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Prolate Spheroidal Wave Functions, Fourier Analysis, and Uncertainty— V: The Discrete Case

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A discrete time series has associated with it an amplitude spectrum which is a periodic function of frequency. This paper investigates the extent to which a time series can be concentrated on a finite index set and also have its spectrum concentrated on a subinterval of the fundamental period of the spectrum. Key to the analysis are certain sequences, called discrete prolate spheroidal sequences, and certain functions of frequency called discrete prolate spheroidal functions. Their mathematical properties are investigated in great detail, and many applications to signal analysis are pointed out.

I. INTRODUCTION

In many branches of technology, such as sampled-data theory, time-series analysis, etc., doubly infinite sequences of complex numbers, $\{h_n\} = \ldots, h_{-1}, h_0, h_1, \ldots$ play an important role. Associated with such a sequence is its amplitude spectrum

$$H(f) \equiv \sum_{-\infty}^{\infty} h_n e^{2\pi i n f}. \tag{1}$$

In this paper we attempt to elucidate certain features of the complex relationship between $\{h_n\}$ and its amplitude spectrum $H(f)$.

Of prime importance in the analysis we present are some special sequences, here called discrete prolate spheroidal sequences (DPSS's), and some related special functions called discrete prolate spheroidal wave functions (DPSWF's). Much of the paper is devoted to a study of their mathematical properties. They are fundamental tools for understanding the extent to which sequences and their spectra can be simultaneously concentrated: they have many potential applications in communications technology.

We motivate our work by discussing a simple problem. But first some notation is needed. We adopt the abbreviation

$$E(n_1,n_2) \equiv \sum_{n=n_1}^{n_2} |h_n|^2 \tag{2}$$

and refer to this quantity as *the energy of the sequence* $\{h_n\}$ *in the index range* (n_1,n_2). Throughout this paper we restrict our attention to sequences whose *total energy* $E \equiv E(-\infty,\infty)$ is finite. Associated with a sequence is its *amplitude spectrum* defined in (1). It is periodic in f with period 1 and we shall generally consider it only for $|f| \leq 1/2$. From the theory of Fourier series, we then have the representation

$$h_n = \int_{-1/2}^{1/2} H(f)e^{-2\pi inf}df, \quad n = 0, \pm 1, \ldots \tag{3}$$

for the sequence, and from Parseval's theorem we have that

$$E = \sum_{-\infty}^{\infty} |h_n|^2 = \int_{1/2}^{1/2} |H(f)|^2 df. \tag{4}$$

If $H(f)$ is given, we say that the sequence $\{h_n\}$ defined by (3) is the sequence *belonging* to H and we write $\{h_n\} \leftrightarrow H(f)$.

Now let W be a positive number less than $1/2$. If the amplitude spectrum of $\{h_n\}$ vanishes for $W < |f| \leq 1/2$, we say that the sequence is *bandlimited* and that it has *bandwidth* W. The elements of a bandlimited sequence can be written in the form

$$h_n = \int_{-W}^{W} H(f)e^{-2\pi inf}df, \quad 0 < W < \frac{1}{2}, \quad n = 0, \pm 1, \ldots \tag{5}$$

Analogously, given two finite integers, $n_1 \leq n_2$, we shall say that a sequence $\{h_n\}$ is *indexlimited to the index interval* (n_1,n_2) if h_n vanishes whenever $n > n_2$ or $n < n_1$. It is not hard to see that, except for the trivial all-zero sequence, a bandlimited sequence cannot be indexlimited and that an indexlimited sequence cannot be bandlimited.

It is natural now to ask just how nearly indexlimited a bandlimited sequence can be. Specifically, we seek the maximum value of the *concentration*

$$\lambda \equiv \frac{E(N_0,N_0+N-1)}{E(-\infty,\infty)} = \left(\sum_{N_0}^{N_0+N-1} |h_n|^2\right) \Big/ \left(\sum_{-\infty}^{\infty} |h_n|^2\right) \tag{6}$$

for all sequences of bandwidth W, and ask for which bandlimited sequences the concentration attains this maximal value. The answers to these questions are simply stated in terms of the discrete prolate spheroidal wave functions $U_k(N,W;f)$, the discrete prolate spheroidal sequences $\{v_n^{(k)}(N,W)\}$, and their associated eigenvalues $\lambda_k(N,W)$, $k = 0,1,2, \ldots, N-1$. The bandlimited sequence $\{h_n\}$ of bandwidth W most concentrated in the sense of (6) is proportional to the DPSS $\{v_{n-N_0}^{(0)}(N,W)\}$, its amplitude spectrum is proportional to $e^{i\pi(2N_0+N-1)f}U_0(N,W;f)$ in the interval $|f| \le W$, and its concentration is given by $\lambda_0(N,W)$.

In earlier papers in this series[1–4] we treated the analogous problem of the maximal time-concentration of a *continuous* signal $f(t)$ of limited bandwidth. The optimal signals in that case, prolate spheroidal wave functions (PSWF's), were found to have many interesting and useful properties that were explored in related papers.[5–8] We here borrow freely from the techniques used in these earlier works and extend many of those results to the present case of discrete time series. Details of derivations that parallel closely ones to be found in Refs. 1–8 are sometimes omitted.

Part of the material presented here has been anticipated by others. As early as 1964 C. L. Mallows in an unpublished work defined versions of the DPSWF's and DPSS's. He showed that the former satisfy a second-order differential equation and that the latter satisfy a second-order difference equation, and described a number of their other properties as well. Tufts and Francis[16] in 1970 showed the importance of the DPSS's in the optimal design of digital filters. Independently, Papoulis and Bertran[12] in 1972 made a similar application. Eberhard[17] in 1973 showed that the DPSS provide optimal design of a discrete window for the calculation of power spectra under a natural criterion. All of these later authors present some numerical values of the functions and of $\lambda_0(N,W)$ for a few isolated values of N and W. None of them treat the subject as intensively as is done here. See Ref. 18 for some comments on these applications. An interesting application in optics to the theory of image formation was made by Gori and Guattari[23] in 1974.

Regarding the organization of this paper, in Section II we state without proof some of the more useful and interesting properties of the DPSWF's and the DPSS's. Included are curves and asymptotic formulae. In Section III we discuss some extremal properties and some applications of the functions. Of particular interest, perhaps, is the prediction problem of Section 3.2. In Section IV, proofs are given or outlined for the less obvious statements found in Sections II and III.

II. THE DPSWF's, THE DPSS's, AND SOME OF THEIR PROPERTIES

Throughout the remainder of this paper, unless otherwise explicitly stated, N is a positive integer and W a positive real number less than $\frac{1}{2}$.

2.1 The discrete prolate spheroidal wave functions

Since its kernel is degenerate, the integral equation

$$\int_{-W}^{W} \frac{\sin N\pi(f-f')}{\sin \pi(f-f')}\psi(f')df' = \lambda\psi(f), \quad -\infty < f < \infty \tag{7}$$

has only N non-zero eigenvalues, $\lambda_0(N,W), \lambda_1(N,W), \ldots, \lambda_{N-1}(N,W)$. They are distinct, real and positive and we order them so that

$$\lambda_0(N,W) > \lambda_1(N,W) > \ldots > \lambda_{N-1}(N,W) > 0. \tag{8}$$

There are N linearly independent real eigenfunctions of (7) associated with these eigenvalues and we denote them by $U_0(N,W;f)$, $U_1(N,W;f)$, $\ldots$, $U_{N-1}(N,W;f)$. When these are normalized so that

$$\int_{-1/2}^{1/2} |U_k(N,W;f)|^2 df = 1,$$

$$U_k(N,W;0) \geq 0, \quad \frac{dU_k(N,W;0)}{df} \geq 0, \tag{9}$$

$$k = 0,1, \ldots, N-1,$$

they are the DPSWF's. Thus, *the discrete prolate spheroidal wave functions* $U_k(N,W;f)$ *and their associated eigenvalues* $\lambda_k(N,W;f)$ *are defined by*

$$\int_{-W}^{W} \frac{\sin N\pi(f-f')}{\sin \pi(f-f')} U_k(N,W;f')df' = \lambda_k(N,W) U_k(N,W;f)$$

$$-\infty < f < \infty, \quad k = 0,1, \ldots, N-1, \tag{10}$$

along with (8) *and* (9) *and the requirement that the* U_k *be real.*

The DPSWF's are doubly orthogonal:

$$\int_{-W}^{W} U_i(N,W;f) U_j(N,W;f) df$$

$$= \lambda_i \int_{-1/2}^{1/2} U_i(N,W;f) U_j(N,W;f) df = \lambda_i \delta_{ij} \tag{11}$$

$$i,j = 0,1, \ldots, N-1.$$

For $k = 0,1, \ldots, N-1$, the function $U_k(N,W;f)$ is periodic in f. It has period 1 if N is odd and period 2 if N is even. In either case we have

$$U_k(N,W;f+1) = (-1)^{N-1} U_k(N,W;f), \tag{12}$$

while

$$U_k\left(N,W;\frac{1}{2}-f\right) = K(N,k) U_{N-1-k}\left(N,\frac{1}{2}-W;f\right)$$

$$\lambda_k\left(N,\frac{1}{2}-W\right) = 1 - \lambda_{N-1-k}(N,W) \tag{13}$$

$$K(N,k) = \begin{cases} (-1)^{(N-1)/2+k}, & N \text{ odd} \\ (-1)^{(N/2)-1}, & N \text{ even.} \end{cases}$$

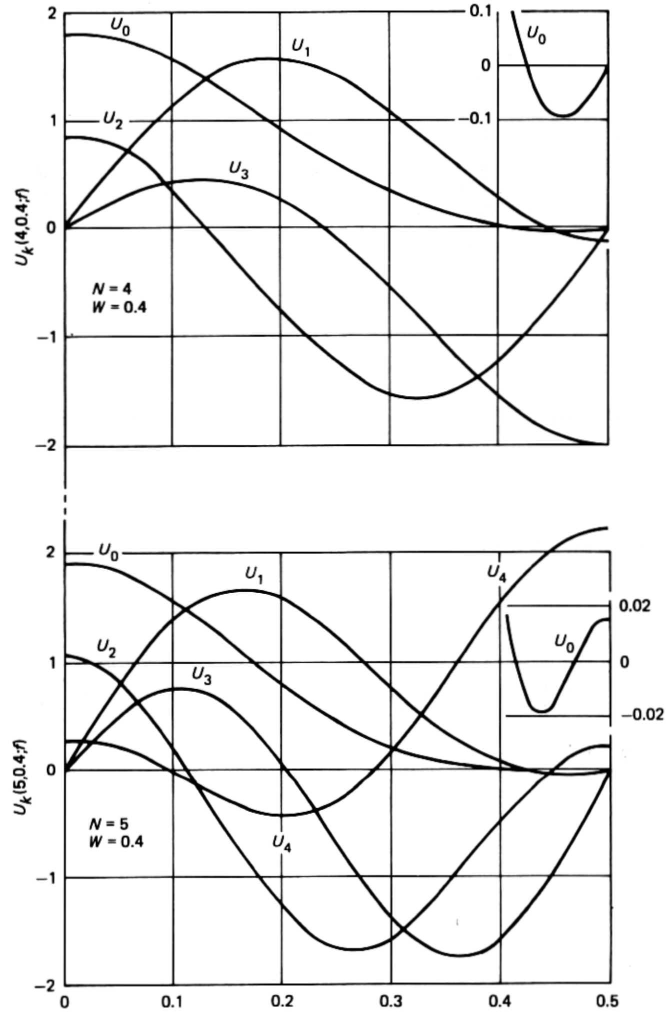

Fig. 1—$U_k(4,0.4;f)$ for $k = 0,1,2,3$ and $U_k(5,0.4;f)$ for $k = 0,1,2,3,4$ for $0 \le f \le 0.5$.

The DPSWF $U_k(N,W;f)$ has exactly k zeros in the open interval $-W < f < W$ and exactly $N - 1$ zeros in $-½ < f \le ½$. It is an even or odd function of f according to the parity of k. Plots of some selected DPSWF's are given in Figures 1 and 2. Note the inserts with changed scales needed to show detail of U_0 for $0.4 \le f \le 0.5$ in Fig. 1 and for $U_4(5,0.2;f)$ in $0 \le f \le 0.1$ in Fig. 2. Values of some $\lambda_k(N,W)$ can be obtained from the ordinates of the curves of Figures 3, 4, 5 and 6 corresponding to integer abscissa values. Figure 7 shows the dependence of some $\lambda_k(N,W)$ on W.

