and set $y = \sqrt{1 - x^2}$. Equation (25) now becomes

$$(1 - y^2)\frac{d^2\psi}{dy^2} + \left[\frac{1}{y} - (2N + 3)y\right]\frac{d\psi}{dy} + \left[\chi_{N,n} - c^2 - \left(N + \frac{1}{2}\right)\left(N + \frac{3}{2}\right) + c^2y^2\right]\psi = 0. \quad (62)$$

Into this equation, substitute $\chi_{N,n}(c)$ as given by (60) and set

$$\psi = \frac{e^{cy}(1 - y)^n}{\sqrt{y}(1 + y)^{N+n+1}}\, v.$$

One finds then for v,

$$\frac{dv}{dy} + O\left(\frac{1}{c}\right) = 0.$$

Accordingly we write

$$\varphi_{N,n}^{2}(x) \sim \frac{x^{N+\frac{1}{2}}(1 - y)^n e^{cy}}{\sqrt{y}(1 + y)^{N+n+1}} \quad (63)$$

$$y = \sqrt{1 - x^2}, \qquad c^{-\frac{1}{4}} \leqq x \leqq 1 - \frac{1}{c}.$$

To obtain an asymptotic form for $\varphi_{N,n}(x)$ valid near $x = 1$, set $y = s/c$ in (62) and again use (60) for $\chi_{N,n}$. There results

$$\frac{d^2\psi}{dy^2} + \frac{1}{s}\frac{d\psi}{ds} - \psi + O\left(\frac{1}{c}\right) = 0.$$

Accordingly we write

$$\begin{aligned} \varphi_{N,n}{}^3(x) &\sim x^{N+\frac{1}{2}} I_o(cy) \\ y &= \sqrt{1-x^2}, \qquad 1 - (1/c) \leqq x \leqq 1 \end{aligned} \tag{64}$$

where $I_o(x)$ is the modified Bessel function. (See Ref. 8, Vol. II, p. 5).

When $x > 1$, we set $z = \sqrt{x^2 - 1}$, and have $y = iz$. The solutions $\varphi_{N,n}{}^2$ and $\varphi_{N,n}{}^3$ then give rise to two more asymptotic forms. We write

$$\varphi_{N,n}{}^4(x) = x^{N+\frac{1}{2}} J_o(cz), \qquad 1 \leqq x \leqq 1 + \frac{1}{c} \tag{65}$$

$$\varphi_{N,n}{}^5(x) = x^{N+\frac{1}{2}} \operatorname{Re} \frac{e^{icz}(1-iz)^n}{\sqrt{iz}(1+iz)^{N+n+1}}, \qquad 1 + \frac{1}{c} \leqq x, \tag{66}$$

$$z = \sqrt{x^2 - 1}.$$

We now determine the joining factors for these five solutions. In $\varphi_{N,n}{}^1$ and $\varphi_{N,n}{}^2$ we set $x = u/c^{\frac{1}{4}}$ and let c become large for fixed u. One finds

$$\varphi_{N,n}{}^1(u/c^{\frac{1}{4}}) \sim \frac{(-1)^n c^{(2n+N+\frac{1}{2})/4}}{n!} u^{2n+N+\frac{1}{2}} e^{-u^2\sqrt{c}/2}$$

$$\varphi_{N,n}{}^2(x = u/c^{\frac{1}{4}}) \sim \frac{e^c c^{-(2n+N+\frac{1}{2})/4}}{2^{N+2n+1}} u^{2n+N+\frac{1}{2}} e^{-u^2\sqrt{c}/2}$$

where we have used the fact that

$$L_n{}^N(u^2\sqrt{c}) \sim (-1)^n u^{2n} c^{n/2}/n!.$$

When $y = v/\sqrt{c}$, one finds for fixed v and large c

$$\varphi_{N,n}{}^2(y = v/\sqrt{c}) \sim c^{\frac{1}{4}} e^{v\sqrt{c}}/\sqrt{v}$$

$$\varphi_{N,n}{}^3(y = v/\sqrt{c}) \sim \frac{1}{\sqrt{2\pi c^{\frac{1}{2}}}} e^{v\sqrt{c}}/\sqrt{v}$$

where to obtain this last expression we have used the known asymptotic formula $I_o(x) \sim e^x/\sqrt{2\pi x}$ (see Ref. 8, Vol. II, p. 86). Finally, when $z = v/\sqrt{c}$ we find

$$\varphi_{N,n}{}^{4}(z = v/\sqrt{c}) \sim \sqrt{\frac{2}{\pi}}\, c^{-\frac{1}{4}} \frac{\cos (v\sqrt{c} - \pi/4)}{\sqrt{v}}$$

$$\varphi_{N,n}{}^{5}(z = v/\sqrt{c}) \sim c^{\frac{1}{4}} \frac{\cos (v\sqrt{c} - \pi/4)}{\sqrt{v}}$$

where we have made use of the formula (see Ref. 8, Vol. II, p. 85) $J_o(z) \sim (\pi z/2)^{-\frac{1}{2}} \cos (z - \pi/4)$.

All these results can be summarized in the following statement:

$$\hat{\varphi}_{N,n}(x) \sim \begin{cases} e^{-t^2/2} t^{N+\frac{1}{2}} L_n^{(N)}(t^2), & 0 \leqq x \leqq c^{-\frac{1}{4}} \\ k_2 \dfrac{x^{N+\frac{1}{2}} e^{cy}(1-y)^n}{\sqrt{y}(1+y)^{N+n+1}}, & c^{-\frac{1}{4}} \leqq x \leqq 1 - c^{-1} \\ k_3 x^{N+\frac{1}{2}} I_o(cy), & 1 - c^{-1} \leqq x \leqq 1 \\ k_4 x^{N+\frac{1}{2}} J_o(cz), & 1 \leqq x \leqq 1 + c^{-1} \\ k_5 x^{N+\frac{1}{2}} \operatorname{Re} \dfrac{e^{icz}(1-iz)^n}{\sqrt{iz}(1+iz)^{N+n+1}}, & 1 + c^{-1} \leqq x \end{cases} \tag{67}$$

where

$$t = x\sqrt{c}, \qquad y = \sqrt{1 - x^2}, \qquad z = \sqrt{x^2 - 1}$$

$$k_2 = \frac{(-1)^n 2^{N+2n+1} c^{n+N/2+\frac{1}{4}} e^{-c}}{n!}$$

$$k_3 = k_4 = \frac{(-1)^n \sqrt{\pi} 2^{N+2n+3/2} c^{n+N/2+\frac{3}{4}} e^{-c}}{n!}$$

$$k_5 = \frac{(-1)^n 2^{N+2n+2} c^{n+N/2+\frac{1}{4}} e^{-c}}{n!}$$

is the asymptotic form for large c of a bounded continuous solution of (25) belonging to the eigenvalue (60).

We next calculate the normalization constant

$$\frac{1}{N_{N,n}{}^2} = \int_0^1 [\hat{\varphi}_{N,n}(x)]^2\, dx.$$

For the contribution due to $\varphi_{N,n}{}^{1}$ we find

$$\int_0^{c^{-\frac{1}{4}}} dx\, [e^{-t^2/2} t^{N+\frac{1}{2}} L_n^{(N)}(t^2)]^2 = \frac{1}{\sqrt{c}} \int_0^{c^{\frac{1}{4}}} dt\, e^{-t^2} t^{2N+1} [L_n^{(N)}(t^2)]^2$$

$$= \frac{1}{2\sqrt{c}} \int_0^{\sqrt{c}} du\, e^{-u} u^N [L_n^{(N)}(u)]^2$$

$$= \frac{\Gamma(n+N+1)}{2\sqrt{c}\,\Gamma(n+1)} [1 + O(c^{N+2n} e^{-\sqrt{c}})]$$

where we have used the fact that

$$\int_0^\infty e^{-x}x^\alpha[L_n^\alpha(x)]^2\,dx = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}$$

(see Ref. 5, p. 99). It is not hard to show that the contribution to $1/N_{N,n}^2$ from integration over the region $c^{-\frac{1}{4}} \leqq x \leqq 1$ is $O(c^p e^{-\sqrt{c}})$ for some $p > 0$. We have then

$$N_{N,n}^2 \sim \frac{2\sqrt{c}\,\Gamma(n+1)}{\Gamma(n+N+1)}. \tag{68}$$

5.3 *Asymptotics for n and c Both Large*

The techniques employed here again follow very closely those used in Ref. 7. We accordingly give a minimum of detail.

We assume that when n and c are both large χ can be written

$$\chi_{N,n} \sim c^2 + 2\delta c + b_0 + b_1/c + \cdots. \tag{69}$$

The ranges of n and c for which this is valid will appear in the analysis to follow.

In (25) make the substitution $x = t/c$ and replace χ by (69). One finds

$$\frac{d^2\varphi}{dt^2} + \left(1 + \frac{\frac{1}{4} - N^2}{t^2}\right)\varphi + O\left(\frac{1}{c}\right) = 0$$

and hence for large c, $\varphi(t) \sim \sqrt{t}J_N(t)$. We write

$$\varphi_{N,n}^6(x) = \sqrt{x}J_N(cx), \qquad 0 \leqq x \leqq \frac{1}{\sqrt{c}}. \tag{70}$$

Returning to (25) with χ replaced by (69), we observe that the substitution

$$\varphi = \frac{\exp\left[i\left(cx - \frac{\delta}{2}\log\frac{1-x}{1+x}\right)\right]}{\sqrt{1-x^2}}\,v$$

yields $\dfrac{dv}{dx} + O\left(\dfrac{1}{c}\right) = 0$, so that for large c, v becomes constant. After multiplying this solution by a complex constant, we take its real part for the next section of φ. Explicitly we define

$$\varphi_{N,n}^7(x) = \sqrt{\frac{2}{\pi c}}\,\frac{\cos\left[cx - \frac{\delta}{2}\log\frac{1-x}{1+x} - (N+\frac{1}{2})\frac{\pi}{2}\right]}{\sqrt{1-x^2}}. \tag{71}$$

Note that when $x = u/\sqrt{c}$ and c is large (71) becomes

$$\varphi_{N,n}{}^7\left(\frac{u}{\sqrt{c}}\right) \sim \sqrt{\frac{2}{\pi c}} \cos\left[u\sqrt{c} - (N + \tfrac{1}{2})\frac{\pi}{2}\right].$$

The asymptotic formula for J_N (see Ref. 8, Vol. II, p. 85) shows that

$$\varphi_{N,n}{}^6(x = u/\sqrt{c}) \sim \sqrt{\frac{2}{\pi c}} \cos\left[u\sqrt{c} - (N + \tfrac{1}{2})\frac{\pi}{2}\right]$$

also, so that $\varphi_{N,n}{}^6$ and $\varphi_{N,n}{}^7$ agree for large c in the neighborhood of $x = 1/\sqrt{c}$.

To find an appropriate asymptotic form for φ valid near $x = 1$, substitute $\varphi = x^{N+1}e^{ic(1-x)}u$ into (25) with χ given by (69). Now make the substitution $x = 1 - i\xi/2c$. There results

$$\xi \frac{d^2u}{d\xi^2} + (1 - \xi)\frac{du}{d\xi} - \left(\frac{1}{2} - i\frac{\delta}{2}\right)u + O\left(\frac{1}{c}\right) = 0.$$

Accordingly, we are led to define

$$\varphi_{N,n}{}^8(x) = x^{N+\frac{1}{2}}e^{ic(1-x)}\Phi\left[\frac{1}{2} - i\frac{\delta}{2}, 1; -2ic(1 - x)\right] \tag{72}$$

where

$$\Phi(a,b;x) = 1 + \frac{a}{b}\frac{x}{1!} + \frac{a(a+1)}{b(b+1)}\frac{x^2}{2!} + \cdots$$

is the confluent hypergeometric function in the notation of Ref. 8, Vol. I, Chap. 6.

The solution (72) is real. Its asymptotic form for large c when $x = 1 \pm v/\sqrt{c}$ can be found from the known† asymptotics for the Φ function. One finds

$$\varphi_{N,n}{}^8(x = 1 \pm v/\sqrt{c}) \sim \frac{\sqrt{2}e^{\pm\delta(\pi/4)}}{\sqrt{v}c^{\frac{1}{4}}R(\delta)} \cos\left[v\sqrt{c} \mp \frac{\delta}{2}\log(2v\sqrt{c}) \pm \theta(\delta) - \frac{\pi}{4}\right] \tag{73}$$

where the real functions $R(\delta)$ and $\theta(\delta)$ are defined by

$$\Gamma\left(\frac{1}{2} + i\frac{\delta}{2}\right) = R(\delta)e^{i\theta(\delta)}. \tag{74}$$

This latter definition is made precise by requiring $\theta(\delta)$ to be continuous with $\theta(0) = 0$.