Let $\sigma(N,W)$ denote the $N \times N$ tri-diagonal matrix whose element in the ith row and jth column is

$$\sigma(N,W)_{ij} = \begin{cases} \frac{1}{2} i(N-i), & j = i-1 \\ \left(\frac{N-1}{2} - i\right)^2 \cos 2\pi W, & j = i \\ \frac{1}{2}(i+1)(N-1-i), & j = i+1 \\ 0, & |j-i| > 1, \end{cases} \tag{14}$$

$$i,j = 0,1,\ldots,N-1.$$

The N eigenvalues of this matrix are real and distinct. We denote them by

$$\theta_0(N,W) > \theta_1(N,W) > \ldots > \theta_{N-1}(N,W). \tag{15}$$

Then the DPSWF's satisfy the differential equation

$$\frac{d}{d\omega}[\cos\omega - A]\frac{dU_k(N,W;f)}{d\omega} + \left[\frac{1}{4}(N^2-1)\cos\omega - \theta_k(N,W)\right]U_k(N,W;f) = 0 \tag{16}$$

where we write

$$\omega \equiv 2\pi f, \quad A \equiv \cos 2\pi W. \tag{17}$$

2.2 The discrete prolate spheroidal sequences

For each $k = 0,1,2,\ldots,N-1$, the DPSS $\{v_n^{(k)}(N,W)\}$ is defined as the real solution to the system of equations*

$$\sum_{m=0}^{N-1} \frac{\sin 2\pi W(n-m)}{\pi(n-m)} v_m^{(k)}(N,W) = \lambda_k(N,W) v_n^{(k)}(N,W), \tag{18}$$

$$n = 0,\pm1,\pm2,\ldots$$

normalized so that

$$\sum_{j=0}^{N-1} v_j^{(k)}(N,W)^2 = 1, \tag{19}$$

$$\sum_0^{N-1} v_j^{(k)}(N,W) \geq 0, \quad \sum_0^{N-1} (N-1-2j) v_j^{(k)}(N,W) \geq 0. \tag{20}$$

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* It is understood here, of course, that when $n = m$ the expression $[\sin 2\pi W(n-m)]/\pi(n-m)$ has the value $2W$.

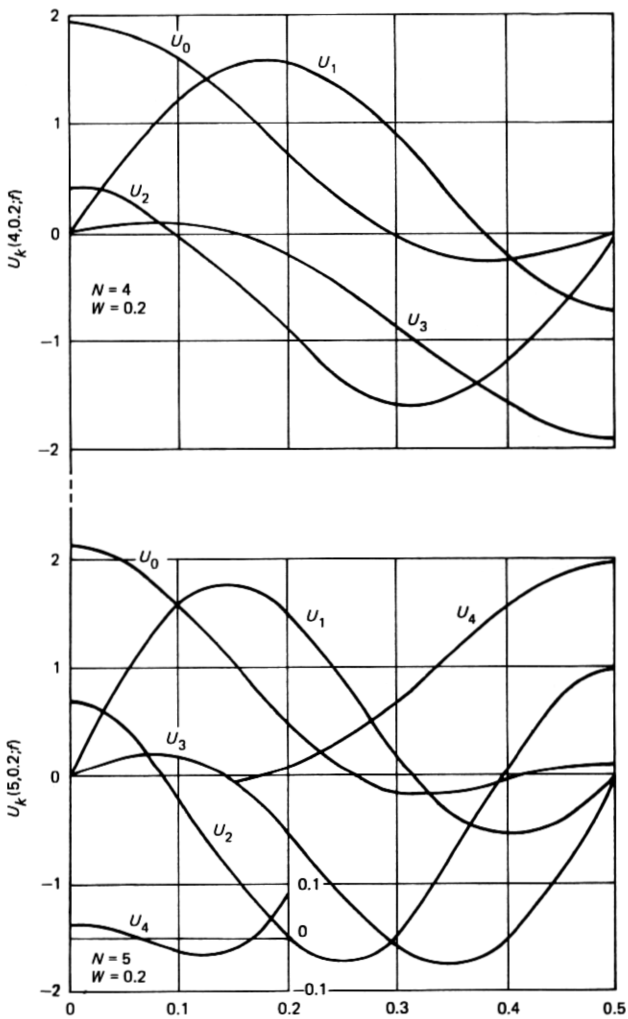

Fig. 2—$U_k(4,0.2;f)$ for $k = 0,1,2,3$ and $U_k(5,0.2;f)$ for $k = 0,1,2,3,4$ for $0 \leq f \leq 0.5$.

The $\lambda_k(N,W)$ here are, as before, the ordered non-zero eigenvalues of the integral equation (7). These quantities are thus seen to be also the eigenvalues of the $N \times N$ matrix $\rho(N,W)$ with elements

$$\rho(N,W)_{mn} = \frac{\sin 2\pi W(m-n)}{\pi(m-n)}, \quad m,n = 0,1,\ldots,N-1, \qquad (21)$$

and the $(N-1)$-vector obtained by indexlimiting the DPSS $\{v_n^{(k)}(N,W)\}$ to the index set $(0,N-1)$ is an eigenvector of $\rho(N,W)$.

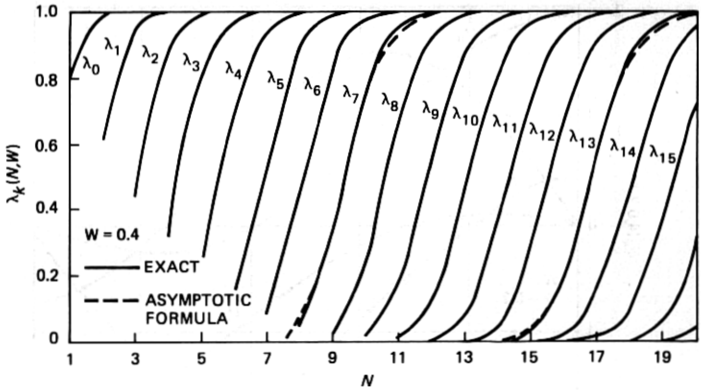

Fig. 3—Values of $\lambda_k(N,0.4)$ for $k = 0, \ldots, 15$ and $N = 0,1, \ldots, 20$.

The DPSS's are doubly orthogonal:

$$\sum_{n=0}^{N-1} v_n^{(i)}(N,W) v_n^{(j)}(N,W) = \lambda_i \sum_{-\infty}^{\infty} v_n^{(i)}(N,W) v_n^{(j)}(N,W) = \delta_{ij} \qquad (22)$$

$$i,j = 0,1, \ldots, N-1.$$

They obey the symmetry laws

$$v_n^{(k)}(N,W) = (-1)^k \, v_{N-1-n}^{(k)}(N,W) \qquad (23)$$

$$v_n^{(k)}(N,W) = (-1)^k v_{N-1-n}^{(N-1-k)}\left(N, \tfrac{1}{2} - W\right), \qquad (24)$$

$$n = 0,\pm 1,\pm 2, \ldots$$

$$k = 0,1, \ldots, N-1.$$

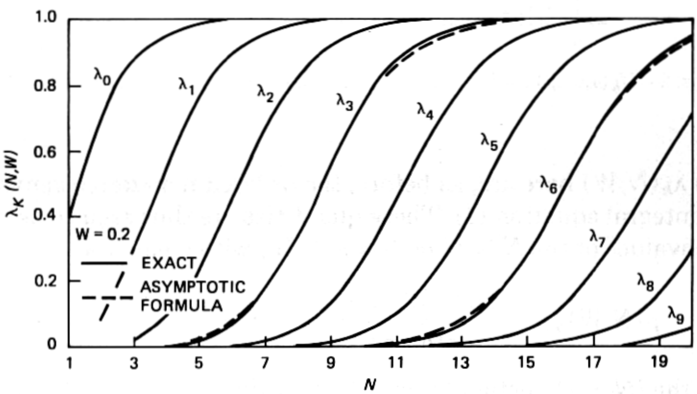

Fig. 4—Values of $\lambda_k(n,0.2)$ for $k = 0, \ldots, 9$ and $N = 0,1, \ldots 20$.

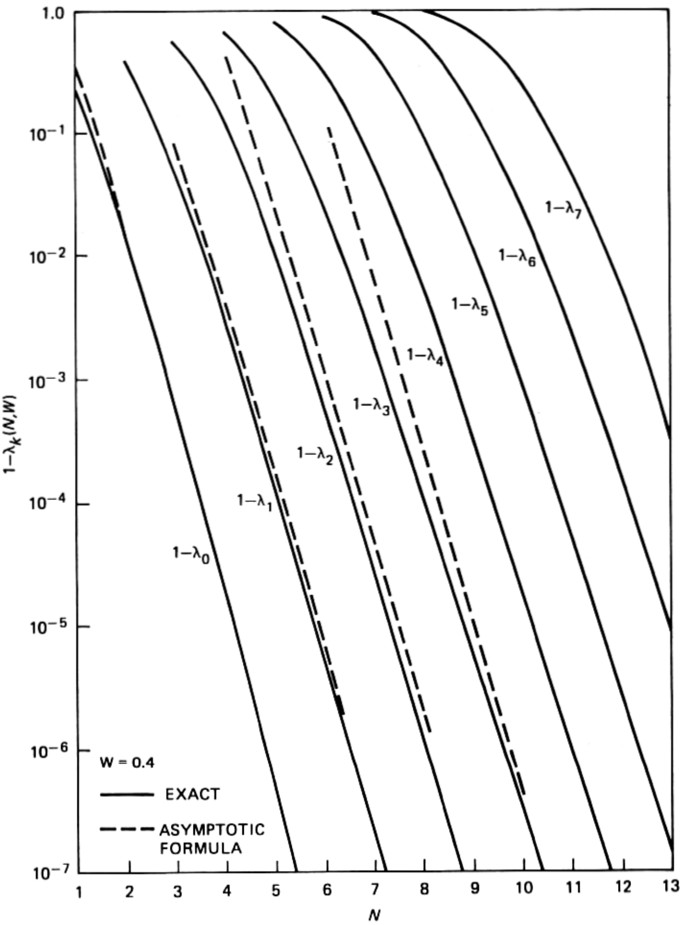

Fig. 5—$1 - \lambda_k(N,0.4)$ for $k = 0, \ldots, 7$ and $N = 1,2, \ldots, 13$.

The DPSS's indexlimited to $(0,N-1)$ satisfy the difference equation

$$\frac{1}{2}n(N-n)v_{n-1}^{(k)}(N,W) + \left[\cos 2\pi W\left(\frac{N-1}{2}-n\right)^2 - \theta_k(N,W)\right]v_n^{(k)}(N,W) + \frac{1}{2}(n+1)[N-1-n]v_{n+1}^{(k)}(N,W) = 0, \quad (25)$$

$$k,n = 0,1,\ldots,N-1.$$

Here the θ's are as in the differential equation (16).

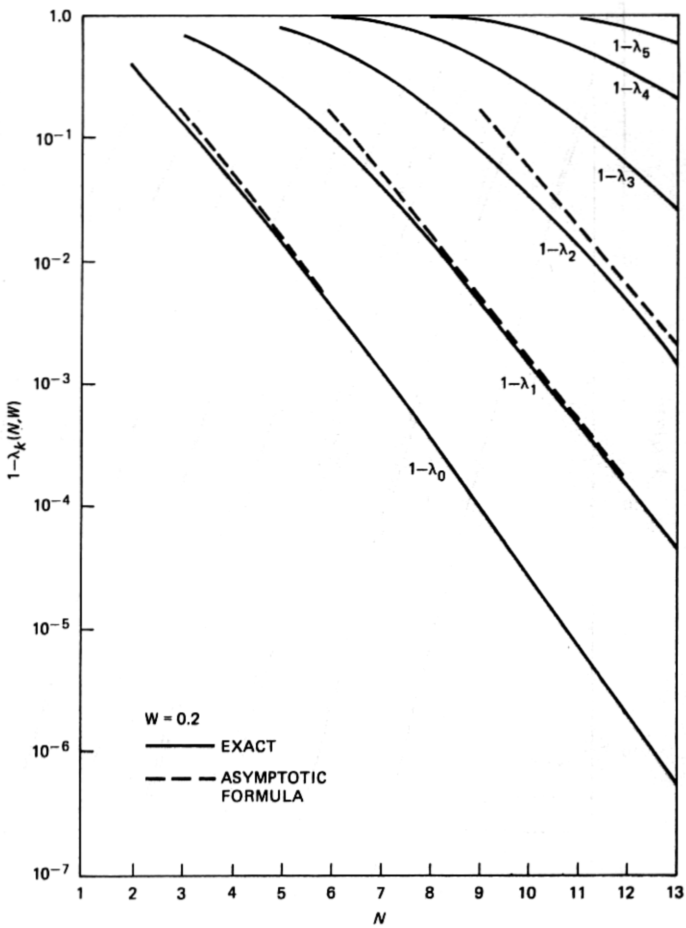

Fig. 6—$1 - \lambda_k(N,0.2)$ for $k = 0, \ldots, 5$ and $N = 1,2, \ldots, 13$.

2.3. Connections between the DPSWF's and the DPSS's

We have

$$U_k(N,W;f) = \epsilon_k \sum_{n=0}^{N-1} v_n^{(k)}(N,W) e^{-i\pi(N-1-2n)f} \tag{26}$$

$$k = 0,1, \ldots, N-1,$$

where

$$\epsilon_k = \begin{cases} 1, & k \text{ even} \\ i, & k \text{ odd.} \end{cases} \tag{27}$$

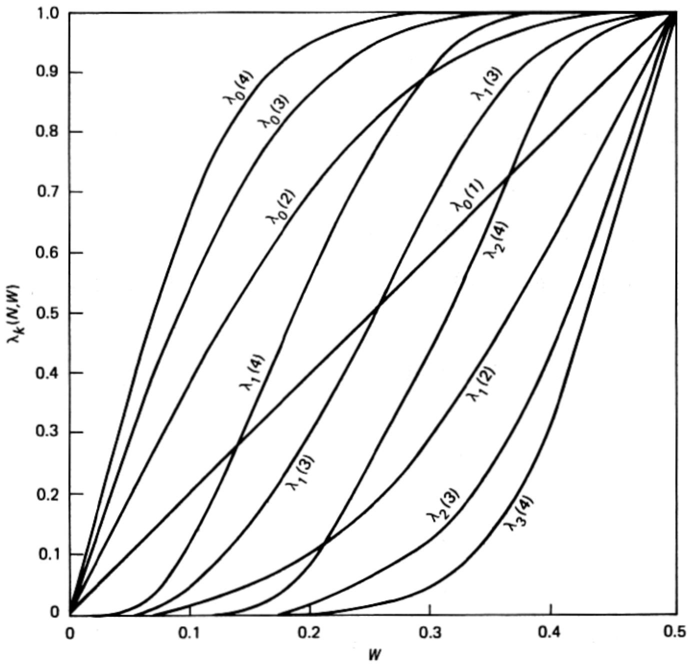

Fig. 7—$\lambda_k(N,W) \equiv \lambda_k(N)$ vs. W for several values of k and N.

Conversely,

$$v_n^{(k)}(N,W) = \frac{1}{\epsilon_k} \int_{-1/2}^{1/2} U_k(N,W;f) e^{i\pi(N-1-2n)f} df \tag{28}$$

$$n,k = 0,1,\ldots,N-1.$$

But, one also has

$$v_n^{(k)}(N,W) = \frac{1}{\epsilon_k \lambda_k(N,W)} \int_{-W}^{W} U_k(N,W;f) e^{i\pi(N-1-2n)f} df \tag{29}$$

$$k = 0,1,\ldots,N-1$$

for all values of n.

It is convenient now to introduce the *bandlimiting operator* B_W defined by

$$B_W H(f) = \begin{cases} H(f), & |f| \le W \\ 0, & |f| > W \end{cases} \tag{30}$$

and the *indexlimiting operator* $I_{N_1}^{N_2}$ defined by

$$I_{N_1}^{N_2}\{h_n\} = \{g_n\}$$

where

$$g_n = \begin{cases} h_n, & N_1 \le n \le N_2 \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

In terms of these operators, (28) and (29) can be stated

$$\epsilon_k I_0^{N-1}\{v_n^{(k)}(N,W)\} \leftrightarrow U_k(N,W;f)e^{i\pi(N-1)f} \tag{32}$$

$$\epsilon_k \lambda_k(N,W)\{v_n^{(k)}(N,W)\} \leftrightarrow B_W U_k(N,W;f)e^{i\pi(N-1)f}. \tag{33}$$

For the sequence $\{u_n^{(k)}(N,W)\}$ belonging to the DPSWF $U_k(N,W;f)$ we have

$$u_n^{(k)}(N,W) = \int_{-1/2}^{1/2} U_k(N,W;f)e^{-2\pi inf}df$$

$$= \epsilon_k \sum_{j=0}^{N-1} \frac{\sin \pi\left(\frac{N-1}{2} + n - j\right)}{\pi\left(\frac{N-1}{2} + n - j\right)} v_j^{(k)}(N,W)$$

$$n = 0,\pm 1,\pm 2,\ldots \tag{34}$$

When $N = 2M + 1$ is odd, this reduces simply to

$$u_n^{(k)}(2M+1,W) = \begin{cases} \epsilon_k v_{n+M}^{(k)}(2M+1,W), & |n| \le M \\ 0, & |n| > M, \end{cases} \tag{35}$$

a multiple of the indexlimited shifted DPSS. Equation (29) shows that conversely the spectrum of the shifted DPSS is in this case a multiple of the bandlimited DPSWF,

$$\epsilon_k \lambda_k(2M+1;W)\{v_{n+M}^{(k)}(2M+1,W)\} \leftrightarrow B_W U_k(2M+1,W;f). \tag{36}$$

2.4. *Asymptotics of DPSWF's*

In what follows, in addition to (17) we adopt the abbreviation

$$\alpha = 1 - A = 1 - \cos 2\pi W. \tag{37}$$

A. $U_k(N,W;f)$ for fixed k and large N

When N is large and W and k are fixed,

$$U_k(N,W;f) \sim \begin{cases} c_1 f_1(\omega), & 0 \le \omega \le N^{-1/3} \\ c_2 f_2(\omega), & N^{-1/3} \le \omega \le \arccos\,[A + N^{-3/2}] \\ c_3 f_3(\omega), & \arccos\,[A + N^{-3/2}] \le \omega \le 2\pi W \\ c_3 f_4(\omega), & 2\pi W \le \omega \le \arccos\,[A - N^{-3/2}] \\ c_5 f_5(\omega), & \arccos\,[A - N^{-3/2}] \le \omega \le \pi. \end{cases} \tag{38}$$

Here

$$f_1(\omega) = D_k\left(\left(\frac{N^2}{2\alpha}\right)^{1/4}\omega\right)$$

$$f_2(\omega) = \frac{[\sqrt{1+\cos\omega} + \sqrt{\cos\omega - A}]^N}{[(1-\cos\omega)(\cos\omega - A)]^{1/4}} \times \left[\frac{\sqrt{1-\cos\omega}}{\sqrt{\alpha(1+\cos\omega)} + \sqrt{2(\cos\omega - A)}}\right]^{k+1/2}$$

$$f_3(\omega) = I_0\left(\frac{N}{\sqrt{2-\alpha}}\sqrt{\cos\omega - A}\right) \tag{39}$$

$$f_4(\omega) = J_0\left(\frac{N}{\sqrt{2-\alpha}}\sqrt{A-\cos\omega}\right)$$

$$f_5(\omega) = \frac{\cos\left[\frac{N}{2}\arcsin\theta(\omega) + \frac{1}{2}\left(k+\frac{1}{2}\right)\arcsin\phi(\omega) + (k-N)\frac{\pi}{4} + \frac{3\pi}{8}\right]}{[(A-\cos\omega)(1-\cos\omega)]^{1/4}}$$

$$\theta(\omega) \equiv \frac{\alpha + 2\cos\omega}{2-\alpha}, \quad \phi(\omega) \equiv \frac{(2-3\alpha)-(2+\alpha)\cos\omega}{(2-\alpha)(1-\cos\omega)}$$

where $D_k(\cdot)$ is the Weber function (Ref. 9, Vol. II, Chapter VIII), and I_0 and J_0 are the usual Bessel functions. The constants in (38) are given by

$$c_i = (-1)^{[k/2]}(k!)^{-1/2}\pi^{Y_1}2^{Y_2}\alpha^{Y_3}N^{Y_4}[\sqrt{2}+\sqrt{\alpha}]^{Y_5}[2-\alpha]^{Y_6}$$

i	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
1	$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{1}{8}$	$\frac{1}{4}$	0	0
2	$\frac{1}{4}$	$\frac{7}{4}k+\frac{7}{8}$	$\frac{k}{4}+\frac{3}{8}$	$\frac{k}{2}+\frac{1}{4}$	$-N$	0
3	$\frac{3}{4}$	$\frac{7}{4}k+\frac{11}{8}$	$\frac{k}{4}+\frac{1}{8}$	$\frac{k}{2}+\frac{3}{4}$	$-N$	$(N-k-1)/2$
5	$\frac{1}{4}$	$\frac{7}{4}k+\frac{15}{8}$	$\frac{k}{4}+\frac{3}{8}$	$\frac{k}{2}+\frac{1}{4}$	$-N$	$\frac{1}{4}+(N-k-1)/2.$

(40)

B. $U_k(N,W;f)$ for large N and $k = \lfloor 2WN(1-\epsilon)\rfloor$

When k and N become large together with

$$k = \lfloor 2WN(1-\epsilon) \rfloor, \quad 0 < \epsilon < 1 \tag{41}$$

and ϵ fixed, then

$$U_k(N,W;f) \sim \begin{cases} d_1 g_1(\omega), & 0 \le \omega \le \arccos\,(B + N^{-1/2}) \\ d_2 g_2(\omega), & \arccos\,(B + N^{-1/2}) \le \omega \le \arccos\,(B - N^{-1/2}) \\ d_3 g_3(\omega), & \arccos\,(B - N^{-1/2}) \le \omega \le \arccos\,(A + N^{-1}) \\ d_4 g_4(\omega), & \arccos\,(A + N^{-1}) \le \omega \le 2\pi W \\ d_4 g_5(\omega), & 2\pi W \le \omega \le \arccos\,(A - N^{-1}) \\ d_6 g_6(\omega), & \arccos\,(A - N^{-1}) \le \omega \le \pi. \end{cases} \tag{42}$$

Here B is determined so that

$$\int_B^1 \sqrt{\frac{\xi - B}{(\xi - A)(1 - \xi^2)}}\, d\xi = \frac{k}{N}\pi \tag{43}$$

and

$$g_1(\omega) = R(\omega) \cos\left[\frac{N}{2}\int_0^\omega \sqrt{\frac{\cos t - B}{\cos t - A}}\, dt - \frac{C}{4}\int_0^\omega \frac{dt}{\sqrt{(\cos t - B)(\cos t - A)}} - (1 - (-1)^k)\frac{\pi}{4}\right]$$

$$g_2(\omega) = Ai\left(-\frac{N^{2/3}(\cos\omega - B)}{[4(1 - B^2)(B - A)]^{1/3}}\right)$$

$$g_3(\omega) = R(\omega) \exp\left[-\frac{N}{2}\int_{\arccos B}^\omega \sqrt{\frac{B - \cos t}{\cos t - A}}\, dt - \frac{C}{4}\int_{\arccos B}^\omega \frac{dt}{\sqrt{(B - \cos t)(\cos t - A)}}\right]$$

$$g_4(\omega) = I_0\left(N\sqrt{\frac{(B - A)}{(1 - A^2)}(\cos\omega - A)}\right) \tag{44}$$

$$g_5(\omega) = J_0\left(N\sqrt{\frac{(B - A)}{(1 - A^2)}(A - \cos\omega)}\right)$$

$$g_6(\omega) = R(\omega) \cos\left[\frac{N}{2}\int_\omega^\pi \sqrt{\frac{B - \cos t}{A - \cos t}}\, dt + \frac{C}{4}\int_\omega^\pi \frac{dt}{\sqrt{(B - \cos t)(A - \cos t)}} + \theta\right]$$

$$R(\omega) \equiv |(B - \cos\omega)(A - \cos\omega)|^{-1/4}.$$

The function $Ai(x)$ is the Airy function defined in Ref. 10, page 446. The parameter C here is given by

$$C = \frac{4}{L_2}\left[\frac{N}{2}L_1 + (2 + (-1)^k)\frac{\pi}{4}\right]_{\text{rem } 2\pi} \tag{45}$$

where $[x]_{\text{rem } 2\pi} = x - 2\pi\lfloor x/2\pi \rfloor$ is the number between zero and 2π congruent to x modulo 2π, and the parameter θ in g_6 is

$$\theta = \left[\frac{\pi}{4} - \frac{N}{2}L_5 - \frac{C}{4}L_6\right]_{\text{rem } 2\pi}. \tag{46}$$

The L's are given by

$$\begin{aligned} L_1 &= \int_B^1 P(\xi)d\xi \qquad & L_2 &= \int_B^1 Q(\xi)d\xi \\ L_3 &= \int_A^B P(\xi)d\xi \qquad & L_4 &= \int_A^B Q(\xi)d\xi \\ L_5 &= \int_{-1}^A P(\xi)d\xi \qquad & L_6 &= \int_{-1}^A Q(\xi)d\xi = L_2 \end{aligned} \tag{47}$$

$$P(\xi) \equiv \left|\frac{\xi - B}{(\xi - A)(1 - \xi^2)}\right|^{1/2}, \qquad Q(\xi) \equiv |(\xi - B)(\xi - A)(1 - \xi^2)|^{-1/2}.$$