Now when $x = 1 - v/\sqrt{c}$, (71) shows that

† See Ref. 8, Vol. I, p. 278, Eq. (2).

$$\varphi_{N,n}{}^7(x = v/\sqrt{c}) \sim \frac{1}{c^{\frac{1}{4}}\sqrt{2\pi v}} \cos\left[v\sqrt{c} + \frac{\delta}{2}\log\frac{v}{2\sqrt{c}} + (N + \tfrac{1}{2})\frac{\pi}{2} - c\right]. \tag{75}$$

Comparison of this expression with (73) shows that $\varphi_{N,n}{}^7$ and $(-1)^q \cdot R(\delta)e^{\delta\pi/4}\varphi_{N,n}{}^8/2\sqrt{\pi}$ are asymptotically the same for $x = 1 - v/\sqrt{c}$ provided

$$c + \delta\log(2\sqrt{c}) - \theta(\delta) = (N + 1)(\pi/2) + \pi q \tag{76}$$

with q an integer to be determined shortly.

Quite analogous to (71) is the solution for $x > 1$,

$$\varphi_{N,n}{}^9(x) = \frac{e^{\delta(\pi/2)}}{\sqrt{\pi c}} \frac{\cos\left[cx - \frac{\delta}{2}\log\frac{x-1}{x+1} - (N+1)\frac{\pi}{2} - \frac{\pi}{4}\right]}{\sqrt{x^2 - 1}}. \tag{77}$$

When $x = 1 + v/\sqrt{c}$ and c is large, this solution becomes

$$\varphi_{N,n}{}^9(x = 1 + v/\sqrt{c}) \sim \frac{e^{\delta(\pi/2)}}{c^{\frac{1}{4}}\sqrt{2\pi v}} \cos\left[c + v\sqrt{c} - \frac{\delta}{2}\log\frac{v}{2\sqrt{c}} - (N+1)\frac{\pi}{2} - \frac{\pi}{4}\right].$$

Comparison with (73) shows that this is the same as $(-1)^q(R(\delta)e^{\delta\pi/4}/2\cdot\sqrt{\pi})\varphi_{N,n}{}^8(x = 1 + v/\sqrt{c})$ when account is taken of (76).

Our results thus far can be summarized as follows:

$$\varphi_{N,n}(x) \sim \begin{cases} \sqrt{x}J_N(cx), & 0 \leqq x \leqq c^{-\frac{1}{2}} \\[2ex] \sqrt{\dfrac{2}{\pi c}}\, \dfrac{\cos\left[cx - \frac{\delta}{2}\log\frac{1-x}{1+x} - (N + \frac{1}{2})\frac{\pi}{2}\right]}{\sqrt{1 - x^2}}, & c^{-\frac{1}{2}} \leqq x \leqq 1 - c^{-\frac{1}{2}} \\[2ex] \dfrac{(-1)^q R(\delta)e^{\delta(\pi/4)}}{2\sqrt{\pi}}\, x^{N+\frac{1}{2}}e^{ic(1-x)}\, \Phi\left[\frac{1}{2} - i\frac{\delta}{2}, 1; -2ic(1-x)\right], & |x - 1| \leqq c^{-\frac{1}{2}} \\[2ex] \dfrac{e^{\delta(\pi/2)}}{\sqrt{\pi c}}\, \dfrac{\cos\left[cx - \frac{\delta}{2}\log\frac{x-1}{x+1} - (N+1)\frac{\pi}{2} - \frac{\pi}{4}\right]}{\sqrt{x^2 - 1}} & x \geqq 1 + c^{-\frac{1}{2}} \end{cases} \tag{78}$$

is the asymptotic form for large n and c of a continuous solution of (25) provided δ and q are chosen to satisfy (76) and the requirement that φ as given by (78) has n zeros in the open x-interval (0,1). The corresponding eigenvalue is given by $\chi_{N,n} \sim c^2 + 2\delta c + O(1)$. Higher-order terms can be found by methods analogous to those presented in Ref. 7.

When c becomes large and δ remains fixed, i.e., $\delta = O(1)$, the number of zeros of $\varphi_{N,n}(x)$ in $0 < x \leqq 1$ can be estimated roughly from (78). Using the asymptotic expansion for J_N, we find that $\varphi_{N,n}{}^6(x)$ contributes

$$z_6 = (\sqrt{c}/\pi) + O(1)$$

zeros as x ranges from zero to $1/\sqrt{c}$. From $\varphi_{N,n}{}^7(x)$ we find

$$z_7 = (1/\pi)[c - 2\sqrt{c} + (\delta/2) \log \sqrt{c}] + O(1)$$

zeros for $1/\sqrt{c} \leqq x \leqq 1 - 1/\sqrt{c}$. Finally, by using the asymptotic form (73) for $\varphi_{N,n}{}^8$, the number of zeros of φ for $1 - 1/\sqrt{c} \leqq x \leqq 1$ is estimated as

$$z_8 = (1/\pi)[\sqrt{c} + (\delta/2) \log \sqrt{c}] + O(1).$$

Since we must have $n = z_6 + z_7 + z_8$, the last three equations show that

$$n\pi = c + \delta \log 2\sqrt{c} + O(1).$$

Combined with (76) this implies that as $c \to \infty$,

$$\theta(\delta) + (N + 1)(\pi/2) + \pi q - n\pi = O(1). \tag{79}$$

The equation just established can be used to obtain a limiting result. Let N be fixed and suppose that n grows with c according to

$$n = (1/\pi)[c + b \log (2\sqrt{c})] \tag{80}$$

where b is a fixed number (independent of c). Multiply this equation by π, add to (76) and rearrange to obtain

$$(\delta - b) \log (2\sqrt{c}) = \theta(\delta) + (N + 1)(\pi/2) + \pi q - n\pi = O(1) \tag{81}$$

where the last equality comes from (79). Divide (81) by $\log(2\sqrt{c})$. We then obtain the limit result: *if n grows with c according to* (80), *then*

$$\lim_{c\to\infty} \delta = b. \tag{82}$$

VI. ASYMPTOTICS OF $\gamma_{N,n}$ AND $\lambda_{N,n}$

6.1 *Fixed N and n. Large c*

The asymptotic solution $\hat{\varphi}_{N,n}(x)$ given in (67) has the values

$$\hat{\varphi}_{N,n}(1) = k_3$$

$$\hat{\varphi}_{N,n}(x \to \infty) \sim (-1)^n k_5 \frac{\cos [cx - (N+1)(\pi/2) - (\pi/4)]}{x} .$$

On recalling the definition given in (49), we see that for large c

$$\varphi_{N,n}^*(x) \sim (-1)^n \hat{\varphi}_{N,n}(x)/ck_5 ,$$

so that for fixed n and N as c becomes large

$$\varphi_{N,n}^*(1) \sim \frac{(-1)^n k_3}{ck_5} = (-1)^n \sqrt{\frac{\pi}{2c}} . \tag{83}$$

Equation (50) then gives

$$\gamma_{N,n} \sim \frac{(-1)^n}{\sqrt{c}} . \tag{84}$$

We now proceed to use (84) and the useful formula (to be established)

$$\frac{\partial \gamma_{N,n}}{\partial c} = \frac{\gamma_{N,n}}{2c} [\phi_{N,n}^2(1) - 1] \tag{85}$$

where

$$\int_0^1 \varphi_{N,n}^2(x)\, dx = 1 \tag{86}$$

to get a much stronger statement regarding the asymptotic behavior of $\gamma_{N,n}$. First we establish (85)–(86).

For simplicity of notation let us write (45) as

$$\gamma_n \varphi_n(x) = \int_0^1 K(cxx')\varphi_n(x')\, dx' \tag{87}$$

where we have suppressed dependences on N. Differentiating, we find

$$\frac{\partial \gamma_n}{\partial c} \varphi_n(x) + \gamma_n \frac{\partial \varphi_n(x)}{\partial c} = \int_0^1 xx'K'(cxx')\varphi_n(x')\, dx' + \int_0^1 K(cxx') \frac{\partial \varphi_n(x')}{\partial c}\, dx'. \tag{88}$$

Differentiating (87) with respect to x gives

$$\gamma_n \varphi_n'(x) = \frac{c}{x} \int_0^1 xx'K'(cxx')\varphi_n(x')\, dx',$$

so that (88) becomes

$$\frac{\partial \gamma_n}{\partial c} \varphi_n(x) + \gamma_n \frac{\partial \varphi_n(x)}{\partial c} = \frac{x}{c} \gamma_n \varphi_n'(x) + \int_0^1 K(cxx') \frac{\partial \varphi_n(x')}{\partial c} dx'.$$

Multiply this equation by $\varphi_n(x)$ and integrate. One finds

$$\frac{\partial \gamma_n}{\partial c} \int_0^1 \varphi_n^2(x)\, dx + \gamma_n \int_0^1 \varphi_n(x) \frac{\partial \varphi_n(x)}{\partial c} dx$$
$$= \gamma_n \int_0^1 \frac{x}{2c} \frac{d}{dx} \varphi_n^2(x)\, dx + \gamma_n \int_0^1 \varphi_n(x') \frac{\partial \varphi_n(x')}{\partial c} dx',$$

where the last term has been obtained by interchange of orders of integration and use of (87). Equation (85) then follows by integrating the first term on the right by parts and by using (86).

To use effectively (85)–(86) it is convenient to introduce $\kappa_{N,n} \equiv (-1)^n \cdot \sqrt{c}\gamma_{N,n}$. We then have

$$\frac{1}{\kappa_{N,n}} \frac{\partial \kappa_{N,n}}{\partial c} = \frac{1}{2c} \varphi_{N,n}^2(1) \tag{89}$$

$$\lim_{c\to\infty} \kappa_{N,n} = 1 \tag{90}$$

from (85) and (84) respectively. From (67) and (68) we see that

$$\varphi_{N,n}^2(1) \sim k_3^2 N_{N,n}^2 = \frac{\pi 2^{2N+4n+4} c^{N+2n+2} e^{-2c}}{\Gamma(n+1)\Gamma(n+N+1)}.$$

Using this expression in (89) and integrating, we obtain

$$\log \kappa_{N,n} \Big|_c^\infty = \frac{\pi 2^{2N+4n+3}}{\Gamma(n+1)\Gamma(n+N+1)} \int_c^\infty t^{N+2n+1} e^{-2t}\, dt.$$

Integrating by parts and using (90), we finally find

$$\gamma_{N,n} = \frac{(-1)^n}{\sqrt{c}} - \frac{(-1)^n \pi 2^{2N+4n+2} c^{N+2n+\frac{1}{2}} e^{-2c}}{\Gamma(n+1)\Gamma(n+N+1)} \left[1 + O\left(\frac{1}{c}\right)\right]. \tag{91}$$

In terms of λ of (13), we find from (17) and (19)

$$\lambda_{N,n} = c\gamma_{N,n}^2 \tag{92}$$

so that

$$\lambda_{N,n} = 1 - \frac{\pi 2^{2N+4n+3} c^{N+2n+1} e^{-2c}}{\Gamma(n+1)\Gamma(n+N+1)} \left[1 + O\left(\frac{1}{c}\right)\right]. \tag{93}$$

6.2 *Fixed N and n. Small c.*

We use (46) to obtain an expression for $\gamma_{N,n}$ for small c. From (42), it follows that

$$d_o^{N,n}(c) = \sum_{l=n}^{\infty} A_{-n}^{\ l}(N,n)c^{2l} = A_{-n}^{\ n}(N,n)c^{2n}[1 + O(c^2)]$$
$$= \frac{(-1)^n\Gamma(n+1)\Gamma(n+N+1)\Gamma(N+2)}{2^{2n}\Gamma(2n+N+1)\Gamma(2n+N+2)} c^{2n}[1 + O(c^2)] \tag{94}$$

where we have used (124), (26) and (32). From (42) one has

$$\sum_{j=0} d_j^{N,n} = d_n^{N,n} + O(c^2) = 1 + O(c^2). \tag{95}$$

Equations (94), (95) and (46) now give

$$\gamma_{N,n} = \frac{(-1)^n\Gamma(n+1)\Gamma(n+N+1)}{2^{2n+N+1}\Gamma(2n+N+1)\Gamma(2n+N+2)} c^{2n+N+\frac{1}{2}}. \tag{96}$$

Higher-order terms could be obtained in a similar manner. An alternative route, however, is to use (85) and (86). From (39) and (30), one sees that $[2(2n+N+1)]^{\frac{1}{2}}\binom{n+N}{n}(-1)^n[1+O(c^4)]$ is the normalization factor for (39). Using (39) one then finds for a normalized solution

$$\varphi_{N,n}(1) = (-1)^n\sqrt{2(2n+N+1)}\left[1 + \frac{N^2c^2}{4(2n+N)^2(2n+N+2)^2}\right] + O(c^4).$$

Inserting this expression in (85) and integrating, we find

$$\gamma_{N,n} = \frac{(-1)^n\Gamma(n+1)\Gamma(n+N+1)c^{2n+N+\frac{1}{2}}}{2^{2n+N+1}\Gamma(2n+N+1)\Gamma(2n+N+2)} \cdot\left[1 + \frac{N^2c^2}{4(2n+N)^2(2n+N+2)^2} + O(c^4)\right]. \tag{97}$$

6.3 *Asymptotics for n and c Both Large*

To obtain an expression for $\gamma_{N,n}$ valid for n and c both large, we use (77) and (49–50). For the asymptotic solution (77) we have

$$\varphi_{N,n}(1) \sim \frac{R(\delta)e^{\delta(\pi/4)}(-1)^q}{2\sqrt{\pi}} \tag{98}$$

and for very large x

$$\varphi_{N,n}(x \to \infty) \sim \frac{e^{\delta(\pi/2)}}{\sqrt{\pi c}} \frac{\cos[cx - (N+1)(\pi/2) - (\pi/4)]}{x}.$$