The L's can be expressed simply in terms of complete elliptic integrals of the first and third kind. (See Ref. 13, pages 242 and 265.) (The integrals in (44) can also be expressed in terms of elliptic functions, but the resulting expressions shed no light on the nature of the solution.) Finally, the d's in (42) are

$$\begin{aligned} d_1 &= L_2^{-1/2}\,\pi^{1/2}2^{1/2} \\ d_2 &= L_2^{-1/2}\,\pi 2^{1/3}(1 - B^2)^{-1/12}(B - A)^{-1/3}N^{1/6} \\ d_3 &= L_2^{-1/2}\,\pi^{1/2}2^{-1/2} \\ d_4 &= L_2^{-1/2}\,\pi(1 - A^2)^{-1/4}e^{-CL_4/4}e^{-NL_3/2}N^{1/2} \\ d_6 &= L_2^{-1/2}\,\pi^{1/2}2^{1/2}e^{-CL_4/4}e^{-NL_3/2} \end{aligned} \tag{48}$$

C. $U_k(N,W;f)$ for large N and $k = \lfloor 2WN + (b/\pi)\log N \rfloor$

When $N \to \infty$ and

$$k = \lfloor 2WN + (b/\pi)\log N \rfloor \tag{49}$$

with b and W fixed, we have asymptotically in N

$$U_k(N,W;f) \sim \begin{cases} e_1h_1(\omega), & 0 \le \omega \le \arccos\,[A + N^{-2/3}] \\ e_2h_2(\omega), & |\cos\omega - A| \le N^{-2/3} \\ e_3h_3(\omega), & \arccos\,[A - N^{-2/3}] \le \omega \le \pi. \end{cases} \tag{50}$$

Here

$$h_1(\omega) = \frac{1}{\sqrt{\cos\omega - A}} \cos\left[\frac{N}{2}\omega + \frac{E\beta}{2}\log\left|\frac{\beta(1+A)\tan\frac{\omega}{2}+1}{\beta(1+A)\tan\frac{\omega}{2}-1}\right| - k\frac{\pi}{2}\right]$$

$$h_2(\omega) = e^{i(\beta/2)N(\cos\omega - A)}\Phi\left(\frac{1}{2} - i\frac{E\beta}{2},\quad 1; -i\beta N(\cos\omega - A)\right) \tag{51}$$

$$h_3(\omega) = \frac{1}{\sqrt{A - \cos\omega}}$$

$$\times \cos\left[\frac{N}{2}\omega + \frac{E\beta}{2}\log\left|\frac{\beta(1+A)\tan\frac{\omega}{2}+1}{\beta(1+A)\tan\frac{\omega}{2}-1}\right| - (k+1)\frac{\pi}{2}\right]$$

$$\beta \equiv |\csc 2\pi W| \tag{52}$$

and

$$\Phi(a,c;x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \dots$$

is the confluent hypergeometric function in the notation of Ref. 9, Vol. 1, Chapter 6. The constant E is to be determined as the root of smallest absolute value of

$$N\pi W + \frac{E\beta}{2}\log\frac{2N}{\beta} + \psi(E\beta) - k\frac{\pi}{2} - \frac{\pi}{4} = 0. \tag{53}$$

Here we have written

$$\Gamma\left(\frac{1}{2} - \frac{1}{2}is\right) = r(s)e^{i\psi(s)} \tag{54}$$

where r, ψ, and s are real and Γ is the usual gamma function.

The constants in (50) are given by

$$e_1 = (-1)^{\lfloor k/2 \rfloor}\left[\frac{3\pi}{\beta[1 + e^{\pi E\beta}]\log N}\right]^{1/2}$$

$$e_2 = r(E\beta)\sqrt{\beta N}e^{(\pi/4)E\beta}e_1 \tag{55}$$

$$e_3 = e^{(\pi/2)E\beta}e_1.$$

D. $U_k(N,W;f)$ **for large** N **and** $k = \lfloor 2WN(1+\epsilon)\rfloor$

The case of large N with $k = \lfloor 2WN(1+\epsilon)\rfloor$, $0 < \epsilon < 1/2W - 1$ can be reduced to case B above by means of the formula (13). One finds

$$U_k(N,W;f) = K(N,k)U_{k'}\left(N,W';\frac{1}{2}-f\right) \quad (56)$$

where $U_{k'}(N,W';½-f)$ can be obtained from (41)–(48). Here

$$W' = \frac{1}{2} - W$$

$$k' = N - 1 - k \sim 2W'N(1-\epsilon') \quad (57)$$

$$\epsilon' \sim \left(1 - \frac{1}{2W}\right)\epsilon.$$

E. $U_{N-\ell}(N,W;f)$ for fixed ℓ and large N

Formula (13) reduces this case to case A above:

$$U_{N-\ell}(N,W;f) = K(N,N-\ell)U_{\ell-1}\left(N,\frac{1}{2}-W;\frac{1}{2}-f\right)$$

where formulas (38)–(40) can be used to obtain asymptotic values for $U_{\ell-1}(N,½-W;½-f)$.

2.5 Asymptotics of the eigenvalues $\lambda_k(N,W)$

For fixed k and large N, one has

$$1-\lambda_k(N,W) \sim \pi^{1/2}(k!)^{-1}2^{(14k+9)/4}\alpha^{(2k+1)/4}[2-\alpha]^{-(k+1/2)}N^{k+1/2}e^{-\gamma N}$$

$$\alpha = 1-\cos 2\pi W, \quad \gamma = \log\left[1+\frac{2\sqrt{\alpha}}{\sqrt{2}-\sqrt{\alpha}}\right]\cdot \quad (58)$$

Some values computed from this expression are shown as dotted lines on Figs. 5 and 6. The fit with λ_0 is very good for $N \geq 2$ when $W = 0.4$ and for $N \geq 6$ when $W = 0.2$.

For large N and k with

$$k = \lfloor 2WN(1-\epsilon)\rfloor, \qquad 0 < \epsilon < 1,$$

$$1-\lambda_k(N,W) \sim e^{-CL_4/2}e^{-L_3 N}. \quad (59)$$

Here the L's are given by (47) with B and C determined from (43) and (45).

For large N and k with

$$k = \lfloor 2WN + (b/\pi)\log N\rfloor$$

$$\lambda_k(N,W) \sim \frac{1}{1+e^{\pi b}}\cdot \quad (60)$$

A good approximation to $\lambda_k(N,W)$ when $0.2 < \lambda < 0.8$ is given by

$$\lambda_k(N,W) \sim [1+e^{\pi\hat{b}}]^{-1} \quad (61)$$

where

$$\delta = -\frac{2\pi[NW - k/2 - 1/4]}{\log [8N|\sin 2\pi W|] + \gamma} \tag{62}$$

where $\gamma = 0.5772156649$ is the Euler-Mascheroni constant. Some values of (61)–(62) are shown on Fig. 3 for $k = 7$ and 13 and on Fig. 4 for $k = 3$ and 6. Near $\lambda = 1/2$ the discrepancy between the true value and the formula (61)–(62) cannot be seen on the scale of Figs. 3 and 4.

Asymptotic values for $\lambda_k(N, W)$ with N large and $N - k = \ell$ fixed can be obtained directly from (13) and (58). In a similar way, (13) and (59) provide an asymptotic formula for $\lambda_k(N, W)$ when $k = \lfloor 2WN(1 + \epsilon) \rfloor$, $0 < \epsilon < 1/2W - 1$. One has in this case

$$\lambda_k(N, W) \sim e^{-CL_4/2} e^{-L_3 N} \tag{63}$$

where C, L_3, and L_4 are to be computed from (43), (45) and (47) with W replaced by $1/2 - W$ and k replaced by $N - k - 1$.

For fixed k and N, but W small, we find

$$\lambda_k(N, W) = \frac{1}{\pi} (2\pi W)^{2k+1} G(k,N)[1 + 0(W)] \tag{64}$$

where, for example

$$G(0,N) = N$$

$$G(1,N) = \frac{1}{36}(N-1)N(N+1)$$

$$G(2,N) = \frac{1}{8100}(N-2)(N-1)N(N+1)(N+2)$$

$$G(N-1,N) = \frac{2^{2N-2}}{(2N-1)\binom{2N-2}{N-1}^3}. \tag{65}$$

The general term is

$$G(k,N) = \frac{2^{2k}(k!)^6}{(2k+1)^2[(2k)!]^4} \prod_{j=-k}^{k} (N - j). \tag{66}$$

For fixed k and N, but W near $1/2$, i.e., $1/2 - W > 0$ small, (13) combined with (64) gives

$$1 - \lambda_k(N, W) = \frac{1}{\pi} G(N - 1 - k, N)[\pi(1 - 2W)]^{2(N-k)-1}[1 + 0(1 - 2W)]. \tag{67}$$

2.6 Relationship to PSWF's: $W \to 0$, $N \to \infty$, $\pi NW \to c > 0$

The prolate spheriodal wave functions (PSWF's) $\psi_i(c;x)$ and their associated eigenvalues $\lambda_i(c)$, $i = 0,1,2, \ldots$ are defined by

$$\int_{-1}^{1} \frac{\sin c(x - x')}{\pi(x - x')} \psi_i(c;x')dx' = \lambda_i(c)\psi_i(c;x), \tag{68}$$

$$-\infty < x < \infty$$

$$\lambda_0 > \lambda_1 > \lambda_2 \ldots$$

$$\int_{-\infty}^{\infty} \psi_i^2(c;x)dx = 1, \quad \psi_i(0) \geq 0, \quad \psi_i'(0) \geq 0 \tag{69}$$

$$i = 0,1,2, \ldots .$$

For each $i = 0,1,2, \ldots$ the PSWF $\psi_i(c;x)$ satisfies the differential equation

$$\frac{d}{dx}(1 - x^2)\frac{d\psi_i}{dx} + [\chi - c^2x^2]\psi_i = 0 \tag{70}$$

for a special value

$$\chi = \chi_i(c) \tag{71}$$

of the parameter χ. The PSWF's and the quantities $\lambda_i(c)$ and $\chi_i(c)$ are discussed in detail in Refs. 1–6.

Now let $c > 0$ and y, a real number, be given. If, as

$$W \to 0, N = \left\lfloor \frac{c}{\pi W} \right\rfloor \quad \text{and } n = \left\lfloor \frac{N}{2}(1 + y) \right\rfloor \tag{72}$$

then

$$\lambda_i(N,W) \sim \lambda_i(c)$$

$$\sqrt{W}\, U_i(N,W;Wf) \sim \psi_i(c,f) \tag{73}$$

$$\sqrt{\frac{N}{2}}\, v_n^{(i)}(N,W) \sim \frac{\pm 1}{\sqrt{\lambda_i(c)}} \psi_i(c;y) \tag{74}$$

$$N^2 - 1 - 2\theta_i(N,W) \sim \chi_i(c). \tag{75}$$

In (74) when i is even the plus sign is taken when $\int_{-1}^{1} \psi_i(c;x)dx > 0$; if i is odd, the plus sign is taken when $\int_{-1}^{1} x\psi_i(c,x)dx > 0$; otherwise, the negative sign is to be used.

III. APPLICATIONS

3.1 Extremal properties

3.1.1 Most concentrated bandlimited sequence

To maximize (6) over the bandlimited sequences, we replace h_n

there by its representation (5) and use (4) to obtain

$$\lambda = \frac{\sum_{n=N_0}^{N_0+N-1} \int_{-W}^{W} df \int_{-W}^{W} df' H(f)\overline{H}(f') e^{-2\pi i n(f-f')}}{\int_{-W}^{W} |H(f)|^2 df}$$

$$= \frac{\int_{-W}^{W} df \int_{-W}^{W} df' e^{-i\pi(2N_0+N-1)(f-f')} \frac{\sin N\pi(f-f')}{\sin \pi(f-f')} H(f)\overline{H}(f')}{\int_{-W}^{W} |H(f)|^2 df}$$

$$= \frac{\int_{-W}^{W} df \int_{-W}^{W} df' \frac{\sin N\pi(f-f')}{\sin \pi(f-f')} \psi(f)\overline{\psi}(f')}{\int_{-W}^{W} |\psi(f)|^2 df}. \qquad (76)$$

Here we have written

$$\psi(f) \equiv e^{-i\pi(2N_0+N-1)f} H(f) \qquad (77)$$

and used the fact that

$$\sum_{N_0}^{N_0+N-1} e^{-2\pi i n(f-f')} = e^{-i\pi(2N_0+N-1)(f-f')} \frac{\sin N\pi(f-f')}{\sin \pi(f-f')}.$$

A simple variational argument applied to (76) shows that λ is stationary when ψ satisfies (7) and hence the maximum value of λ is $\lambda_0(N,W)$, attained when $\psi(f) = cU_0(N,W;f)$, $|f| \le W$. Equation (77) then shows that the most concentration bandlimited sequence is

$$\{h_n\} \leftrightarrow H(f) = \begin{cases} ce^{i\pi(2N_0+N-1)f} & U_0(N,W;f), \quad |f| \le W \\ 0, & W < |f| \le ½. \end{cases} \qquad (78)$$

For the sequence itself, we then find from (5)

$$h_n = c \int_{-W}^{W} U_0(N,W;f) e^{i\pi[N-1-2(n-N_0)f]} df$$

whence from (29)

$$\{h_n\} = d\{v_{n-N_0}^{(0)}(N,W)\} \qquad (79)$$

where d is independent of n. The results (78) and (79) were stated in Section I after eq. (6).