Comparison with (49) shows that $\varphi_{N,n}^* = \sqrt{\pi/c}e^{-\delta(\pi/2)}\varphi_{N,n}$, and (98) and (50) now give

$$\gamma_{N,n} \sim \frac{(-1)^q R(\delta) e^{-\delta(\pi/4)}}{\sqrt{2\pi c}}. \tag{99}$$

Now (91) and (97) show that for large and small c the sign of $\gamma_{N,n}$ is the same as the sign of $(-1)^n$. As c varies, $\gamma_{N,n}$ cannot change sign, for by (92) if $\gamma_{N,n}$ were to vanish for some value of $c \neq 0$, so would $\lambda_{N,n}$. Since, as we have noted in Sections II and III, the kernel K_c of (12) is positive definite, this is impossible. We can therefore replace q by n in (99) and we have

$$q = n (\text{mod } 2). \tag{100}$$

From the definition (74) of $R(\delta)$, one has

$$[R(\delta)]^2 = \Gamma\left(\frac{1}{2} + i\frac{\delta}{2}\right) \Gamma\left(\frac{1}{2} - i\frac{\delta}{2}\right) = \frac{\pi}{\cosh \delta(\pi/2)}. \tag{101}$$

Here we have used the functional relation [Ref. 8, Vol. I, p. 3, Eq. (7)] for the gamma function

$$\Gamma(\tfrac{1}{2} + z)\Gamma(\tfrac{1}{2} - z) = \pi \sec \pi z.$$

Equations (99), (100), (101) and (92) combined are

$$\gamma_{N,n} \sim \frac{(-1)^n}{\sqrt{c(1 + e^{\pi\delta})}}, \qquad \lambda_{N,n} \sim \frac{1}{1 + e^{\pi\delta}}. \tag{102}$$

Finally from (80), (82) and (102) we have the limiting result: if

$$n = [(1/\pi)(c + b \log 2\sqrt{c})]$$

where the brackets denote "largest integer in" and b is a fixed number, then

$$\lim_{c\to\infty} \lambda_{N,n} = \frac{1}{1 + e^{\pi b}}. \tag{103}$$

VII. THE CASE $D > 2$, R THE UNIT SPHERE

In the previous sections, we have treated the important special case $D = 2$, R the unit circle in considerable detail. Most of the analysis there was concerned with solving the integral equation (20). Fortunately, as we shall now see, the solution of that equation also affords a complete solution of the case R the unit sphere centered at the origin in E_D, $D = 3, 4, \ldots$. In treating this general case, we shall draw freely on the theory of D-dimensional spherical harmonics as given, for example, in Ref. 8, Vol. II, Chap. XI. We follow the notation of this work and set

$$D = p + 2, \qquad p = 1, 2, \ldots. \tag{104}$$

Let $\mathbf{x} = r\boldsymbol{\xi}$ and $\mathbf{y} = r'\mathbf{n}$ where $\boldsymbol{\xi}$ and $\mathbf{n}$ are unit vectors in E_{p+2}. Equation (8) now becomes

$$\alpha\psi(r,\boldsymbol{\xi}) = \int_0^1 dr' r'^{p+1} \int_\Omega \exp\,(icrr'\boldsymbol{\xi}\cdot\mathbf{n})\psi(r',\mathbf{n})\, d\Omega(\mathbf{n}) \tag{105}$$

where Ω is the surface of the unit sphere in E_{p+2}.

Now let

$$h(N,p) = (2N + p)\,\frac{(N + p - 1)!}{p!N!}, \qquad N = 0, 1, 2, \ldots, \tag{106}$$

and let $S_N{}^l(\boldsymbol{\xi})$, $l = 1, 2, \cdots, h(N,p)$, be a complete set of orthonormal surface harmonics of degree N. The Funk-Hecke theorem (Ref. 8, Vol. II, pp. 247–248) asserts that

$$\int_\Omega \exp\,(icrr'\boldsymbol{\xi}\cdot\mathbf{n})S_N{}^l(\mathbf{n})\, d\Omega(\mathbf{n}) = H_N(crr')S_N{}^l(\boldsymbol{\xi}) \tag{107}$$

where

$$H_N(crr') = \frac{2\pi^{(p+1)/2}N!(p - 1)!}{\Gamma\left(\frac{p+1}{2}\right)(N + p - 1)!}\int_{-1}^1 e^{icrr'u}C_N{}^{p/2}(u)(1 - u^2)^{(p-1)/2}\, du \tag{108}$$

is independent of l and $C_N{}^\eta(u)$ is a Gegenbauer polynomial (Ref. 8, Vol. II, p. 235). By expanding ψ in surface harmonics,

$$\psi(r,\boldsymbol{\xi}) = \sum_{N=0}^{\infty}\sum_{l=0}^{h(N,p)} R_{N,l}(r)S_N{}^l(\boldsymbol{\xi}),$$

we find from (105) and (107)

$$\alpha_{N,l}R_{N,l}(r) = \int_0^1 dr' r'^{p+1}H_N(crr')R_{N,l}(r'), \tag{109}$$

from which it is seen that $R_{N,l}(r)$ and $\alpha_{N,l}$ are independent of l. We have the expected degeneracy of eigenvalues due to spherical symmetry.

Now [Ref. 8, Vol. II, p. 236, Eq. (25)]

$$C_N{}^{p/2}(u) = \frac{(-1)^N}{2^N}\frac{(p)_N}{\left(\frac{p+1}{2}\right)_N N!}(1 - u^2)^{-(p-1)/2}\frac{d^N}{du^N}(1 - u^2)^{N+(p-1)/2},$$

where $(a)_N = a(a + 1)\cdots(a + N - 1)$, so that from (108)

$$H_N(crr') = \frac{2\pi^{(p+1)/2}(-1)^N}{\Gamma\left(\frac{p+1}{2}\right)2^N\left(\frac{p+1}{2}\right)_N}\int_{-1}^1 du\, e^{icrr'u}\frac{d^N}{du^N}(1 - u^2)^{N+(p-1)/2}.$$

Integration by parts gives for the integral here

$$(-1)^N \int_{-1}^{1} du(1 - u^2)^{N+(p-1)/2} \frac{d^N}{du^N} e^{icrr'u}$$

$$= (-icrr')^N \int_{-1}^{1} du\, e^{icrr'u}(1 - u^2)^{N+(p-1)/2}$$

$$= (-i)^N \sqrt{\pi}\Gamma\left(N + \frac{p+1}{2}\right) 2^{N+p/2}(crr')^{-p/2} J_{N+p/2}(crr')$$

where we have used the Poisson formula

$$\Gamma(\nu + \tfrac{1}{2})J_\nu(z) = \pi^{-\frac{1}{2}}(z/2)^\nu \int_{-1}^{1} e^{izu}(1 - u^2)^{\nu-\frac{1}{2}}\, du$$

[Ref. 8, Vol. II, p. 81, Eq. (7)]. We have then, finally

$$H_N(crr') = i^N(2\pi)^{1+p/2} J_{N+p/2}(crr')/(crr')^{p/2}.$$

We see now from (109) that the eigenfunctions and eigenvalues of (105) are

$$\begin{aligned} \psi_{N,l,n}(r,\xi) &= R_{N,n}(r) S_N{}^l(\xi), \qquad l = 1, 2, \ldots, h(N,p) \\ \alpha_{N,n} &= i^N (2\pi)^{1+p/2} \beta_{N,n} \\ N,n &= 0, 1, 2, \ldots \end{aligned} \tag{110}$$

where

$$\beta_{N,n} R_{N,n}(r) = \int_0^1 \frac{J_{N+p/2}(crr')}{(crr')^{p/2}} r'^{p+1} R_{N,n}(r')\, dr'. \tag{111}$$

These equations are the analogues of (16), (17) and (18). Set

$$\gamma = \beta c^{(p+1)/2}, \qquad \varphi = r^{(p+1)/2} R. \tag{112}$$

Equation (111) becomes

$$\gamma\varphi(r) = \int_0^1 J_{N+p/2}(crr')\sqrt{crr'}\varphi(r')\, dr'. \tag{113}$$

This, however, is (20) with N replaced by $N + p/2$. The formulae of Section IV for the solutions of (20) can be taken over exactly replacing N by $N + p/2$ throughout. (Expressions involving factorials must be replaced by the appropriate ones in terms of Γ functions when p is an odd integer.) Together with (110), (111) and (112), they provide solution of (105) for all $D \geqq 2$.

It is interesting to note that the one-dimensional case treated in Refs. 1 and 2 can be obtained as a special case of the present theory by appropriate interpretation. The parameter N of this section is the degree of the homogeneous polynomial solution to Laplace's equation in D dimension afforded by the spherical harmonic $S_N^{\,l}$ when expressed in rectangular coordinates. When $D = 1$, Laplace's equation $d^2\psi/dx^2 = 0$ has only two homogeneous solutions, $\psi = k$ and $\psi = x$, respectively of degrees zero and one. For $D = 1$, i.e., $p = -1$ from (104), we have

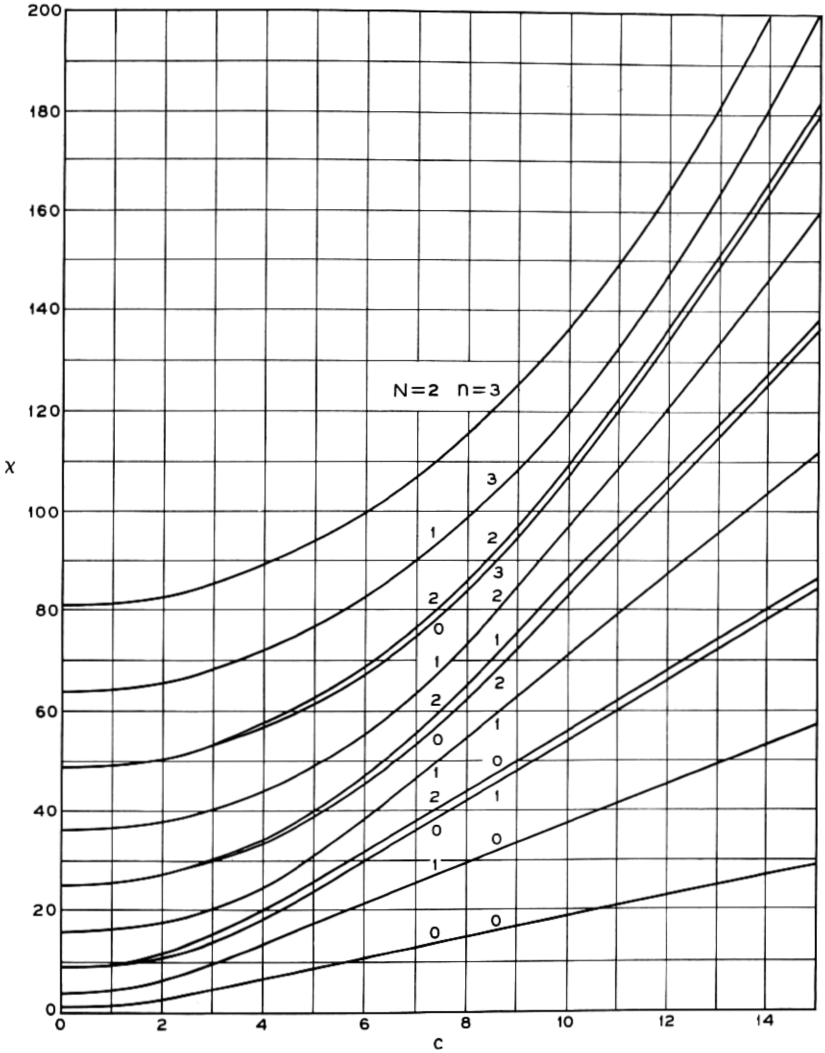

Fig. 1 — Curves of $\chi_{N,n}$ of (25) vs c.

only two allowed values, $N = 0$ and $N = 1$. The quantity $N + p/2$ occurring in (113) then has values $-\frac{1}{2}$ and $\frac{1}{2}$. The kernel becomes $\sqrt{2/\pi} \cos crr'$ and $\sqrt{2/\pi} \sin crr'$ respectively in these two cases, and we retrieve the integral equations for the even and odd prolate spheroidal functions of zero order. Note that when $N = \pm\frac{1}{2}$, (25) reduces to the prolate spheroidal equation.

VIII. NUMERICAL RESULTS

A program for the IBM 7090 has been written to compute generalized prolate spheroidal functions using formulae (40) and (44). Trial values for the $\chi_{N,n}$ were obtained from (34) and (55) and the recurrences (36)–(37) and (59). The method of Bouwkamp[6] was then used to correct these estimates and obtain the $d_j^{N,n}$. Values of $\gamma_{N,n}$ were obtained from (46) and these were converted to values of λ by $\lambda_{N,n} = c\gamma_{N,n}^2$.

Fig. 1 shows plots of $\chi_{N,n}$ versus c. Fig. 2 gives the behavior of the first few $\lambda_{N,n}$. By definition of the labels, $\chi_{N,n+1} \geqq \chi_{N,n}$ for $N,n = 0, 1, \ldots$ and if $c > 0$ the inequality is strict. From Sturmian theory, it follows that $\chi_{N+1,n} > \chi_{N,n}$. For the λ's, one can show correspondingly that $\lambda_{N,n+1} < \lambda_{N,n}$ and $\lambda_{N+1,n} < \lambda_{N,n}$ for $N,n = 0, 1, \ldots$. The problem of ordering the λ's and χ's for all N and n appears to be a difficult one. Some values are listed on Table I.

Figs. 3 and 4 show plots of $\varphi_{N,n}(x)$ versus x for $N = 0,2$, $n = 0,1,2,3$ and $c = 2,10$. Values of the $\varphi_{N,n}$ for a larger set of parameter values are given in Table II. Normalization is as in (86).

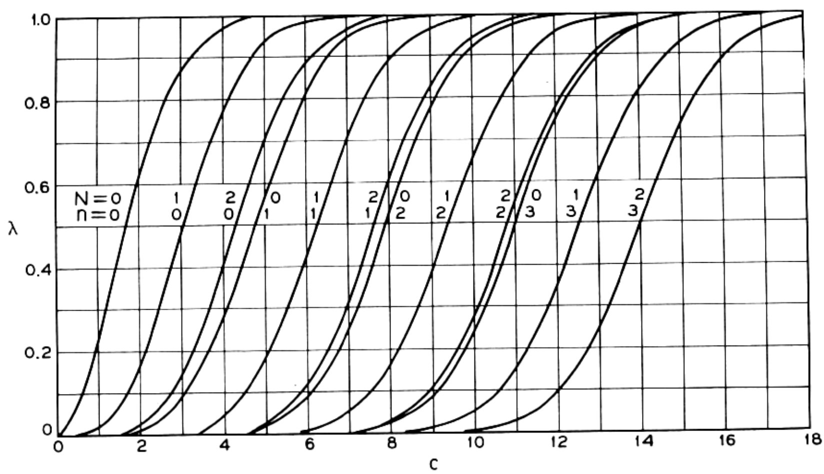

Fig. 2 — Curves of $\lambda_{N,n}$ of (13) and (15) vs c.

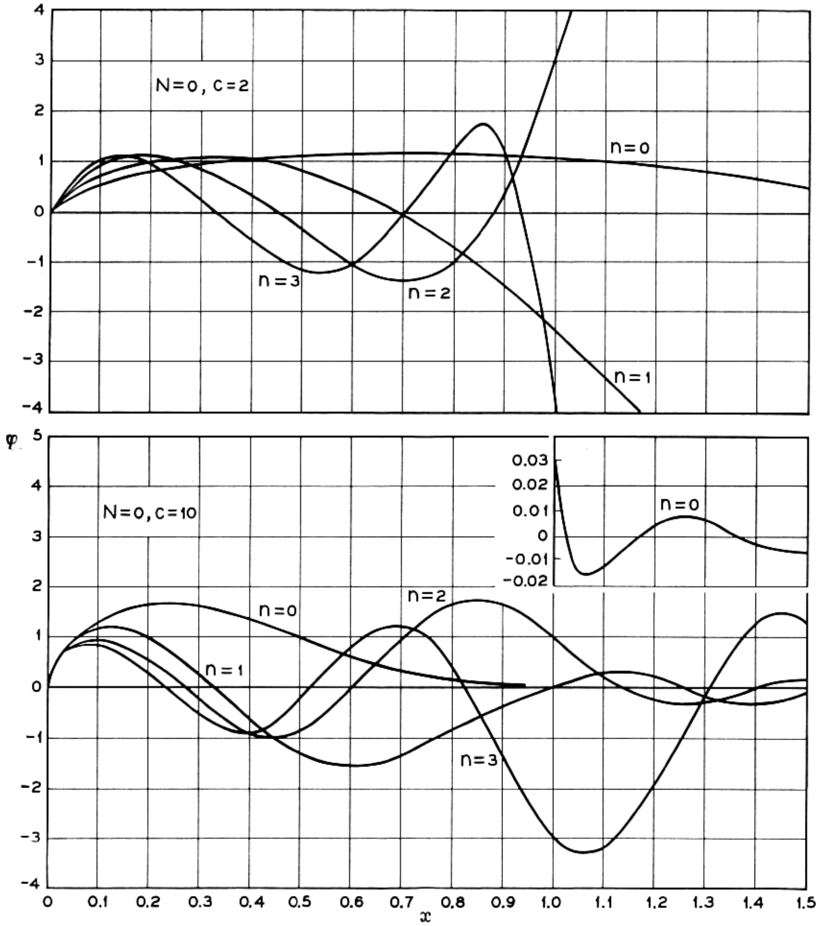

Fig. 3 — Some generalized prolate spheroidal functions, $\varphi_{N,n}(x)$.

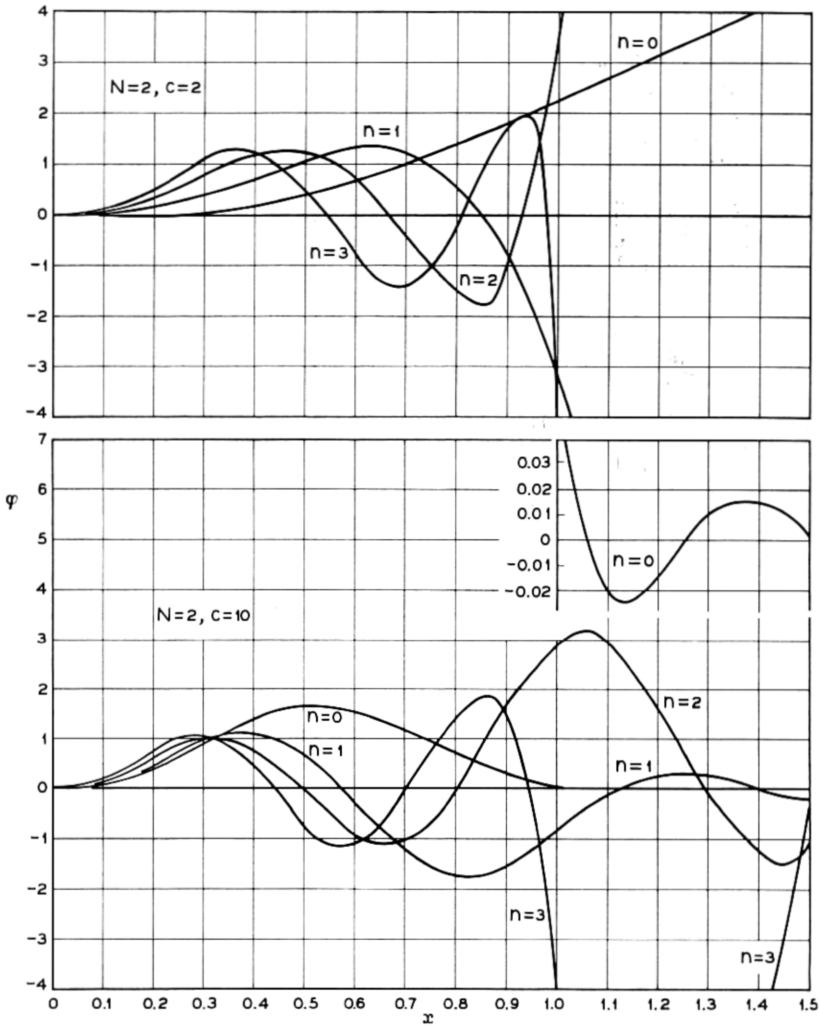

Fig. 4 — More generalized prolate spheroidal functions, $\varphi_{N,n}(x)$.

TABLE I — NUMERICAL VALUES OF $\chi_{N,n}$ AND $\lambda_{N,n}$

$N = 0 \quad n = 0$

c	χ	λ
0.1	7.5499895 − 1	2.4968775 − 3
0.5	8.7434899 − 1	6.0585348 − 2
1.0	1.2395933 + 0	2.2111487 − 1
1.5	1.8225178 + 0	4.2951906 − 1
2.0	2.5857968 + 0	6.2963045 − 1
3.0	4.4622709 + 0	8.8705036 − 1
4.0	6.5208586 + 0	9.7495117 − 1
5.0	8.5869176 + 0	9.9534230 − 1
10.0	1.8690110 + 1	9.9999957 − 1

$N = 0 \quad n = 1$

c	χ	λ
1	9.2562398 + 0	1.0829815 − 4
2	1.0847476 + 1	6.7214485 − 3
3	1.3698728 + 1	6.6745424 − 2
4	1.7898720 + 1	2.6742780 − 1
5	2.3241561 + 1	5.7877057 − 1
6	2.9277622 + 1	8.3060712 − 1
7	3.5550580 + 1	9.4973850 − 1
8	4.1805821 + 1	9.8782700 − 1
9	4.7985976 + 1	9.9738554 − 1
10	5.4108072 + 1	9.9947801 − 1

$N = 0 \quad n = 2$

c	χ	λ
1	2.5751488 + 1	1.8834675 − 9
2	2.6773866 + 1	1.9235204 − 6
5	3.8241737 + 1	1.6017987 − 2
6	4.4846367 + 1	8.1254764 − 2
7	5.3021146 + 1	2.5847455 − 1
8	6.2527715 + 1	5.3544699 − 1
9	7.2854528 + 1	7.8635574 − 1
10	8.3461406 + 1	9.2600949 − 1
11	9.4019226 + 1	9.7915064 − 1
12	1.0443896 + 2	9.9484586 − 1
13	1.1474313 + 2	9.9882732 − 1
14	1.2496987 + 2	9.9974793 − 1
15	1.3514611 + 2	9.9994820 − 1
16	1.4528810 + 2	9.9998984 − 1

$N = 0 \quad n = 3$

c	χ	λ
1	4.9250694 + 1	6.0066949 − 15
2	5.0761114 + 1	9.8333952 − 11
5	6.1688709 + 1	3.5422330 − 5
7	7.4995083 + 1	3.5278392 − 3
8	8.3823340 + 1	2.0130790 − 2
9	9.4336396 + 1	8.2918248 − 2
10	1.0659367 + 2	2.4212641 − 1
11	1.2034708 + 2	4.9658387 − 1
12	1.3504432 + 2	7.4660703 − 1
13	1.5007176 + 2	9.0244395 − 1
14	1.6502439 + 2	9.6944048 − 1
15	1.7977291 + 2	9.9165294 − 1
16	1.9433894 + 2	9.9791376 − 1

$N = 1 \quad n = 0$

c	χ	λ
0.5	3.9163765 + 0	9.4982658 − 4
1.0	4.4119661 + 0	1.3986168 − 2
2.0	6.3394615 + 0	1.6123183 − 1
3.0	9.3427678 + 0	4.8326866 − 1
4.0	1.3086855 + 1	7.8473505 − 1
5.0	1.7170130 + 1	9.3671678 − 1
6.0	2.1310500 + 1	9.8534266 − 1
10.0	3.7555900 + 1	9.9998314 − 1

$N = 1 \quad n = 1$

c	χ	λ
1	1.6255011 + 1	7.4672551 − 7
2	1.7912353 + 1	1.8549511 − 4
4	2.4832293 + 1	3.8313651 − 2
5	3.0401459 + 1	1.6818804 − 1
6	3.7326440 + 1	4.2912557 − 1
7	4.5219234 + 1	7.1473948 − 1
8	5.3565692 + 1	8.9618892 − 1
9	6.1976089 + 1	9.7041388 − 1
10	7.0297509 + 1	9.9279210 − 1

$N = 1 \quad n = 2$

c	χ	λ
1	3.6265101 + 1	4.6976877 − 12
2	3.7820310 + 1	1.9082396 − 8
5	4.9160037 + 1	1.0305314 − 3
6	5.5464880 + 1	8.3652966 − 3
7	6.3286568 + 1	4.4641026 − 2
8	7.2759605 + 1	1.6080875 − 1
9	8.3789365 + 1	3.9082845 − 1
10	9.5955815 + 1	6.6399691 − 1
11	1.0867089 + 2	8.6120123 − 1
12	1.2145589 + 2	9.5495434 − 1
13	1.3409696 + 2	9.8759852 − 1
14	1.4657506 + 2	9.9692509 − 1

$N = 1 \quad n = 3$

c	χ	λ
1	6.4258409 + 1	7.6540787 − 18
2	6.5789319 + 1	4.9988893 − 13
5	7.6749767 + 1	1.1190662 − 6
9	1.0834214 + 2	1.0298512 − 2
10	1.2001776 + 2	4.5573797 − 2
11	1.3350648 + 2	1.5017502 − 1
12	1.4872078 + 2	3.5789707 − 1
13	1.6522672 + 2	6.1969392 − 1
14	1.8237982 + 2	8.2818634 − 1
15	1.9961900 + 2	9.3866345 − 1
16	2.1666412 + 2	9.8150785 − 1
17	2.3347382 + 2	9.9500835 − 1

TABLE I — *Continued*

$N = 2 \quad n = 0$

c	χ	λ
1	9.4976317 + 0	3.9517707 − 4
2	1.1710916 + 1	1.9088335 − 2
3	1.5291960 + 1	1.3627864 − 1
4	2.0048498 + 1	4.0298411 − 1
5	2.5667098 + 1	7.0317221 − 1
6	3.1747966 + 1	8.9417071 − 1
7	3.7952889 + 1	9.7092917 − 1
8	4.4125829 + 1	9.9324606 − 1
10	5.6324064 + 1	9.9972026 − 1