More generally, for $k = 1,2,\ldots,N-1$ we have that $d\{v_{n-N_0}^{(k)}(N,W)\}$ is the bandlimited sequence most concentrated in (N_0,N_0+N-1) that is orthogonal to $\{v_{n-N_0}^{(i)}(N,W)\}$, $i = 0,1,\ldots,k-1$. The fraction of its energy in the range (N_0,N_0+N-1) is $\lambda_k(N,W)$.

3.1.2 Indexlimited sequence with most concentrated spectrum

If $\{h_n\}$ is indexlimited, so that

$$h_n = \begin{cases} 0, & n < N_0 \\ h_n, & N_0 \le n \le N_0 + N - 1 \\ 0, & n > N_0 + N - 1, \end{cases}$$

and if $H(f) \leftrightarrow \{h_n\}$, then

$$\mu \equiv \frac{\int_{-W}^{W} |H(f)|^2 df}{\int_{-1/2}^{1/2} |H(f)|^2 df} = \frac{\sum_{n=N_0}^{N_0+N-1} \sum_{m=N_0}^{N_0+N-1} \frac{\sin 2\pi W(n-m)}{\pi(n-m)} h_n \bar{h}_m}{\sum_{N_0}^{N_0+N-1} |h_n|^2}$$

$$= \frac{\sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \frac{\sin 2\pi W(n-m)}{\pi(n-m)} h_{n+N_0} \bar{h}_{m+N_0}}{\sum_{0}^{N-1} |h_{n+N_0}|^2}. \qquad (80)$$

Here we have used (2) to replace $H(f)$ in the numerator, and have used (4) to rewrite the denominator. The quantity μ is a natural measure of the extent to which $H(f)$ is concentrated in the frequency interval $(-W, W)$. Comparison of the right member of (80) with (18) shows that μ will be a maximum when $h_{n+N_0} = cv_n^{(0)}(N, W)$, $n = 0, 1, \ldots, N-1$. Thus the indexlimited sequence with most concentrated spectrum in $-W \le f \le W$ is

$$\{h_n\} = \begin{cases} 0, & n < N_0 \\ v_{n-N_0}^{(0)}(N, W), & N_0 \le n \le N_0 + N - 1 \\ 0, & n > N_0 + N - 1. \end{cases} \qquad (81)$$

The concentration of its spectrum $H(f)$ is $\mu = \lambda_0(N, W)$ and

$$H(f) = dU_0(N, W; f) e^{i\pi(2N_0+N-1)f}, \quad \forall f \qquad (82)$$

with d independent of f.

More generally, for $k = 1, 2, \ldots, N-1$, $I_{N_0}^{N_0+N-1}\{v_{n-N}^{(k)}(N, W)\}$ is the indexlimited sequence with most concentrated spectrum in $-W \le f \le W$ that is orthogonal to

$$I_{N_0}^{N_0+N-1}\{v_{n-N}^{(i)}(N, W)\} \quad i = 0, 1, \ldots, k-1.$$

The fraction of its spectral energy in $|f| \le W$ is $\lambda_k(N, W)$.

3.1.3 Simultaneously achievable concentrations

Let $\{h_n\} \leftrightarrow H(f)$ and consider the two measures of concentration

$$\alpha^2 \equiv \frac{\sum_{N_0}^{N_0+N-1} |h_n|^2}{\sum_{-\infty}^{\infty} |h_n|^2}, \qquad \beta^2 \equiv \frac{\int_{-W}^{W} |H(f)|^2 df}{\int_{-1/2}^{1/2} |H(f)|^2 df}.$$

What values of α and β are possible?

Just as in Ref. 2, pp. 74–77, one finds the attainable nonnegative values of α and β are given by the intersection of the unit square $0 \le \alpha \le 1$, $0 \le \beta \le 1$ and the elliptical region

$$\alpha^2 - 2\alpha\beta\sqrt{\lambda_0(W,N)} + \beta^2 \le 1 - \lambda_0(W,N).$$

The elliptical boundary cuts the square at $\alpha = 1$, $\beta = \sqrt{\lambda_0(W,N)}$ and $\alpha = \sqrt{\lambda_0(W,N)}$, $\beta = 1$. As either N gets large, or as $W \to \frac{1}{2}$, $\lambda_0(W,N) \to 1$ as seen by (58) and (67), and the attainable region becomes the unit square.

3.1.4 Minimum energy bandlimited extension of a finite sequence

Let numbers $h_0, h_1, \ldots, h_{N-1}$ be given. There are infinitely many ways that one can choose numbers $h_N, h_{N+1}, \ldots$ and $h_{-1}, h_{-2}, \ldots$ so that the infinite sequence $\{h_n\}$ is bandlimited. Which of these sequences has least energy?

The answer is

$$h_n = \sum_{j=0}^{N-1} \alpha_j v_n^{(j)}(N,W) \tag{83}$$

$$n = 0, \pm 1, \pm 2, \ldots$$

where

$$\alpha_j = \sum_{n=0}^{N-1} h_n v_n^{(j)}(N,W) \tag{84}$$

$$j = 0, 1, \ldots, N-1.$$

The energy of this bandlimited sequence is

$$E = \sum_{-\infty}^{\infty} |h_n|^2 = \sum_{j=0}^{N-1} \frac{|\alpha_j|^2}{\lambda_j(N,W)}. \tag{85}$$

The dual to this problem is the following: Let $H(f)$ be given for $|f| \le W$. Consider extensions of H to the interval $|f| \le \frac{1}{2}$ that correspond to sequences $\{h_n\} \leftrightarrow H(f)$ that are indexlimited to the index set $(N_0, N_0 + N - 1)$. Which such extension has least energy?

The situation is quite different here from the dual just discussed. Given an arbitrary $H(f)$, $|f| \le W$, in general there is *no* way to extend it so that the corresponding sequence will be indexlimited. The extension can be accomplished only if for $|f| \le 1/2$

$$H(f) = \sum_{N_0}^{N_0+N-1} h_n e^{2\pi i n f}$$

or stated another way, only if for $|f| \le W$ we have

$$H(f) = e^{2\pi i (N_0+N-1/2)f} \sum_{j=0}^{N-1} \alpha_j U_j(N;W;f). \tag{86}$$

Then, of course,

$$\alpha_j = \frac{1}{\lambda_j} \int_{-W}^{W} H(f) e^{-2\pi i (N_0+(N-1)/2)f} U_j(N,W;f)df$$

by (11). But (86) for $|f| \le 1/2$ is then the extension sought of minimum energy. Its energy is

$$\int_{-1/2}^{1/2} |H(f)|^2 df = \sum_{j=0}^{N-1} \alpha_j^2. \tag{87}$$

The distinction between the two cases just treated arises, of course, because the Hilbert space of indexlimited sequences is finite dimensional while the space of bandlimited sequences is of infinite dimension.

3.1.5 Trigonometric polynomial with greatest fractional energy in an interval—optimal windows

Let $g(f)$ be a function of the form

$$g(f) = \sum_{k=0}^{N-1} g_k e^{-i\pi(N-1-2k)f}. \tag{88}$$

If $N = 2M + 1$ is odd, this can be written

$$g(f) = \sum_{n=-M}^{M} \hat{g}_n e^{2\pi i n f} \tag{89}$$

and if $N = 2M$ is even it can be written

$$g(f) = \sum_{n=-(M-1)}^{M} \hat{\hat{g}}_n e^{i\pi(2n-1)f} \tag{90}$$

where $\hat{g}_n$ and $\hat{\hat{g}}_n$ are suitably defined. In either case $g(f)$ can be called a trigonometric polynomial.

For functions of form (88) one readily computes

$$\lambda = \frac{\int_{-W}^{W} |g(f)|^2 df}{\int_{-1/2}^{1/2} |g(f)|^2 df} = \frac{\sum_{n,m=0}^{N-1} \rho(N,W)_{mn} g_m \bar{g}_n}{\sum_{0}^{N-1} |g_n|^2} \tag{91}$$

with $\rho(N,M)$ given by (21). Comparison of (91) with (18) and (26) shows that $U_0(N,W;f)$ *is the trigonometric polynomial of form* (88) *having the largest fractional concentration of energy in* $(-W,W)$.

Applications of this fact have been made to digital filtering,[12,16,18] to spectral estimation,[17] and to the definition of an essentially band-limited process by Balakrishnan[19] in 1965. In most of these applications, N is odd, and the $\hat{g}_k$ of (89) are required to be real and even in k. Thus for functions of form

$$g(f) = 2a_o + \sum_1^M a_j \cos 2\pi jf$$

with the a's real, one desires to choose the a's to maximize the fraction of the energy of g in $(-W,W)$. The answer is

$$a_j = v_{M-j}^{(o)}(2M+1,W) \quad j = 0,1,\ldots,M$$

and

$$g(f) = cU_o(2M+1,W;f).$$

This basic property of U_o can clearly make it of special interest in many fields.

3.2 A prediction problem

N successive samples spaced T_0 seconds apart are taken from a stationary white noise $X(t)$ of bandwidth W_0 and mean zero. The linear predictor formed from these samples that has minimum mean-squared error is used to estimate the next sample value of $X(t)$. What is the mean-squared error of this prediction, and how fast does it decrease with N?

We write $X_j = X(jT_0)$. Let the observed samples of $X(t)$ be X_0, X_1, $\ldots$, X_{N-1}. Then the predicted value $\hat{X}$ of X_N is to be of the form

$$\hat{X} = \sum_0^{N-1} a_j X_j$$

where the a's are chosen to minimize $\eta \equiv E(\hat{X} - X_N)^2$. The solution to this problem is well known. (See Ref. 11, pp. 302–305, for example.) The least value possible for η is

$$\eta_0 \equiv \min_{a's} \eta = \frac{\Delta_{N+1}}{\Delta_N} \tag{92}$$

where Δ_ℓ is the $\ell \times \ell$ determinant whose entry in row i and column j is EX_iX_j, $i,j = 0,1,\ldots,\ell-1$. For the white noise case at hand this entry is

$$\sigma^2 \frac{\sin 2\pi W_0T_0(i-j)}{2\pi W_0T_0(i-j)} = \frac{\sigma^2}{2W_0T_0}\rho(N,W_0T_0)_{ij}$$

in the notation of (21). Here $\sigma^2 = EX(t)^2$ is the noise power. Since the determinant of a matrix is the product of its eigenvalues, it follows that

$$\eta_0 = \frac{\sigma^2}{2W_0T_0} \frac{\prod_0^N \lambda_k(N+1, W_0T_0)}{\prod_0^{N-1} \lambda_k(N, W_0T_0)}. \tag{93}$$

We can now use our knowledge (Section 2.5) of the asymptotics of the $\lambda_k(N,W)$ to find the behavior of η_0 for large N. It is shown in Appendix A that for

$$0 < W_0T_0 < \frac{1}{2},$$

$$\lim_{N\to\infty} \frac{1}{N} \log \eta_0 = \log(\sin \pi W_0T_0)^2. \tag{94}$$

Thus the mean-squared error of the best linear prediction vanishes exponentially in N when the sampling rate $1/T_0$ is greater than the Nyquist rate $2W_0$. The exponent decreases in absolute value towards the limit zero as the sampling rate is decreased to the Nyquist rate.

The situation is very different when $W_0T_0 > 1/2$. Then η_0 approaches a limiting positive value, η_∞, as N gets large. We find (see Appendix A) that

$$\lim_{N\to\infty} \eta_0 \equiv \eta_\infty = \frac{\sigma^2 n}{2W_0T_0}\left(1 + \frac{1}{n}\right)^{2W_0T_0 - n} \tag{95}$$

$$\frac{n}{2} \le W_0T_0 \le \frac{n+1}{2}, \quad n = 1,2,\ldots$$

A plot of η_∞ for $1/2 \le W_0T_0 \le 5/2$ is shown in Fig. 8. Examination of (95) shows that $\eta_\infty = \sigma^2$ for $W_0T_0 = n/2$, $n = 1,2,\ldots$, and that the loops between these values shown in Fig. 8 get smaller and smaller as W_0T_0 increases. Thus, while η_∞ is zero for all sampling rates greater than the Nyquist rate, $\eta_\infty > 0.94$ for rates less than $1/2W_0$.