$N = 2 \quad n = 1$

c	χ	λ
1	2.5333581 + 1	4.1659113 − 9
2	2.7088321 + 1	4.0530517 − 6
5	3.9788041 + 1	2.5787154 − 2
6	4.6842565 + 1	1.1592915 − 1
7	5.5371030 + 1	3.2573165 − 1
8	6.5067655 + 1	6.0754452 − 1
9	7.5425367 + 1	8.3161633 − 1
10	8.5969115 + 1	9.4455397 − 1
11	9.6442775 + 1	9.8484078 − 1

$N = 2 \quad n = 2$

c	χ	λ
1	4.9292042 + 1	1.0614368 − 14
2	5.0922802 + 1	1.7074109 − 10
5	6.2566028 + 1	5.5396960 − 5
7	7.6509011 + 1	5.0165514 − 3
8	8.5676381 + 1	2.7077307 − 2
9	9.6519700 + 1	1.0455606 − 1
10	1.0904728 + 2	2.8443961 − 1
11	1.2295597 + 2	5.4626700 − 1
12	1.3768257 + 2	7.8216583 − 1
13	1.5265435 + 2	9.1919613 − 1
14	1.6752489 + 2	9.7528766 − 1
15	1.8220066 + 2	9.9334500 − 1

$N = 2 \quad n = 3$

c	χ	λ
1	8.1275289 + 1	9.2115325 − 21
2	8.2854667 + 1	2.3940665 − 15
5	9.4065006 + 1	3.2594658 − 8
10	1.3679420 + 2	5.8064778 − 3
11	1.4973956 + 2	2.6682390 − 2
12	1.6450899 + 2	9.5057491 − 2
13	1.8113152 + 2	2.5380401 − 1
14	1.9931616 + 2	4.9785798 − 1
15	2.1846670 + 2	7.3845601 − 1
16	2.3793070 + 2	8.9389256 − 1
17	2.5727411 + 2	9.6462917 − 1
18	2.7635393 + 2	9.8968676 − 1

TABLE II — VALUES OF $\varphi_{N,n}(x)$

$N = 0 \quad n = 0$

x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
0.1	4.74638 − 1	5.55421 − 1	9.15662 − 1	1.31455 + 0
0.2	6.68776 − 1	7.74706 − 1	1.22032 + 0	1.62247 + 0
0.3	8.14070 − 1	9.27095 − 1	1.35165 + 0	1.57689 + 0
0.4	9.31948 − 1	1.03607 + 0	1.35103 + 0	1.30428 + 0
0.5	1.03044 + 0	1.11011 + 0	1.24626 + 0	9.31637 − 1
0.6	1.11351 + 0	1.15353 + 0	1.06660 + 0	5.70325 − 1
0.7	1.18341 + 0	1.16921 + 0	8.43474 − 1	2.91331 − 1
0.8	1.24157 + 0	1.15957 + 0	6.07845 − 1	1.17077 − 1
0.9	1.28896 + 0	1.12695 + 0	3.86969 − 1	3.19741 − 2
1.0	1.32627 + 0	1.07383 + 0	2.01532 − 1	3.00159 − 3
1.1	1.35405 + 0	1.00285 + 0	6.38588 − 2	−1.09501 − 3
1.2	1.37278 + 0	9.16840 − 1	−2.24980 − 2	4.23236 − 4
1.3	1.38285 + 0	8.18791 − 1	−6.18395 − 2	6.58696 − 4
1.4	1.38464 + 0	7.11797 − 1	−6.43066 − 2	−2.21883 − 4
1.5	1.37853 + 0	5.98995 − 1	−4.32142 − 2	−5.95391 − 4
1.6	1.36489 + 0	4.83499 − 1	−1.22744 − 2	−1.21194 − 4
1.7	1.34410 + 0	3.68328 − 1	1.68342 − 2	4.20886 − 4
1.8	1.31655 + 0	2.56332 − 1	3.61314 − 2	3.79898 − 4
1.9	1.28264 + 0	1.50130 − 1	4.20554 − 2	−8.00341 − 5

TABLE II — *Continued*

$N = 0 \quad n = 0$				
x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
2.0	1.24281 + 0	5.20429 − 2	3.52268 − 2	−3.67577 − 4
2.1	1.19748 + 0	−3.59472 − 2	1.94309 − 2	−2.42458 − 4
2.2	1.14713 + 0	−1.12245 − 1	1.30129 − 4	1.44876 − 4
2.3	1.09221 + 0	−1.75666 − 1	−1.71274 − 2	3.08117 − 4
2.4	1.03322 + 0	−2.25459 − 1	−2.80628 − 2	1.76068 − 4
2.5	9.70655 − 1	−2.61300 − 1	−3.05323 − 2	−1.11428 − 4
2.6	9.05010 − 1	−2.83289 − 1	−2.47697 − 2	−3.05239 − 4
2.7	8.36800 − 1	−2.91927 − 1	−1.30522 − 2	−6.14275 − 5
2.8	7.66537 − 1	−2.88082 − 1	1.07933 − 3	1.53741 − 4
2.9	6.94735 − 1	−2.72956 − 1	1.38275 − 2	6.59394 − 5
3.0	6.21906 − 1	−2.48005 − 1	2.20781 − 2	1.15331 − 4

$N = 0 \quad n = 1$				
x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
0.1	7.57682 − 1	7.49125 − 1	7.51850 − 1	1.11517 + 0
0.2	1.00189 + 0	9.78062 − 1	8.84348 − 1	9.64135 − 1
0.3	1.08562 + 0	1.03485 + 0	7.51864 − 1	2.39045 − 1
0.4	1.02660 + 0	9.38690 − 1	4.08540 − 1	−6.68891 − 1
0.5	8.24694 − 1	6.94825 − 1	−7.50154 − 2	−1.35733 + 0
0.6	4.76162 − 1	3.08610 − 1	−6.10130 − 1	−1.58691 + 0
0.7	−2.29984 − 2	−2.11235 − 1	−1.10302 + 0	−1.36404 + 0
0.8	−6.75867 − 1	−8.51754 − 1	−1.47075 + 0	−8.87374 − 1
0.9	−1.48414 + 0	−1.59571 + 0	−1.65550 + 0	−4.07419 − 1
1.0	−2.44790 + 0	−2.41297 + 0	−1.63388 + 0	−9.25172 − 2
1.1	−3.56556 + 0	−3.30607 + 0	−1.41956 + 0	2.63841 − 2
1.2	−4.83382 + 0	−4.22107 + 0	−1.05901 + 0	2.11679 − 2
1.3	−6.24769 + 0	−5.13836 + 0	−6.20954 − 1	−1.29202 − 2
1.4	−7.80051 + 0	−6.02868 + 0	−1.82563 − 1	−2.10428 − 2
1.5	−9.48401 + 0	−6.86308 + 0	1.85537 − 1	−2.91179 − 3
1.6	−1.12884 + 1	−7.61390 + 0	4.31871 − 1	1.47146 − 2
1.7	−1.32026 + 1	−8.25574 + 0	5.32289 − 1	1.36618 − 2
1.8	−1.52139 + 1	−8.76628 + 0	4.92313 − 1	−1.33175 − 3
1.9	−1.73086 + 1	−9.12711 + 0	3.43349 − 1	−1.28433 − 2
2.0	−1.94720 + 1	−9.32424 + 0	1.34214 − 1	−9.24495 − 3
2.1	−2.16882 + 1	−9.34869 + 0	−8.02848 − 2	2.21756 − 3
2.2	−2.39406 + 1	−9.19674 + 0	−2.51019 − 1	1.09934 − 2
2.3	−2.62120 + 1	−8.87007 + 0	−3.44099 − 1	7.19680 − 3
2.4	−2.84846 + 1	−8.37588 + 0	−3.46476 − 1	−2.21224 − 3
2.5	−3.07405 + 1	−7.72616 + 0	−2.66729 − 1	−6.65003 − 3
2.6	−3.29612 + 1	−6.93789 + 0	−1.31077 − 1	−7.12403 − 3
2.7	−3.51286 + 1	−6.03209 + 0	2.39279 − 2	8.00041 − 4
2.8	−3.72245 + 1	−5.03334 + 0	1.60443 − 1	8.27644 − 3
2.9	−3.92310 + 1	−3.96899 + 0	2.48208 − 1	1.58822 − 3
3.0	−4.11308 + 1	−2.86785 + 0	2.70290 − 1	−1.20582 − 3

$N = 0 \quad n = 2$				
x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
0.1	9.39351 − 1	9.35161 − 1	8.84844 − 1	9.05937 − 1
0.2	1.08208 + 0	1.06262 + 0	9.01678 − 1	5.16232 − 1

TABLE II — *Continued*

$N = 0 \quad n = 2$				
x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
0.3	8.66349 − 1	8.22287 − 1	5.05390 − 1	−3.42126 − 1
0.4	3.64401 − 1	2.96652 − 1	−1.35886 − 1	−9.82679 − 1
0.5	−3.05506 − 1	−3.80932 − 1	−7.81957 − 1	−8.93819 − 1
0.6	−9.56032 − 1	−1.00914 + 0	−1.16120 + 0	−7.44970 − 2
0.7	−1.32240 + 0	−1.31928 + 0	−1.03567 + 0	9.85253 − 1
0.8	−1.05832 + 0	−9.83017 − 1	−2.64525 − 1	1.65961 + 0
0.9	2.65717 − 1	3.74968 − 1	1.15147 + 0	1.60911 + 0
1.0	3.16217 + 0	3.16046 + 0	3.05638 + 0	9.70106 − 1
1.1	8.22453 + 0	7.79020 + 0	5.16083 + 0	2.03097 − 1
1.2	1.61238 + 1	1.46703 + 1	7.09600 + 0	−2.55403 − 1
1.3	2.76035 + 1	2.41735 + 1	8.48753 + 0	−2.68864 − 1
1.4	4.34731 + 1	3.66166 + 1	9.03589 + 0	−2.22585 − 2
1.5	6.46001 + 1	5.22386 + 1	8.58154 + 0	1.83653 − 1
1.6	9.19010 + 1	7.11812 + 1	7.14579 + 0	1.76198 − 1
1.7	1.26331 + 2	9.34716 + 1	4.93120 + 0	9.12279 − 3
1.8	1.68874 + 2	1.19010 + 2	2.28768 + 0	−1.36988 − 1
1.9	2.20527 + 2	1.47564 + 2	−3.55354 − 1	−1.39402 − 1
2.0	2.82291 + 2	1.78757 + 2	−2.57167 + 0	−1.18170 − 2
2.1	3.55155 + 2	2.12083 + 2	−4.02160 + 0	1.05884 − 1
2.2	4.40084 + 2	2.46904 + 2	−4.51700 + 0	1.18832 − 1
2.3	5.38002 + 2	2.82468 + 2	−4.05482 + 0	1.72440 − 2
2.4	6.49779 + 2	3.17933 + 2	−2.81239 + 0	−8.73616 − 2
2.5	7.76218 + 2	3.52375 + 2	−1.10632 + 0	−7.36407 − 2
2.6	9.18038 + 2	3.84832 + 2	6.76904 − 1	−2.63630 − 2
2.7	1.07586 + 3	4.14324 + 2	2.16111 + 0	3.13871 − 2
2.8	1.25021 + 3	4.39892 + 2	3.05658 + 0	9.47047 − 2
2.9	1.44146 + 3	4.60641 + 2	3.21902 + 0	7.91312 − 3
3.0	1.64988 + 3	4.75731 + 2	2.66304 + 0	−5.44166 − 2

$N = 0 \quad n = 3$				
x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
0.1	1.04335 + 0	1.03904 + 0	1.00396 + 0	8.16002 − 1
0.2	9.41806 − 1	9.22225 − 1	7.82101 − 1	2.80879 − 1
0.3	2.91804 − 1	2.54987 − 1	1.09411 − 2	−5.85742 − 1
0.4	−5.66503 − 1	−6.04220 − 1	−8.19616 − 1	−9.01249 − 1
0.5	−1.16122 + 0	−1.17035 + 0	−1.15387 + 0	−2.96465 − 1
0.6	−1.04537 + 0	−1.00493 + 0	−6.55699 − 1	7.15663 − 1
0.7	−6.94405 − 2	3.87818 − 3	5.00576 − 1	1.12898 + 0
0.8	1.23639 + 0	1.27745 + 0	1.45426 + 0	2.99477 − 1
0.9	1.17030 + 0	1.12044 + 0	6.85065 − 1	−1.44767 + 0
1.0	−3.74163 + 0	−3.74119 + 0	−3.72277 + 0	−2.98825 + 0
1.1	−1.94508 + 1	−1.86263 + 1	−1.36118 + 1	−3.23654 + 0
1.2	−5.51898 + 1	−5.13331 + 1	−3.01767 + 1	−1.98417 + 0
1.3	−1.24365 + 2	−1.12506 + 2	−5.34231 + 1	−6.65948 − 2
1.4	−2.45530 + 2	−2.15895 + 2	−8.18229 + 1	1.27315 + 0
1.5	−4.43417 + 2	−3.78480 + 2	−1.12269 + 2	1.31011 + 0
1.6	−7.50023 + 2	−6.20420 + 2	−1.40431 + 2	3.14866 − 1
1.7	−1.20572 + 3	−9.64812 + 2	−1.61419 + 2	−7.44769 − 1
1.8	−1.86039 + 3	−1.43724 + 3	−1.70733 + 2	−1.00264 + 0
1.9	−2.77451 + 3	−2.06513 + 3	−1.65204 + 2	−4.10716 − 1
2.0	−4.02027 + 3	−2.87683 + 3	−1.43828 + 2	4.24001 − 1
2.1	−5.68257 + 3	−3.90062 + 3	−1.08200 + 2	8.04224 − 1

TABLE II — *Continued*

$N = 0 \quad n = 3$

x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
2.2	−7.85998 + 3	−5.16340 + 3	−6.24706 + 1	4.53989 − 1
2.3	−1.06656 + 4	−6.68939 + 3	−1.27832 + 1	−2.40332 − 1
2.4	−1.42276 + 4	−8.49881 + 3	3.37407 + 1	−6.90334 − 1
2.5	−1.86902 + 4	−1.06060 + 4	7.02754 + 1	−3.30234 − 1
2.6	−2.42135 + 4	−1.30186 + 4	9.15596 + 1	1.36373 − 1
2.7	−3.09743 + 4	−1.57357 + 4	9.49226 + 1	2.58923 − 1
2.8	−3.91655 + 4	−1.87469 + 4	8.08334 + 1	4.73302 − 1
2.9	−4.89962 + 4	−2.20322 + 4	5.29160 + 1	−7.99752 − 3
3.0	−6.06911 + 4	−2.55591 + 4	1.70562 + 1	−4.71948 − 1

$N = 1 \quad n = 0$

x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
0.1	6.67799 − 2	7.82376 − 2	1.75066 − 1	3.92683 − 1
0.2	1.88413 − 1	2.19147 − 1	4.70668 − 1	9.77051 − 1
0.3	3.44707 − 1	3.96102 − 1	7.93680 − 1	1.44397 + 0
0.4	5.27637 − 1	5.96043 − 1	1.08116 + 0	1.62479 + 0
0.5	7.31901 − 1	8.08692 − 1	1.28498 + 0	1.49159 + 0
0.6	9.53337 − 1	1.02496 + 0	1.37488 + 0	1.13769 + 0
0.7	1.18837 + 0	1.23655 + 0	1.34097 + 0	7.13592 − 1
0.8	1.43380 + 0	1.43586 + 0	1.19359 + 0	3.52409 − 1
0.9	1.68661 + 0	1.61605 + 0	9.60016 − 1	1.21545 − 1
1.0	1.94398 + 0	1.77112 + 0	6.78651 − 1	1.73276 − 2
1.1	2.20321 + 0	1.89606 + 0	3.91685 − 1	−6.46094 − 3
1.2	2.46172 + 0	1.98689 + 0	1.37762 − 1	−6.17021 − 4
1.3	2.71702 + 0	2.04079 + 0	−5.42516 − 2	4.36117 − 3
1.4	2.96675 + 0	2.05612 + 0	−1.69153 − 1	1.77204 − 3
1.5	3.20863 + 0	2.03244 + 0	−2.06531 − 1	−2.50040 − 3
1.6	3.44053 + 0	1.97051 + 0	−1.79111 − 1	−2.92887 − 3
1.7	3.66041 + 0	1.87221 + 0	−1.08794 − 1	5.03281 − 5
1.8	3.86640 + 0	1.74049 + 0	−2.14572 − 2	2.47164 − 3
1.9	4.05675 + 0	1.57927 + 0	5.85005 − 2	1.94403 − 3
2.0	4.22984 + 0	1.39322 + 0	1.12680 − 1	−5.40072 − 4
2.1	4.38426 + 0	1.18771 + 0	1.31527 − 1	−2.13282 − 3
2.2	4.51871 + 0	9.68543 − 1	1.15060 − 1	−1.41516 − 3
2.3	4.63210 + 0	7.41814 − 1	7.15483 − 2	6.26047 − 4
2.4	4.72350 + 0	5.13706 − 1	1.46127 − 2	1.92158 − 3
2.5	4.79216 + 0	2.90272 − 1	−4.05001 − 2	7.95172 − 4
2.6	4.83753 + 0	7.72847 − 2	−8.05860 − 2	−6.86022 − 4
2.7	4.85923 + 0	−1.19954 − 1	−9.73081 − 2	−7.63503 − 4
2.8	4.85708 + 0	−2.96762 − 1	−8.86612 − 2	−9.86627 − 4
2.9	4.83107 + 0	−4.49226 − 1	−5.89386 − 2	1.47764 − 4
3.0	4.78140 + 0	−5.74265 − 1	−1.69513 − 2	1.50302 − 3

$N = 1 \quad n = 1$

x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
0.1	1.78797 − 1	1.86209 − 1	2.26456 − 1	4.69291 − 1
0.2	4.81623 − 1	4.98465 − 1	5.77313 − 1	1.01396 + 0

TABLE II — *Continued*

$N = 1 \quad n = 1$

x	c = 1	c = 2	c = 5	c = 10
0.3	8.11322 − 1	8.30625 − 1	8.81124 − 1	1.10226 + 0
0.4	1.09167 + 0	1.09943 + 0	1.01214 + 0	5.69582 − 1
0.5	1.24498 + 0	1.22371 + 0	8.87650 − 1	−3.54754 − 1
0.6	1.19004 + 0	1.12484 + 0	4.82831 − 1	−1.21477 + 0
0.7	8.42479 − 1	7.29666 − 1	−1.63619 − 1	−1.61645 + 0
0.8	1.15744 − 1	−2.61055 − 2	−9.56512 − 1	−1.44879 + 0
0.9	−1.07772 + 0	−1.19460 + 0	−1.76308 + 0	−9.06520 − 1
1.0	−2.82516 + 0	−2.81285 + 0	−2.43994 + 0	−3.26420 − 1
1.1	−5.21224 + 0	−4.90037 + 0	−2.86233 + 0	3.12685 − 2
1.2	−8.32175 + 0	−7.45742 + 0	−2.94990 + 0	1.08983 − 1
1.3	−1.22323 + 1	−1.04640 + 1	−2.68284 + 0	2.38728 − 2
1.4	−1.70172 + 1	−1.38798 + 1	−2.10636 + 0	−6.19490 − 2
1.5	−2.27431 + 1	−1.76444 + 1	−1.32097 + 0	−6.25964 − 2
1.6	−2.94693 + 1	−2.16792 + 1	−4.62707 − 1	−1.89305 − 4
1.7	−3.72461 + 1	−2.58893 + 1	3.24280 − 1	5.11365 − 2
1.8	−4.61145 + 1	−3.01663 + 1	9.15552 − 1	4.39191 − 2
1.9	−5.61051 + 1	−3.43921 + 1	1.22972 + 0	−3.52397 − 3
2.0	−6.72374 + 1	−3.84424 + 1	1.24232 + 0	−3.87077 − 2
2.1	−7.95192 + 1	−4.21912 + 1	9.87598 − 1	−3.49613 − 2
2.2	−9.29459 + 1	−4.55151 + 1	5.48435 − 1	2.81473 − 3
2.3	−1.07501 + 2	−4.82978 + 1	3.67863 − 2	3.09193 − 2
2.4	−1.23154 + 2	−5.04353 + 1	−4.30799 − 1	3.14921 − 2
2.5	−1.39863 + 2	−5.18371 + 1	−7.57370 − 1	−6.17863 − 4
2.6	−1.57573 + 2	−5.24330 + 1	−8.84163 − 1	−2.95268 − 2
2.7	−1.76214 + 2	−5.21739 + 1	−8.00250 − 1	−1.28822 − 2
2.8	−1.95708 + 2	−5.10350 + 1	−5.41499 − 1	−1.34150 − 3
2.9	−2.15961 + 2	−4.90184 + 1	−1.79916 − 1	6.07715 − 3
3.0	−2.36871 + 2	−4.61486 + 1	1.95783 − 1	2.63666 − 2

$N = 1 \quad n = 2$

x	c = 1	c = 2	c = 5	c = 10
0.1	3.17572 − 1	3.23375 − 1	3.58452 − 1	4.61327 − 1
0.2	7.89038 − 1	7.98456 − 1	8.45034 − 1	8.84405 − 1
0.3	1.13929 + 0	1.13924 + 0	1.10027 + 0	6.93291 − 1
0.4	1.16268 + 0	1.13700 + 0	9.09831 − 1	−8.13386 − 2
0.5	7.46367 − 1	6.88194 − 1	2.60402 − 1	−8.60192 − 1
0.6	−6.41935 − 2	−1.40156 − 1	−6.18349 − 1	−9.72981 − 1
0.7	−9.90830 − 1	−1.04501 + 0	−1.27959 + 0	−2.01002 − 1
0.8	−1.44413 + 0	−1.42980 + 0	−1.15587 + 0	1.02436 + 0
0.9	−4.43333 − 1	−3.60476 − 1	2.92752 − 1	1.92471 + 0
1.0	3.46336 + 0	3.46067 + 0	3.41429 + 0	1.94059 + 0
1.1	1.22761 + 1	1.16998 + 1	8.22613 + 0	1.12202 + 0
1.2	2.86165 + 1	2.63729 + 1	1.43419 + 1	5.54149 − 2
1.3	5.57952 + 1	4.98022 + 1	2.09872 + 1	−5.92374 − 1
1.4	9.78715 + 1	8.45483 + 1	2.71147 + 1	−5.45072 − 1
1.5	1.59705 + 2	1.33319 + 2	3.15908 + 1	−5.23392 − 2
1.6	2.46994 + 2	1.98856 + 2	3.34313 + 1	3.78477 − 1
1.7	3.66308 + 2	2.83810 + 2	3.20166 + 1	4.06284 − 1
1.8	5.25100 + 2	3.90597 + 2	2.72591 + 1	7.68751 − 2
1.9	7.31713 + 2	5.21256 + 2	1.96583 + 1	−2.68863 − 1
2.0	9.95363 + 2	6.77296 + 2	1.02468 + 1	−3.08683 − 1

TABLE II — *Continued*

$N = 1 \quad n = 2$

x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
2.1	1.32612 + 3	8.59566 + 2	4.17984 − 1	−9.67834 − 2
2.2	1.73484 + 3	1.06812 + 3	−8.32684 + 0	1.95130 − 1
2.3	2.23316 + 3	1.30213 + 3	−1.46613 + 1	2.53304 − 1
2.4	2.83336 + 3	1.55981 + 3	−1.76896 + 1	1.16918 − 1
2.5	3.54832 + 3	1.83831 + 3	−1.71224 + 1	−9.81521 − 2
2.6	4.39138 + 3	2.13379 + 3	−1.33267 + 1	−2.49959 − 1
2.7	5.37625 + 3	2.44140 + 3	−7.25156 + 0	−5.82649 − 2
2.8	6.51688 + 3	2.75538 + 3	−2.34462 − 1	9.65074 − 2
2.9	7.82726 + 3	3.06921 + 3	6.27542 + 0	5.32562 − 2
3.0	9.32135 + 3	3.37554 + 3	1.10027 + 1	1.36992 − 1

$N = 1 \quad n = 3$

x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
0.1	4.70292 − 1	4.74792 − 1	5.03663 − 1	5.60653 − 1
0.2	1.03640 + 0	1.03944 + 0	1.05094 + 0	9.52509 − 1
0.3	1.15261 + 0	1.13873 + 0	1.02612 + 0	4.75116 − 1
0.4	5.87098 − 1	5.49119 − 1	2.80849 − 1	−5.29321 − 1
0.5	−4.34815 − 1	−4.78943 − 1	−7.50431 − 1	−1.03082 + 0
0.6	−1.22557 + 0	−1.23658 + 0	−1.23879 + 0	−3.74929 − 1
0.7	−9.49593 − 1	−9.03408 − 1	−5.28507 − 1	8.87232 − 1
0.8	5.95761 − 1	6.56832 − 1	1.04428 + 0	1.26903 + 0
0.9	1.58393 + 0	1.56393 + 0	1.35005 + 0	−3.80514 − 1
1.0	−3.99973 + 0	−3.99876 + 0	−3.98415 + 0	−3.66768 + 0
1.1	−2.94218 + 1	−2.82967 + 1	−2.13760 + 1	−6.63805 + 0
1.2	−9.93028 + 1	−9.31025 + 1	−5.82859 + 1	−7.12958 + 0
1.3	−2.54877 + 2	−2.33198 + 2	−1.21470 + 2	−4.48170 + 0
1.4	−5.60512 + 2	−5.00166 + 2	−2.14807 + 2	−1.91044 − 1
1.5	−1.11157 + 3	−9.66261 + 2	−3.37164 + 2	3.09018 + 0
1.6	−2.04369 + 3	−1.72812 + 3	−4.80966 + 2	3.47603 + 0
1.7	−3.54344 + 3	−2.91000 + 3	−6.31861 + 2	1.24407 + 0
1.8	−5.86049 + 3	−4.66630 + 3	−7.70049 + 2	−1.54042 + 0
1.9	−9.32112 + 3	−7.18302 + 3	−8.72930 + 2	−2.73125 + 0
2.0	−1.43431 + 4	−1.06778 + 4	−9.19104 + 2	−1.43583 + 0
2.1	−2.14517 + 4	−1.53984 + 4	−8.92696 + 2	7.02031 − 1
2.2	−3.12971 + 4	−2.16194 + 4	−7.87269 + 2	2.14894 + 0
2.3	−4.46725 + 4	−2.96368 + 4	−6.08384 + 2	1.44329 + 0
2.4	−6.25326 + 4	−3.97614 + 4	−3.74068 + 2	−1.90745 − 1
2.5	−8.60136 + 4	−5.23076 + 4	−1.12896 + 2	−1.17683 + 0
2.6	−1.16452 + 5	−6.75841 + 4	1.40286 + 2	−1.61055 + 0
2.7	−1.55406 + 5	−8.58796 + 4	3.50030 + 2	−7.92157 − 2
2.8	−2.04672 + 5	−1.07450 + 5	4.86619 + 2	1.29119 + 0
2.9	−2.66305 + 5	−1.32504 + 5	5.32282 + 2	3.73378 − 1
3.0	−3.42633 + 5	−1.61182 + 5	4.83375 + 2	4.32692 − 1