The foregoing is, of course, an unrealistic model of a physical prediction scheme in that it assumes perfect knowledge of the samples. If one assumes that to each sample $X(jT)$ an independent observation noise Y_j is added, then the linear predictor takes the form

$$\hat{X} = \sum_0^{N-1} a_j(X_j + Y_j). \tag{96}$$

If we assume that $\hat{X}$ is a prediction of the noisy next measurement $X(NT) + Y_N$, then all proceeds as before with the matrix ρ replaced by $\rho + (2W_0T_0\mu/\sigma^2)\, I$ where $\mu = EY_j^2$ and I is the unit matrix. By replacing

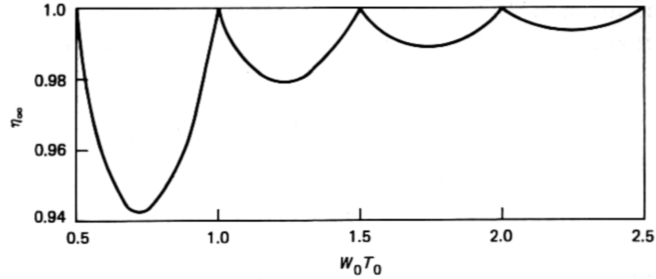

Fig. 8—Best mean squared error η_∞ vs. W_0T_0. The noise variance $\sigma^2 = 1$.

$\lambda_k(N, W_0T_0)$ by $2W_0T_0\mu/\sigma^2 + \lambda_k(N, W_0T_0)$ one finds readily that

$$\eta_\infty = \frac{\sigma^2 s}{2W_0T_0}\left(1 + \frac{1}{s}\right)^{2W_0T_0 - n}$$

$$s = \frac{\mu}{\sigma^2} 2W_0T_0 + n \tag{97}$$

$$\frac{n}{2} < W_0T_0 \le \frac{n+1}{2}, \quad n = 0,1,2,\ldots$$

When $n = 0$, so that sampling takes place faster than the Nyquist rate, η_∞ is positive. Indeed, η_∞ rises monotonically from the value μ at $T_0 = 0$ to the value $\mu + \sigma^2$ when $W_0T_0 = 1/2$, as might be expected; perfect prediction is no longer possible.

A more satisfying model would add independent noise to the observed samples, but require $\hat{X}$ to be a best linear predictor of $X(NT)$ itself, rather than of $X(NT)$ plus noise. The asymptotic behavior of η_0 in this case seems more difficult to obtain. A related problem is readily solved, however.

Let $\hat{X}$ as given by (96) now be a minimum variance estimate of X_{N-1}, where as before the Y_i are independent identically distributed random variables that represent the imprecision of the measurement process. We are now not trying to predict X_N but rather to eliminate the noise and estimate X_{N-1} correctly. One then finds for the mean-squared error

$$\eta_0 = \mu\left[1 - \phi\,\frac{\prod_{k=0}^{N-2}[\phi + \lambda_k(N-2, W_0T_0)]}{\prod_{k=0}^{N-1}[\phi + \lambda_k(N-1, W_0T_0)]}\right] \tag{98}$$

where

$$\phi = \frac{2W_0T_0}{\sigma^2}\mu.$$

Again using the techniques of Appendix A, we find in this case that

$$\eta_\infty = \mu \left[1 - \frac{\phi}{\phi + n} \left(1 + \frac{1}{n + \phi} \right)^{n - 2W_0T_0} \right] \tag{99}$$

$$\frac{n}{2} < W_0T_0 \le \frac{n+1}{2}, \quad n = 0,1,2,\ldots$$

Equation (99) can be obtained as a special case of a filtering problem solved by Viterbi.[20] He uses the result of Szegö that

$$\lim_{N\to\infty} \frac{Q_{N+1}}{Q_N} = \exp \int_{-1/2}^{1/2} \log H(f)df \tag{100}$$

where Q_N is the determinant of the $N \times N$ Toeplitz matrix having h_{i-j} as the entry in the ith row and jth column. Here, as usual, $\{h_n\} \leftrightarrow H(f)$ and we require that $h_{-n} = \bar{h}_n$, so that $H(f)$ is real. Szegö's result can indeed be applied to the ratio of determinants in (92). The Poisson summation formula, Ref. 14, p. 466, gives

$$H(f) = \frac{\sigma^2}{2W_0T_0} \sum_k \chi\left(\frac{f-k}{W}\right)$$

for this case where $\chi(f) = 1$ if $|f| \le 1$ and zero otherwise. Carrying out the details one finds (95) again, but finds only that

$$\lim_{N\to\infty} \eta_0 = 0$$

when $W_0T_0 < 1/2$. Our detailed knowledge of the λ's has permitted calculation of the rate at which η_0 approaches zero as expressed in eq. (94).*

3.3 The approximate dimension of signal space

The DPSS's $\{v_n^{(k)}\}$, $k = 0,1,\ldots,N-1$ are bandlimited to $(-W,W)$ (see (33)). The concentration of $\{v_n^{(k)}\}$ is given by

$$\lambda_k(N,W) = \frac{E(0,N-1)}{E(-\infty,\infty)}, \quad k = 0,1,\ldots,N-1$$

[see (22)]. From the results of Section 2.5 we have seen that as $N \to \infty$, $\lambda_k \to 1$ if $k = 2WN(1-\epsilon)$, while if $k = 2WN(1+\epsilon)$, $\lambda_k \to 0$. And this is true for any ϵ satisfying $1 > \epsilon > 0$. Thus a fraction arbitrarily close to $2W$ of the bandlimited DPSS's are confined almost entirely to the index set

* Note: After the work in Section 3.2 was completed, it was called to my attention that Widom[22] has derived an important extension of Szegö's theorem which applies to the case at hand here and gives the stronger result $\eta_0 \sim k[\sin \pi W_0T_0]^{2N}$ with k given explicitly to replace (94). The derivations of (94) and (95) given in the present paper are felt to be of interest in their own right and serve to verify the accuracy of the results given in Section 2.5.

$0 \leq n \leq N-1$. The remaining DPSS's have almost none of their energy in this index set.

The facts just noted can be summarized loosely in the statement that "for large N the set of sequences of bandwidth W that are confined to an index set of length about N has dimension approximately $2WN$." This basic intuitive notion can be made precise in a number of ways. We prefer the following method which treats bandlimiting and indexlimiting symmetrically.

Denote by I the index set $I = \{0,1,\ldots,N-1\}$. Now let $\epsilon > 0$ be given. Denote by G_ϵ the set of finite-energy sequences $\{h_n\} \leftrightarrow H(f)$ for which

$$E_{\bar{I}} \equiv \sum_{n \notin I} |h_n|^2 \leq \epsilon \tag{101}$$

and

$$E_{\overline{W}} \equiv \int_{1/2 \geq |f| > W} |H(f)|^2 df \leq \epsilon. \tag{102}$$

If ϵ is small, members of G_ϵ have little energy outside the index set $(0, N-1)$ or outside the frequency range $(-W, W)$. Now let $M = M(N, W, \epsilon, \epsilon')$ be the smallest integer such that there exist fixed sequences $\{g_n^{(1)}\}, \{g_n^{(2)}\}, \ldots, \{g_n^{(M)}\}$ such that for every $\{g\} \in G_\epsilon$ a's can be found for which

$$\sum_{n=0}^{N-1} \left[g_n - \sum_{1}^{M} a_j g_n^{(j)} \right]^2 \leq \epsilon'. \tag{103}$$

In words, M is the dimension of the smallest linear space of sequences that approximates G_ϵ on the index set $(0, N-1)$ with "energy" error less than ϵ'.

With these definitions out of the way, the key theorem on the dimension of signal space can be stated as follows.

Theorem: *If* $\frac{1}{2} \geq W > 0$ *and* $\epsilon' > \epsilon > 0$, *then*

$$\lim_{N \to \infty} \frac{M(N, W, \epsilon, \epsilon')}{N} = 2W. \tag{104}$$

Proof of this theorem follows very closely that given in the Appendix of Ref. 7 and will be omitted here.

For applications of this theorem it is important to note that the DPSS's $\{v_n^{(k)}\}$, for $k = 0,1,\ldots, 2NW(1-\eta)$ for suitable choice of η, can be used as an orthogonal basis for the M-dimensional space of sequences that best approximates G_ϵ in the sense of (103). Thus if N is large and one is dealing with sequences known to be approximately of bandwidth W and very small outside the index set $(0, N-1)$, $2WN$ numbers suffice to describe the sequence—namely, the first $2WN$ coefficients a_i in the expansion of the sequence $\{h_n\}$ on the appropriate DPSS's. We have then

$$h_n \approx \sum_{0}^{2WN-1} a_i v_n^{(i)}(N,W) \tag{105}$$

and

$$a_i = \sum_{0}^{N-1} h_n v_n^{(i)}(N,W). \tag{106}$$

Of course when N is large and $W < \frac{1}{2}$, $2WN \ll N$ so that the savings can be considerable.

Possible applications of the foregoing ideas to picture processing, cryptography, bandwidth compression and other sampled data systems should be evident. In many such applications, one starts with a signal $x(t) \in L^2(-\infty,\infty)$ defined for all time. A sequence $\{h_n\}$ is derived from $x(t)$ by sampling at rate $1/T_0$ so that

$$h_n = x(nT_0), \quad n = 0,1,\ldots. \tag{107}$$

If $X(f)$ is the spectrum of $x(t)$, so that

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{2\pi i f t} df, \tag{108}$$

then

$$H(f) = \sum_{-\infty}^{\infty} h_n e^{2\pi i n f} = \sum_{-\infty}^{\infty} e^{2\pi i n f} \int_{-\infty}^{\infty} X(f') e^{2\pi i n T_0 f'} df$$

$$= \frac{1}{T_0} \sum_{-\infty}^{\infty} X\left(\frac{f-n}{T_0}\right) \tag{109}$$

by the Poisson summation formula (Ref. 14, p. 466). If now $X(f)$ vanishes for $|f| > W_0$ and if $T_0 < 1/2\, W_0$, then $H(f) = 0$ for $W' < |f| \le \frac{1}{2}$ where $W' = W_0 T_0 < \frac{1}{2}$. Thus when signals are sampled at rates greater than the Nyquist rate, the DPSS are of particular value in providing a succinct method of describing N-vectors of samples.

An interesting application of these ideas forms part of a digital transmission scheme invented by Wyner to be described in a forthcoming paper by him.

IV. DERIVATIONS

4.1 Basic facts of Section 2.1–2.3

An orderly development of this subject is facilitated by a few comments about the operators

$$L \equiv \int_{-W}^{W} df' \frac{\sin N\pi(f-f')}{\sin \pi(f-f')} \tag{110}$$

and

$$M \equiv \frac{1}{4\pi^2} \frac{d}{df} (\cos 2\pi f - A) \frac{d}{df} + \frac{1}{4} (N^2 - 1) \cos 2\pi f \tag{111}$$

that appear in (10) and (16). As before, we take $0 < W < ½$, but now allow N to be an arbitrary real number. Operators of the type (110) and (111) have been well studied in the past and we borrow freely from the literature.

The kernel

$$K(f - f') = \frac{\sin N\pi(f - f')}{\sin \pi(f - f')} \tag{112}$$

is real, symmetric and square-integrable over the region $|f| \leq W$, $|f'| \leq W$. The characteristic equation, $L\psi = \lambda\psi$, therefore has as solutions a set of real eigenfunctions $\psi_0, \psi_1, \psi_2, \ldots$ that are orthogonal on $|f| \leq W$ and complete in $L^2(-W,W)$. The corresponding eigenvalues are real, and those eigenvalues that are different from zero have a finite degeneracy. The eigenfunctions and eigenvalues are continuous functions of the parameter N. The kernel of the operator L in (110) is defined for all values of f. The domain of definition of eigenfunctions of L belonging to non-zero eigenvalues can then be extended to the whole line $-\infty < f < \infty$ by means of

$$\psi \equiv \frac{1}{\lambda} L\psi.$$

These eigenfunctions are readily seen to possess continuous derivatives of all orders. The eigenfunctions belonging to the eigenvalue zero can also be chosen to have derivatives of all orders.

The characteristic equation for M,

$$MU = \theta U, \tag{113}$$

is an example of the well studied Sturm-Liouville equation (Ref. 14, p. 719). Let us denote by $\mathcal{U}$ the class of function continuous on $|f| \leq W$ and piecewise twice differentiable there. Then (113) has solutions in $\mathcal{U}$ only for a discrete set of real values of θ, the eigenvalues of M, say $\theta_0 \geq \theta_1 \geq \theta_2 \geq \ldots$ and a corresponding set of real eigenfunctions $U_0, U_1, \ldots$ can be found that are orthonormal, i.e. that satisfy

$$\int_{-W}^{W} U_i(f) U_j(f) df = \delta_{ij}. \tag{114}$$

Furthermore the U_i's are complete in $\mathcal{L}^2(-W,W)$.