$N = 2 \quad n = 0$

x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
0.1	8.11214 − 3	9.30928 − 3	2.21477 − 2	8.00048 − 2
0.2	4.58035 − 2	5.22724 − 2	1.20067 − 1	4.01321 − 1

TABLE II — *Continued*

$N = 2 \quad n = 0$				
x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
0.3	1.25827 − 1	1.42274 − 1	3.07992 − 1	9.01985 − 1
0.4	2.57171 − 1	2.87029 − 1	5.71015 − 1	1.38094 + 0
0.5	4.46741 − 1	4.90296 − 1	8.72575 − 1	1.62971 + 0
0.6	6.99877 − 1	7.52394 − 1	1.16320 + 0	1.54932 + 0
0.7	1.02059 + 0	1.07049 + 0	1.39112 + 0	1.19392 + 0
0.8	1.41171 + 0	1.43889 + 0	1.51294 + 0	7.25292 − 1
0.9	1.87493 + 0	1.84932 + 0	1.50259 + 0	3.16554 − 1
1.0	2.41089 + 0	2.29129 + 0	1.35656 + 0	6.79148 − 2
1.1	3.01924 + 0	2.75254 + 0	1.09462 + 0	−2.02873 − 2
1.2	3.69863 + 0	3.21944 + 0	7.56039 − 1	−1.46045 − 2
1.3	4.44678 + 0	3.67758 + 0	3.91913 − 1	1.04085 − 2
1.4	5.26054 + 0	4.11222 + 0	5.55434 − 2	1.52034 − 2
1.5	6.13588 + 0	4.50887 + 0	−2.07467 − 1	1.29918 − 3
1.6	7.06802 + 0	4.85377 + 0	−3.67156 − 1	−1.11917 − 2
1.7	8.05139 + 0	5.13443 + 0	−4.13815 − 1	−9.67213 − 3
1.8	9.07977 + 0	5.34008 + 0	−3.58514 − 1	1.55354 − 3
1.9	1.01463 + 1	5.46205 + 0	−2.29434 − 1	9.62330 − 3
2.0	1.12436 + 1	5.49406 + 0	−6.51947 − 2	6.50298 − 3
2.1	1.23637 + 1	5.43250 + 0	9.35338 − 2	−2.03792 − 3
2.2	1.34985 + 1	5.27657 + 0	2.12220 − 1	−8.19124 − 3
2.3	1.46391 + 1	5.02833 + 0	2.68832 − 1	−5.06774 − 3
2.4	1.57769 + 1	4.69273 + 0	2.57171 − 1	1.94735 − 3
2.5	1.69026 + 1	4.27721 + 0	1.86697 − 1	4.94597 − 3
2.6	1.80072 + 1	3.79179 + 0	7.90086 − 2	5.04030 − 3
2.7	1.90816 + 1	3.24856 + 0	−3.78705 − 2	−7.03146 − 4
2.8	2.01167 + 1	2.66139 + 0	−1.36199 − 1	−6.15552 − 3
2.9	2.11036 + 1	2.04550 + 0	−1.94874 − 1	−1.13042 − 3
3.0	2.20337 + 1	1.41675 + 0	−2.03218 − 1	1.09442 − 3

$N = 2 \quad n = 1$				
x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
0.1	3.01525 − 2	3.18266 − 2	4.36114 − 2	1.10168 − 1
0.2	1.63412 − 1	1.71721 − 1	2.27802 − 1	5.09403 − 1
0.3	4.17552 − 1	4.35481 − 1	5.46215 − 1	9.75159 − 1
0.4	7.63468 − 1	7.87547 − 1	9.08386 − 1	1.10096 + 0
0.5	1.12468 + 0	1.14274 + 0	1.16775 + 0	6.37139 − 1
0.6	1.37439 + 0	1.36734 + 0	1.16137 + 0	−2.87398 − 1
0.7	1.33292 + 0	1.28261 + 0	7.56169 − 1	−1.22293 + 0
0.8	7.66020 − 1	6.71043 − 1	−1.09012 − 1	−1.69212 + 0
0.9	−6.15776 − 1	−7.14755 − 1	−1.39595 + 0	−1.50113 + 0
1.0	−3.15683 + 0	−3.14077 + 0	−2.96182 + 0	−8.45332 − 1
1.1	−7.25498 + 0	−6.87847 + 0	−4.57844 + 0	−1.47679 − 1
1.2	−1.33591 + 1	−1.21904 + 1	−5.97349 + 0	2.35668 − 1
1.3	−2.19659 + 1	−1.93150 + 1	−6.88402 + 0	2.24248 − 1
1.4	−3.36151 + 1	−2.84518 + 1	−7.11308 + 0	3.10851 − 3
1.5	−4.88847 + 1	−3.97466 + 1	−6.57336 + 0	−1.65861 − 1
1.6	−6.83847 + 1	−5.32791 + 1	−5.31243 + 0	−1.46461 − 1
1.7	−9.27502 + 1	−6.90514 + 1	−3.50808 + 0	2.53438 − 3
1.8	−1.22634 + 2	−8.69800 + 1	−1.43922 + 0	1.23063 − 1
1.9	−1.58698 + 2	−1.06891 + 2	5.66677 − 1	1.16626 − 1
2.0	−2.01606 + 2	−1.28518 + 2	2.19548 + 0	3.46739 − 3
2.1	−2.52012 + 2	−1.51503 + 2	3.20680 + 0	−9.50870 − 2

TABLE II — *Continued*

$N = 2 \quad n = 1$

x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
2.2	−3.10555 + 2	−1.75404 + 2	3.47937 + 0	−1.00215 − 1
2.3	−3.77844 + 2	−1.99703 + 2	3.03214 + 0	−1.01271 − 2
2.4	−4.54453 + 2	−2.23824 + 2	2.01698 + 0	7.86142 − 2
2.5	−5.40910 + 2	−2.47136 + 2	6.84586 − 1	6.25620 − 2
2.6	−6.37687 + 2	−2.68987 + 2	−6.69192 − 1	1.86542 − 2
2.7	−7.45191 + 2	−2.88715 + 2	−1.76387 + 0	−2.83514 − 2
2.8	−8.63756 + 2	−3.05673 + 2	−2.39029 + 0	−8.09642 − 2
2.9	−9.93633 + 2	−3.19263 + 2	−2.45263 + 0	−6.05693 − 3
3.0	−1.13498 + 3	−3.28923 + 2	−1.97549 + 0	4.94337 − 2

$N = 2 \quad n = 2$

x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
0.1	6.92988 − 2	7.12673 − 2	8.51196 − 2	1.31062 − 1
0.2	3.52496 − 1	3.60980 − 1	4.18221 − 1	5.67604 − 1
0.3	8.01544 − 1	8.14659 − 1	8.93117 − 1	9.45356 − 1
0.4	1.20746 + 0	1.21237 + 0	1.21221 + 0	7.67342 − 1
0.5	1.27474 + 0	1.25311 + 0	1.05255 + 0	−3.76902 − 2
0.6	7.57995 − 1	7.02242 − 1	2.84685 − 1	−9.13196 − 1
0.7	−3.27092 − 1	−3.94312 − 1	−8.22806 − 1	−1.05173 + 0
0.8	−1.41364 + 0	−1.43709 + 0	−1.49256 + 0	−7.12440 − 2
0.9	−1.01239 + 0	−9.55456 − 1	−4.82515 − 1	1.57998 + 0
1.0	3.74003 + 0	3.73507 + 0	3.69332 + 0	2.86776 + 0
1.1	1.76108 + 1	1.68770 + 1	1.23963 + 1	2.91426 + 0
1.2	4.78738 + 1	4.45904 + 1	2.64247 + 1	1.66265 + 0
1.3	1.04979 + 2	9.51393 + 1	4.56130 + 1	−6.78936 − 2
1.4	2.03280 + 2	1.79119 + 2	6.85921 + 1	−1.18788 + 0
1.5	3.61809 + 2	3.09537 + 2	9.27854 + 1	−1.12986 + 0
1.6	6.05086 + 2	5.01768 + 2	1.14715 + 2	−2.05881 − 1
1.7	9.63954 + 2	7.73357 + 2	1.30538 + 2	7.08003 − 1
1.8	1.47641 + 3	1.14368 + 3	1.36797 + 2	8.79453 − 1
1.9	2.18846 + 3	1.63343 + 3	1.31147 + 2	3.17527 − 1
2.0	3.15489 + 3	2.26396 + 3	1.12984 + 2	−4.11104 − 1
2.1	4.44006 + 3	3.05647 + 3	8.37540 + 1	−7.13268 − 1
2.2	6.11858 + 3	4.03107 + 3	4.68923 + 1	−3.71699 − 1
2.3	8.27598 + 3	5.20576 + 3	7.34290 + 0	2.40288 − 1
2.4	1.10092 + 4	6.59547 + 3	−2.92519 + 1	6.17233 − 1
2.5	1.44270 + 4	8.21064 + 3	−5.75607 + 1	2.78319 − 1
2.6	1.86502 + 4	1.00565 + 4	−7.35509 + 1	−1.46009 − 1
2.7	2.38122 + 4	1.21320 + 4	−7.52754 + 1	−2.33031 − 1
2.8	3.00582 + 4	1.44287 + 4	−6.32751 + 1	−4.05877 − 1
2.9	3.75458 + 4	1.69309 + 4	−4.05587 + 1	1.20902 − 2
3.0	4.64442 + 4	1.96136 + 4	−1.18748 + 1	4.27288 − 1

$N = 2 \quad n = 3$

x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
0.1	1.26968 − 1	1.29109 − 1	1.44081 − 1	1.94166 − 1
0.2	5.91772 − 1	5.99200 − 1	6.48662 − 1	7.75163 − 1
0.3	1.13308 + 0	1.13787 + 0	1.15991 + 0	1.06828 + 0

TABLE II — *Continued*

		$N = 2 \quad n = 3$		
x	$c = 1$	$c = 2$	$c = 5$	$c = 10$
0.4	1.22242 + 0	1.20735 + 0	1.08268 + 0	4.42410 − 1
0.5	5.02125 − 1	4.61442 − 1	1.72834 − 1	−7.08002 − 1
0.6	−7.28650 − 1	−7.67417 − 1	−9.99672 − 1	−1.12281 + 0
0.7	−1.34994 + 0	−1.33947 + 0	−1.20442 + 0	−6.35200 − 2
0.8	−1.55815 − 1	−9.78895 − 2	3.13440 − 1	1.38983 + 0
0.9	1.72855 + 0	1.73119 + 0	1.69507 + 0	6.83573 − 1
1.0	−4.24197 + 0	−4.23990 + 0	−4.22216 + 0	−4.05639 + 0
1.1	−4.29825 + 1	−4.14973 + 1	−3.22563 + 1	−1.17483 + 1
1.2	−1.69966 + 2	−1.60436 + 2	−1.05765 + 2	−1.81766 + 1
1.3	−4.92189 + 2	−4.54612 + 2	−2.55016 + 2	−1.86221 + 1
1.4	−1.19761 + 3	−1.08177 + 3	−5.12503 + 2	−1.15388 + 1
1.5	−2.59483 + 3	−2.28970 + 3	−9.05339 + 2	−3.41475 − 1
1.6	−5.16494 + 3	−4.44652 + 3	−1.44608 + 3	8.52760 + 0
1.7	−9.62735 + 3	−8.07479 + 3	−2.12354 + 3	1.01259 + 1
1.8	−1.70216 + 4	−1.38880 + 4	−2.89670 + 3	4.31901 + 0
1.9	−2.88067 + 4	−2.28281 + 4	−3.69313 + 3	−3.63918 + 0
2.0	−4.69805 + 4	−3.61016 + 4	−4.41450 + 3	−7.22580 + 0
2.1	−7.42193 + 4	−5.52126 + 4	−4.94883 + 3	−5.31249 + 0
2.2	−1.14040 + 5	−8.19885 + 4	−5.18833 + 3	8.57079 − 1
2.3	−1.70988 + 5	−1.18598 + 5	−5.05014 + 3	5.14940 + 0
2.4	−2.50841 + 5	−1.67557 + 5	−4.49592 + 3	5.60742 + 0
2.5	−3.60853 + 5	−2.31714 + 5	−3.54624 + 3	5.87060 − 1
2.6	−5.10005 + 5	−3.14232 + 5	−2.28432 + 3	−4.18090 + 0
2.7	−7.09289 + 5	−4.18531 + 5	−8.50402 + 2	−2.39443 + 0
2.8	−9.72020 + 5	−5.48229 + 5	5.76599 + 2	−1.90159 + 0
2.9	−1.31415 + 6	−7.07072 + 5	1.80814 + 3	6.87950 − 1
3.0	−1.75464 + 6	−8.98762 + 5	2.67521 + 3	4.80391 + 0

I am indebted to Mrs. E. Sonnenblick for programming and carrying out the computations reported here.