For our particular M, (111), all the eigenvalues are non-degenerate. For, suppose U_i and U_j are linearly independent continuous solutions of (113) belonging to the same eigenvalue θ. From $MU_i = \theta U_i$ and MU_j

$= \theta U_j$ we obtain $U_j M U_i - U_i M U_j = 0$ or

$$U_j \frac{d}{df}(\cos 2\pi f - A)\frac{dU_i}{df} - U_i \frac{d}{df}(\cos 2\pi f - A)\frac{dU_j}{df}$$
$$= \frac{d}{df}(\cos 2\pi f - A)\left[U_j \frac{dU_i}{df} - U_i \frac{dU_j}{df}\right] = 0.$$

Integrate this last equation from $f = -W$ to a generic point f' with $-W < f' < W$. We find that

$$U_j(f')\frac{dU_i(f')}{df} - U_i(f')\frac{dU_j(f')}{df} = 0, \quad -W < f' < W,$$

which contradicts the assumed linear independence of U_i and U_j.

The non-degeneracy of all eigenvalues permits us to write

$$\theta_0 > \theta_1 > \theta_2 > \dots \tag{115}$$

It follows then from well-known theorems that $U_k(f)$ has exactly k zeros in the open interval $|f| < W$. That an eigenfunction U cannot vanish at either $f = W$ or $f = -W$ follows directly from the differential equation. For if U vanishes at $f = W$, for instance,

$$\frac{1}{(2\pi)^2}(\cos 2\pi f - A)\frac{d^2U}{df^2} - \frac{1}{2\pi}\sin 2\pi f \frac{dU}{df}$$
$$+ \left[\frac{1}{4}(N^2 - 1)\cos 2\pi f - \theta\right] U = 0 \tag{116}$$

evaluated at $f = W$ shows that $dU(W)/df = 0$. Differentiate (116) and evaluate at $f = W$ to see that $d^2U(W)/df^2 = 0$. Continued differentiation shows that all derivatives of U vanish at $f = W$. But U possesses a Taylor series about $f = W$ and so the assumption that $U(f) = 0$ leads to the conclusion $U \equiv 0$ which cannot be. Thus $U(W) \neq 0$.

We now know that both L and M possess orthonormal sets of eigenfunctions belonging to $\mathcal{U}$ that separately span $\mathcal{L}^2(-W, W)$. We show in Appendix C that L and M commute, i.e. for all $g(f) \in \mathcal{U}$, $LMg = MLg$. It is not hard to see then[15] that one can find a single set of orthonormal functions in $\mathcal{U}$ complete in $\mathcal{L}^2(-W, W)$ that are simultaneously eigenfunctions of L and M. Because of (115), however, the normalized eigenfunctions U_k, $k = 0, 1, \dots$ of M are unique except for sign. Thus the normalized solutions of (16) in $\mathcal{U}$, ordered by (115), are a complete set of eigenfunctions of L as well.

Any continuous solution to (113) in $|f| \leq \frac{1}{2}$ can be written as a Fourier series

$$U(f) = e^{i\pi(N-1)f} \sum_{-\infty}^{\infty} c_n e^{2\pi i n f}.$$

Substituting this form in (113) we find the 3-term recurrence for the c's

$$\frac{1}{2}n(N-n)c_{n-1} + \left[A\left(\frac{N-1}{2} - n\right)^2 - \theta\right]c_n + \frac{1}{2}(n+1)(N-n-1)c_{n+1} = 0, \quad (117)$$

$$n = 0,\pm 1, \ldots$$

Note that the coefficient of c_{n-1} here vanishes if $n = 0$ or $n = N$, while the coefficient of c_{n+1} vanishes for $n = -1$ and $n = N - 1$. Thus if *N is a positive integer,* which is the case of primary importance to us, the infinite system of equation (117) uncouples and we see that a solution is possible with $0 = c_{-1} = c_{-2} = \ldots = c_N = c_{N+1} = c_{N+2} = \ldots$ provided that

$$\sum_{j=0}^{N-1} \sigma(N,W)_{ij}c_j = \theta c_i \quad (118)$$

$$i = 0,1,\ldots,N-1$$

where the real symmetric matrix $\sigma(N,W)$ is given by (14). Such a matrix has N real eigenvalues, which we now see to be eigenvalues of M as well. We denote them by $\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_N}$. From (115) we know that $i_1, i_2, \ldots, i_N$ are N distinct non-negative integers. We denote the real eigenvector of $\sigma(N,W)$ corresponding to θ_{i_j} by

$$\underline{v}^{(i_j)} = (v_0^{(i_j)}(N,W), v_1^{(i_j)}(N,W), \ldots, v_{N-1}^{(i_j)}(N,W))^T \quad (119)$$

$$j = 1,2,\ldots,N$$

and suppose these vectors normalized so that

$$\sum_{\ell=0}^{N-1} v_\ell^{(i_j)}(N,W)v_\ell^{(i_k)}(N,W) = \delta_{jk}, \quad j,k = 1,2,\ldots,N. \quad (120)$$

We denote the corresponding eigenfunction of M by

$$U_{i_j}(N,W;f) = \sum_{n=0}^{N-1} v_n^{(i_j)}(N,W)e^{i\pi(N-1-2n)f}, \quad j = 1,2,\ldots,N. \quad (121)$$

Now again let N be a positive integer and denote by $\mathcal{G}_N$ the finite-dimensional space of functions of form

$$g(f) = \sum_{n=0}^{N-1} g_n e^{i\pi(N-1-2n)f} \quad (122)$$

where $g_0, g_1, \ldots, g_{N-1}$ are arbitrary complex numbers. We have just seen that if N is a positive integer, M leaves $\mathcal{G}_N$ invariant. Indeed, a simple calculation shows that if g is given by (122), then

$$Mg = g'(f) \equiv \sum_{n=0}^{N-1} g'_n e^{i\pi(N-1-2n)f} \tag{123}$$

where

$$g'_m = \sum_{n=0}^{N-1} \sigma(N,W)_{mn} g_n. \tag{124}$$

With N a positive integer, L also leaves $\mathcal{G}_N$ invariant. (Indeed, in this case L projects all of $\mathcal{L}^2$ onto $\mathcal{G}_N$.) If g is given by (122), one readily finds that

$$Lg = g''(f) \equiv \sum_{n=0}^{N-1} g''_n e^{i\pi(N-1-2n)f} \tag{125}$$

where

$$g''_m = \sum_{n=0}^{N-1} \rho(N,W)_{mn} g_n \tag{126}$$

and the $N \times N$ symmetric matrix $\rho(N,W)$ is given by (21). This is most easily seen from the fact that for integer N the kernel (112) is degenerate. Specifically,

$$\frac{\sin N\pi(f-f')}{\sin \pi(f-f')} = \sum_{n=0}^{N-1} e^{i\pi(N-1-2n)f} e^{-i\pi(N-1-2n)f'}. \tag{127}$$

Since L and M commute, so do the matrices $\rho(N,W)$ and $\sigma(N,W)$.

We now show that for integer N the eigenfunctions of M spanning $\mathcal{G}_N$, namely $U_{i_j}(N,W;f)$, $j = 1,2,\ldots,N$, belong to the N largest eigenvalues of M, namely, $\theta_0, \theta_1, \ldots, \theta_{N-1}$. We order the integers i_j so that $\theta_{i_1} > \theta_{i_2} > \ldots > \theta_{i_N}$, so that our task is to show that $i_j = j-1$, $j = 1,2,\ldots,N$. Now, if θ' and θ'' are two eigenvalues of M with $\theta'' < \theta'$, the eigenfunction belonging to θ'' must have at least one more zero in $|f| < W$ than the eigenfunction belonging to θ' (see Ref. 14, p. 721). It follows then that U_{i_N} must have at least $N-1$ zeros in $|f| < W$, since the smallest number of zeros U_{i_1} could have in $|f| < W$ is zero. But U_{i_N} cannot possibly have more than $N-1$ zeros in this interval, since, from (121), we can write

$$U_{i_N} = e^{i\pi(N-1)f} \sum_{n=0}^{N-1} v_n^{(i_N)} z^n, \quad z = e^{-2\pi i f}$$

which shows U_{i_N} to be a function of modulus unity times a polynomial of degree at most $N-1$. It follows then that U_{i_N} has exactly $N-1$ zeros in $|f| < W$, whence U_{i_j} has precisely $j-1$ such zeros $j = 1,2,\ldots,N$. It then follows that $\theta_{i_1} = \theta_0$ the largest eigenvalue of M, $\theta_{i_2} = \theta_1$, the next largest eigenvalue, $\ldots$ $\theta_{i_N} = \theta_{N-1}$. *Q.E.D.*

We have now shown that when N is an integer, the eigenfunctions of M that span $\mathcal{G}_N$ are

$$U_i(N,W;f) = \sum_{n=0}^{N-1} v_n^{(i)}(N,W)e^{i\pi(N-1-2n)f} \tag{128}$$

where the $\underline{v}^{(i)}$ are normalized eigenvectors of $\sigma(N,W)$:

$$\sum_{m=0}^{N-1} \sigma(N,W)_{nm} v_m^{(i)}(N,W) = \theta_i(N,W)v_n^{(i)}(N,W) \tag{129}$$

$$i,n = 0,1,\ldots, N-1.$$

These U's are also eigenfunctions of L and from (126) it then follows that

$$\sum_{m=0}^{N-1} \frac{\sin 2\pi W(n-m)}{\pi(n-m)} v_m^{(i)}(N,W) = \lambda_i(N,W)v_n^{(i)}(N,W) \tag{130}$$

$$i,n = 0,1,\ldots, N-1.$$

The matrix $\rho(N,W)$ is positive definite, since

$$\sum_{n,m=0}^{N-1} \rho(N,W)_{nm}\xi_n\bar{\xi}_m = \sum \int_{-W}^{W} dt e^{2\pi it(n-m)}\xi_n\bar{\xi}_m$$

$$= \int_{-W}^{W} \left| \sum_{0}^{N-1} \xi_n e^{2\pi int} \right|^2 dt$$

which is positive unless all the ξ's are zero. Thus the $\lambda_i(N,W)$ in (130) are all positive.

We have defined the U_i as eigenfunctions of M and have ordered them so that (115) is true. These same U_i are a complete set of eigenfunctions of L and we define λ_i to be the eigenvalue of L corresponding to U_i. We shall show next that the non-zero eigenvalues of L are non-degenerate and that when N is a positive integer

$$\lambda_0(N,W) > \lambda_1(N,W) > \ldots > \lambda_{N-1}(N,W) > 0. \tag{131}$$

The proof that if $\lambda \neq 0$ then λ is non-degenerate can be made exactly as in Ref. 1, equations (30)–(39). The assumption that two independent eigenfunctions of L, say U_n and U_m, belong to the same eigenvalue $\lambda \neq 0$ leads to the conclusion that $\theta_n = \theta_m$ which we have shown to be false. The reader can find details of the proof in Ref. 1.

We note next that for integer N, $U_k(N,W;fW) \to c_k P_k(f)$, $|f| \leq 1$, as $W \to 0$ where $P_k(f)$ is the Legendre polynomial of degree k. This follows directly from the differential equation (16) which for small W becomes

$$\frac{d}{df}(1-f^2)\frac{dU_k(N,W;fW)}{df} + \chi U_k(N,W;fW) + 0(W^2) = 0$$

where $\chi = \frac{1}{2}(N^2 - 1) - 2\theta$. Thus $\theta_k(N,W) = \frac{1}{4}(N^2 - 1) - \frac{1}{2}k(k + 1) + 0(W)$, $k = 0,1,\ldots,N-1$. Now the argument of Ref. 1, pages 61–62 holds again and it follows that for sufficiently small positive W, (131) holds. Since for integer N and $0 < W < \frac{1}{2}$ these λ's are non-degenerate and are continuous in W, it follows that (131) holds for $0 < W < \frac{1}{2}$ which is stated as (8).

Proofs of the remaining claims of Sections 2.1–2.3 are all of a more elementary nature. Most involve a straightforward calculation. We leave the details of the verification of these claims to the reader.