APPENDIX A

A Perturbation Scheme

We treat briefly the following problem. Eigenfunctions u_n and eigenvalues λ_n of an operator $\mathbf{L}$ are assumed known. That is, we have

$$\mathbf{L}u_n + \lambda_n u_n = 0, \qquad n = 0, 1, 2, \ldots . \tag{114}$$

It is desired to find eigenfunctions ψ_n and eigenvalues χ_n of the perturbed equation

$$(\mathbf{L} - \epsilon\mathbf{M})\psi + \chi\psi = 0. \tag{115}$$

It is assumed that the u_n satisfy the boundary condition to be imposed on the ψ's and that the u_n are complete in some appropriate sense. We proceed further in a purely formal manner.

Substitute the series

$$\psi_n = u_n + \sum_{j=1} \epsilon^j Q_j \tag{116}$$

$$\chi_n = \lambda_n + \sum_{j=1} \epsilon^j a_j \tag{117}$$

into (115). Here in the notation we have suppressed the dependence of the Q_j and a_j on n. By equating to zero the coefficients of distinct powers of ϵ, we find

$$\mathbf{L}u_n + \lambda_n u_n = 0 \tag{118}$$

$$\mathbf{L}Q_j + \lambda_n Q_j = \mathbf{M}Q_{j-1} - \sum_{k=1}^{j} a_k Q_{j-k}, \quad j = 1, 2, \ldots \tag{119}$$

where we define $Q_o = u_n$.

Now it frequently happens that the operator $\mathbf{M}$ is such that $\mathbf{M}u_n$ can be expressed as a finite linear combination of the u's with constant coefficients. We assume this to be the case and write

$$\mathbf{M}u_n = \sum_{i=-l}^{l} \gamma_n{}^i u_{n+i\alpha}, \quad n = 0, 1, 2, \ldots . \tag{120}$$

Here α is a positive integer, l a nonnegative integer, and the superscript i on γ is not an exponent, but a label.

If the u_n are linearly independent, formal solution of system (119) is now straightforward. Set

$$Q_j = \sum_{k=-jl}^{jl} A_k{}^j u_{n+k\alpha}$$
$$A_o{}^j = 0 \tag{121}$$
$$j = 1, 2, \ldots .$$

The A's of course depend on n, but for simplicity we have suppressed this fact in the notation. Again the superscript is a label, not an exponent. Substitute (121) and (120) into (119). Setting the coefficient of u_n equal to zero in the resultant expression yields

$$a_j = \sum_{k=-l}^{l} A_{-k}^{\;j-1}\gamma_{n-k\alpha}^{\;k}, \qquad j = 1, 2, \dots . \tag{122}$$

Requiring the coefficient of $u_{n+m\alpha}$ to vanish gives

$$(\lambda_{n+m\alpha} - \lambda_n)A_m^{\;j} = \sum_{k=1}^{j} a_k A_m^{\;j-k} - \sum_{k=-l}^{l} A_{m-k}^{\;j-1}\gamma_{n+(m-k)\alpha}^{\;k} \tag{123}$$
$$m = -jl, -jl+1, \dots, jl; \quad j = 1, 2, \dots .$$

Here we have adopted the conventions

$$A_k^{\;j} \equiv 0$$

if either

$$|k| > jl, \quad \text{or} \quad \alpha k < -n, \quad \text{or} \quad k = 0 \quad \text{and} \quad j = 1, 2, \dots$$
$$A_o^{\;0} = 1, \qquad A_k^{\;0} = 0, \qquad k \neq 0, a_o = 0.$$

Equations (122) and (123) together with these conventions permit successive determination of the a's and A's. The case $l = 1$ occurs frequently. The first few coefficients for this case are given below where we have set

$$h_j = [\lambda_{n+j} - \lambda_n]^{-1}.$$
$$a_1 = \gamma_n^{\;0}$$
$$A_{-1}^{\;1} = -h_{-\alpha}\gamma_n^{\;-1}$$
$$A_1^{\;1} = -h_\alpha\gamma_n^{\;1}$$
$$a_2 = -[h_\alpha\gamma_n^{\;1}\gamma_{n+\alpha}^{\;-1} + h_{-\alpha}\gamma_n^{\;-1}\gamma_{n-\alpha}^{\;1}]$$
$$A_{-2}^{\;2} = h_{-2\alpha}h_{-\alpha}\gamma_n^{\;-1}\gamma_{n-\alpha}^{\;-1}$$
$$A_{-1}^{\;2} = (h_{-\alpha})^2\gamma_n^{\;-1}[-\gamma_n^{\;0} + \gamma_{n-\alpha}^{\;0}]$$
$$A_1^{\;2} = (h_\alpha)^2\gamma_n^{\;1}[-\gamma_n^{\;0} + \gamma_{n+\alpha}^{\;0}]$$
$$A_2^{\;2} = h_{2\alpha}h_\alpha\gamma_n^{\;1}\gamma_{n+\alpha}^{\;1}$$
$$a_3 = (h_\alpha)^2\gamma_n^{\;1}\gamma_{n+\alpha}^{\;-1}(-\gamma_n^{\;0} + \gamma_{n+1}^{\;0}) + (h_{-\alpha})^2\gamma_n^{\;-1}\gamma_{n-\alpha}^{\;1}(-\gamma_n^{\;0} + \gamma_{n-\alpha}^{\;0})$$
$$A_1^{\;3} = h_\alpha[A_1^{\;2}(\gamma_n^{\;0} - \gamma_{n+\alpha}^{\;0}) + a_2A_1^{\;1} - \gamma_{n+2\alpha}^{\;-1}A_2^{\;2}]$$
$$A_{-1}^{\;3} = h_{-\alpha}[A_{-\alpha}^{\;2}(\gamma_n^{\;0} - \gamma_{n-\alpha}^{\;0}) + a_2A_{-1}^{\;2} - \gamma_{n-2\alpha}^{\;1}A_{-2}^{\;2}]$$
$$a_4 = A_1^{\;3}\gamma_{n+\alpha}^{\;-1} + A_{-1}^{\;3}\gamma_{n-\alpha}^{\;1}.$$

More generally for this case one finds

$$A_{\pm j}{}^{j} = (-1)^j \prod_{k=1}^{j} h_{\pm k\alpha} \gamma_{n\pm(k-1)\alpha}{}^{\pm 1}, \qquad j = 1, 2, \ldots$$

$$A_{\pm(j-1)}{}^{j} = A_{\pm(j-1)}{}^{j-1} \sum_{k=1}^{j-1} h_{\pm k\alpha}[a_1 - \gamma_{n\pm k\alpha}{}^{0}], \qquad j = 2, 3, \ldots$$

$$\begin{aligned} A_{\pm(j-2)}{}^{j} = A_{\pm(j-2)}{}^{j-2} \sum_{l=1}^{j-2} h_{\pm l\alpha} & \\ \times [a_2 + h_{\pm(l+1)\alpha} \gamma_{n\pm(l+1)\alpha}{}^{\mp 1} \gamma_{n\pm l\alpha}{}^{\pm 1} & \\ + (a_1 - \gamma_{n\pm l\alpha}{}^{0}) \sum_{k=1}^{l} h_{\pm k\alpha}(a_1 - \gamma_{n\pm k\alpha}{}^{0})], & \\ j = 3, 4, \ldots . & \end{aligned} \tag{124}$$

APPENDIX B

Evaluation of an Integral

We establish here the formula (43). Let

$$\begin{aligned} F_{N,n}(x) &= \int_0^1 J_N(xy)\sqrt{xy}\, T_{N,n}(y)\,dy \\ &= \int_0^1 K_N(xy)\, T_{N,n}(y)\,dy \end{aligned} \tag{125}$$

on using the notation of (21). Then

$$\begin{aligned} &\left[x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} + (x^2 - \chi)\right] F_{N\,n}(x) \\ &= \int_0^1 T_{N,n}(y)[x^2y^2K_N''(xy) + 2xyK_N'(xy) + (x^2 - \chi)K_N(xy)]dy \\ &= \int_0^1 T_{N,n}(y)[(-x^2y^2 - \tfrac{1}{4} + N^2 + x^2 \\ &\qquad - x)K_N(xy) + 2xyK_N'(xy)]dy \end{aligned} \tag{126}$$

by (23). Here primes denote differentiation with respect to the argument indicated.

Now

$$\frac{d}{dy}(1 - y^2)\frac{dT_{N,n}(y)}{dy} + \left(\frac{\frac{1}{4} - N^2}{y^2} + \chi\right) T_{N,n}(y) = 0$$

with χ given by (26). Multiply this equation by $K_N(xy)$ and integrate from zero to one. There results

$$0 = \int_0^1 K_N(xy)\left[\frac{d}{dy}(1-y^2)\frac{dT_{N,n}}{dy} + \left(\frac{\frac{1}{4}-N^2}{y^2}+\chi\right)T_{N,n}\right]dy$$

$$= \int_0^1 T_{N,n}(y)\left[\frac{d}{dy}(1-y^2)\frac{d}{dy}K_N(xy) + \left(\frac{\frac{1}{4}-N^2}{y^2}+\chi\right)K_N(xy)\right]dy$$

where we have integrated by parts and made use of the fact that $K_N(0) = T_{N,n}(0) = 0$. Carrying out the indicated differentiation, we find

$$0 = \int_0^1 T_{N,n}(y)\left[(1-y^2)x^2K_N''(xy) - 2xyK_N'(xy) + \left(\frac{\frac{1}{4}-N^2}{y^2}+\chi\right)K_N(xy)\right]dy \qquad (127)$$

$$= -\int_0^1 T_{N,n}(y)[(-x^2y^2 - \tfrac{1}{4} + N^2 + x^2 - \chi)K_N(xy) + 2xyK_N'(xy)]dy.$$

Equations (126) and (127) give

$$\left[x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} + \left\{x^2 - \left(N+2n+\frac{1}{2}\right)\left(N+2n+\frac{3}{2}\right)\right\}\right]F_{N,n}(x) = 0.$$

The only solution of this equation that vanishes for $x = 0$ is

$$F_{N,n}(x) = k\,\frac{J_{N+2n+1}(x)}{\sqrt{x}}.$$

Using (125), we then have

$$k\,\frac{J_{N+2n+1}(x)}{\sqrt{x}} = \int_0^1 J_N(xy)\sqrt{xy}\,T_{N,n}(y)\,dy. \qquad (128)$$

To determine k, we have only to compare the coefficients of $x^{N+2n+\frac{1}{2}}$ on both sides of (128). In this way we find

$$\frac{k}{2^{N+2n+1}\Gamma(N+2n+2)} = \frac{(-1)^n}{2^{N+2n}\Gamma(N+n+1)\,n!}\int_0^1 y^{N+2n+\frac{1}{2}}T_{N,n}(y)\,dy. \qquad (129)$$

The integral here can be evaluated by using (27), (28) and known properties of the Jacobi polynomials. We have

$$\int_0^1 y^{N+2n+\frac{1}{2}} T_{N,n}(y)\,dy$$

$$= \binom{n+N}{n}^{-1} \int_0^1 y^{2N+2n+1} P_n^{(N,0)}(1-2y^2)\,dy$$

$$= 2^{-(N+n+2)} \binom{n+N}{n}^{-1} \int_{-1}^1 (1-u)^N (1-u)^n P_n^{(N,0)}(u)\,du.$$

Now the coefficient of u^n in $(1-u)^n$ is $(-1)^n$ and the coefficient of u^n in $P_n^{(N,0)}(u)$ is $\binom{2n+N}{n} \Big/ 2^n$ [Ref. 8, Vol. II, p. 169, Eq. (5)], so

$$(1-u)^n = \frac{(-1)^n 2^n}{\binom{2n+N}{n}} P_n^{(N,0)}(u) + \sum_{j=0}^{n-1} A_j P_j^{(N,0)}(u).$$

It follows, then, that

$$\int_1^1 (1-u)^N (1-u)^n P_n^{(N,0)}(u)\,du$$

$$= \frac{(-1)^n 2^n}{\binom{2n+N}{n}} \int_{-1}^1 (1-u)^N P_n^{(N,0)}(u) P_n^{(N,0)}(u)\,du$$

$$= \frac{(-1)^n 2^{n+N+1}}{\binom{2n+N}{n}(2n+N+1)}$$

where we have used the orthogonality of the Jacobi polynomials and the known normalization integral [see Ref. 5, page 68, Eq. (4.3.3), for example]. Combining these results, (129) yields

$$k = \binom{n+N}{n}^{-1}$$

and together with (128) this establishes (43).

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