4.2 Asymptotics of the differential equation

We now consider solutions of the differential equation

$$\frac{d}{d\omega}[\cos\omega - A]\frac{dU}{d\omega} + \left[\frac{1}{4}(N^2 - 1)\cos\omega - \theta\right]U = 0 \tag{132}$$

for $0 \le \omega \le \pi$ when N is large and

$$4\theta = BN^2 + CN + \sum_{j=0}^{\infty} D_j N^{-j} \tag{133}$$

where B, C and the D's are assumed independent of N. The substitution

$$U = \frac{G}{\sqrt{\cos\omega - A}} \tag{134}$$

gives

$$\frac{d^2G}{d\omega^2} + \frac{N^2(\cos\omega - y_0)(\cos\omega - y_1)}{4(\cos\omega - A)^2}G = 0 \tag{135}$$

or

$$\frac{d^2G}{d\omega^2} + \left[N^2\frac{\cos\omega - B}{4(\cos\omega - A)} - N\frac{C}{4(\cos\omega - A)} + 0(1)\right]G = 0. \tag{136}$$

Here

$$y_0 = B + 0\left(\frac{1}{N}\right), \qquad y_1 = A + 0\left(\frac{1}{N}\right). \tag{137}$$

Case A. $1 > B > A > -1$ or $k = \lfloor 2WN(1 - \epsilon)\rfloor$

If $1 > B > A > -1$, then, as seen from (135) and (137), U is oscillatory for $1 \ge \cos\omega \ge B$ and for $A \ge \cos\omega \ge -1$, but is non-oscillatory in the interval $B \ge \cos\omega \ge A$. We investigate the solutions of (132) separately in each of these regions and also in the vicinity of the turning points y_0 and y_1.

Let $\cos\omega - A = t/N^2$. Then (132) becomes

$$t\frac{d^2U}{dt^2} + \frac{dU}{dt} - \frac{B-A}{4(1-A^2)}U + 0\left(\frac{1}{N}\right) = 0$$

so that near $\cos\omega = A$ we have

$$U \sim \begin{cases} U_4 \equiv d_4 I_0\left(N\sqrt{\frac{(B-A)}{(1-A^2)}(\cos\omega - A)}\right), & \cos\omega \geq A \\ U_5 \equiv d_4 J_0\left(N\sqrt{\frac{(B-A)}{(1-A^2)}(A - \cos\omega)}\right), & \cos\omega \leq A. \end{cases} \tag{138}$$

Here I_0 and J_0 are the usual Bessel functions. We note that when $\cos\omega = A + u/N$

$$U_4 = d_4 I_0\left(N^{1/2}\sqrt{\frac{(B-A)}{(1-A^2)}u}\right)$$

$$\sim \frac{d_4}{\sqrt{2\pi}}\left[N^{1/2}\sqrt{\frac{(B-A)}{(1-A^2)}u}\right]^{-1/2} \exp\left(N^{1/2}\sqrt{\frac{(B-A)}{(1-A^2)}u}\right) \tag{139}$$

(see Ref. 9, Vol. II, eq. 7.13.5, p. 86). When $\cos\omega = A - u/N$

$$U_5 = d_4 J_0\left(N^{1/2}\sqrt{\frac{(B-A)}{(1-A^2)}u}\right)$$

$$\sim d_4\sqrt{\frac{2}{\pi}}\left[N^{1/2}\sqrt{\frac{(B-A)}{(1-A^2)}u}\right]^{-1/2} \cos\left[N^{1/2}\sqrt{\frac{(B-A)}{(1-A^2)}u} - \frac{\pi}{4}\right] \tag{140}$$

(see Ref. 9, Vol. II, eq. 7.13.3, p. 85).

Now, the *WKB* solution of

$$\frac{d^2g}{dx^2} - [n^2E^2(x) + nF(x) + 0(1)]g = 0 \tag{141}$$

for large n is

$$g(x) \sim \frac{1}{\sqrt{E}}[c_1 e^{-n\int E dx - 1/2\int (F/E)dx} + c_2 e^{n\int E dx + 1/2\int (F/E)dx}] \tag{142}$$

provided x is not a zero of $E(x)$. (See Ref. 8, Sec. 7, Lemma 2.) Applying this to (136) and taking account of (134), we find that for $B > \cos\omega > A$

an asymptotic solution of (132) is

$$U \sim U_3 = d_3 R(\omega) \exp\left(-\frac{N}{2}\int_{\arccos B}^{\omega}\sqrt{\frac{B-\cos t}{\cos t - A}}\,dt - \frac{1}{4}\int_{\arccos B}^{\omega}\frac{C\,dt}{\sqrt{(B-\cos t)(\cos t - A)}}\right), \quad (143)$$

$$R(\omega) \equiv |(B-\cos\omega)(A\cos\omega)|^{-1/4}.$$

Here we have chosen $c_2 = 0$ in (142) to obtain a matching of U_3 and U_4 at $\cos\omega = A + u/N$. Indeed, one finds in a straighforward way that

$$U_3\left(\arccos\left(A+\frac{u}{N}\right)\right) \sim \frac{d_3 N^{1/4}}{u^{1/4}[B-A]^{1/4}} \times \exp\left(-\frac{N}{2}L_3 + N^{1/2}\sqrt{\frac{B-A}{1-A^2}}\,u - \frac{1}{4}CL_4\right) \quad (144)$$

where

$$L_3 = \int_A^B \sqrt{\frac{B-\xi}{(\xi-A)(1-\xi^2)}}\,d\xi,$$

$$L_4 = \int_A^B \frac{d\xi}{\sqrt{(B-\xi)(\xi-A)(1-\xi^2)}}. \quad (145)$$

Comparison of (144) and (139) shows that

$$d_4 = \sqrt{2\pi}\,N^{1/2}(1-A^2)^{-1/4}e^{-(NL_3/2)-(CL_4/4)}d_3. \quad (146)$$

An asymptotic solution to (132) near the turning point $\cos\omega = y_0$ is obtained by substituting $\cos\omega - B = t/N^{2/3}$ to obtain

$$\frac{d^2U}{dt^2} + \frac{t}{4(1-B^2)(B-A)}U + 0(N^{-2/3}) = 0. \quad (147)$$

Thus, near $\cos\omega = B$, we find

$$U \sim U_2 \equiv d_2 Ai\left(-\frac{N^{2/3}(\cos\omega - B)}{[4(1-B^2)(B-A)]^{1/3}}\right) \quad (148)$$

(see Ref. 10, 10.4.1, p. 446). Here we have chosen the asymptotic solution of (147) that agrees with U_3 at $\cos\omega = B - u/\sqrt{N}$. Indeed, from Ref. 10, 10.4.59, page 448, we have that

$$U_2\left(\arccos\left(B - \frac{u}{\sqrt{N}}\right)\right) = d_2 Ai\left(\frac{N^{1/6}u}{[4(1-B^2)(B-A)]^{1/3}}\right) \sim \frac{d_2}{2\sqrt{\pi}}\left[\frac{N^{1/6}u}{[4(1-B^2)(B-A)]^{1/3}}\right]^{-1/4} \times \exp\left(-\frac{2}{3}\frac{N^{1/4}u^{3/2}}{[4(1-B^2)(B-A)]^{1/2}}\right). \quad (149)$$

On the other hand, from (143) we find that

$$U_3\left(\arccos\left(B-\frac{u}{\sqrt{N}}\right)\right)\sim\frac{d_3N^{1/8}}{[u(B-A)]^{1/4}}$$
$$\times\exp\left(-\frac{N^{1/4}}{2}\frac{2}{3}\frac{u^{3/2}}{[(1-B^2)(B-A)]^{1/2}}\right) \quad (150)$$

so that on comparison with (149) we must have

$$d_3=\frac{2^{-5/6}}{\sqrt{\pi}}N^{-1/6}(1-B^2)^{1/12}(B-A)^{1/3}d_2. \quad (151)$$

On the other side of this turning point the solution U_2 continues as

$$U_2\left(\arccos\left(B+\frac{u}{\sqrt{N}}\right)\right)=d_2Ai\left(\frac{-N^{1/6}u}{[4(1-B^2)(B-A)]^{1/3}}\right)$$
$$\sim\frac{d_2}{\sqrt{\pi}}\left[\frac{N^{1/6}u}{[4(1-B^2)(B-A)]^{1/3}}\right]^{-1/4}\sin\left[\frac{2}{3}\frac{N^{1/4}u^{3/2}}{\sqrt{4(1-B^2)(B-A)}}+\frac{\pi}{4}\right]$$
$$(152)$$

as seen from Ref. 10, 10.4.60, page 448.

Applying (141)–(142) to (136) for $1\geq\cos\omega>B$, we find that

$$E=i\sqrt{\frac{\cos\omega-B}{4(\cos\omega-A)}}.$$

On recalling (134), we find the asymptotic formula

$$U\sim U_1\equiv d_1R(\omega)\cos\left[\frac{N}{2}\int_0^\omega\sqrt{\frac{\cos t-B}{\cos t-A}}\,dt\right.$$
$$\left.-\frac{C}{4}\int_0^\omega\frac{dt}{\sqrt{(\cos t-B)(\cos t-A)}}+\phi\right] \quad (153)$$

with $R(\omega)$ as in (143). Near the turning point $\cos\omega=y_0$, this becomes

$$U_1\left(\arccos\left(B+\frac{u}{\sqrt{N}}\right)\right)\sim\frac{d_1N^{1/8}}{u^{1/4}(B-A)^{1/4}}$$
$$\times\cos\left[\frac{N}{2}L_1-\frac{\tfrac{2}{3}N^{1/4}u^{3/2}}{\sqrt{4(B-A)(1-B^2)}}-\frac{C}{4}L_2+\phi\right] \quad (154)$$

where

$$L_1=\int_B^1\sqrt{\frac{\xi-B}{(\xi-A)(1-\xi^2)}}\,d\xi,$$
$$L_2=\int_B^1\frac{d\xi}{\sqrt{(\xi-B)(\xi-A)(1-\xi^2)}}. \quad (155)$$

Comparison of (154) and (152) shows that we must have

$$d_2 = \sqrt{\pi}\, N^{1/6}[4(1-B^2)]^{-1/12}(B-A)^{-1/3}d_1 \tag{156}$$

and

$$\frac{N}{2}L_1 - \frac{C}{4}L_2 + \phi = \frac{\pi}{4} \pmod{2\pi}. \tag{157}$$

Turning now to the interval $A > \cos\omega \geq -1$, we find from (141)–(142)

$$\begin{aligned} U \sim U_6 \equiv \frac{d_6}{[(B-\cos\omega)(A-\cos\omega)]^{1/4}} \\ \times \cos\left[\frac{N}{2}\int_\omega^\pi \sqrt{\frac{B-\cos t}{A-\cos t}}\,dt \right. \\ \left. + \frac{C}{4}\int_\omega^\pi \frac{dt}{\sqrt{(B-\cos t)(A-\cos t)}} + \theta\right]. \end{aligned} \tag{158}$$

At $\cos\omega = A - u/N$ this becomes

$$\begin{aligned} U_6\left(\arccos\left(A - \frac{u}{N}\right)\right) \sim \frac{d_6 N^{1/4}}{(B-A)^{1/4}u^{1/4}} \cdot \\ \times \cos\left[\frac{N}{2}L_5 - N^{1/2}\sqrt{\frac{B-A}{1-A^2}}\,u + \frac{C}{4}L_6 + \theta\right] \end{aligned} \tag{159}$$

where

$$\begin{aligned} L_5 &= \int_{-1}^{A}\sqrt{\frac{B-\xi}{(A-\xi)(1-\xi^2)}}\,d\xi, \\ L_6 &= \int_{-1}^{A}\frac{d\xi}{\sqrt{(B-\xi)(A-\xi)(1-\xi^2)}}. \end{aligned} \tag{160}$$

Comparison with (140) shows that

$$d_6 = \sqrt{\frac{2}{\pi}}\,N^{-1/2}(1-A^2)^{1/4}d_4. \tag{161}$$

$$\frac{N}{2}L_5 + \frac{C}{4}L_6 + \theta = \frac{\pi}{4} \pmod{2\pi}. \tag{162}$$

Equations (138) and (148) provide asymptotic solutions to (132) at the two turning points $\cos\omega = B$ and $\cos\omega = A$. Equations (153), (143) and (158) provide asymptotic solutions for the regions away from the turning points. Equations (146), (151), (156)–(157) and (161)–(162) insure that these solutions join together. This solution is summarized in Eqs. (42) where the regions of validity for each piece are shown explicitly, and in (43)–(48). As presented there, the constant ϕ of (153) has been chosen as $-[1-(-1)^k]\pi/4$ as it must to satisfy the inequalities shown in (9).

To normalize the solution (42)–(44) we must compute

$$W_i \equiv d_i^2 \int g_i^2(\omega)d\omega \qquad (163)$$

$$i = 1,2,\ldots,6$$

where the range of integration for each g_i^2 is the range of validity for that g given in (42). We then require that

$$\sum_1^6 W_i = \pi \qquad (164)$$

since $\omega = 2\pi f$.

Asymptotic forms for the W_i are readily worked out. One finds, for example,

$$W_1 = d_1^2 \int_0^{\arccos(B+1/\sqrt{N})} g_1^2(\omega)d\omega$$

$$\sim d_1^2 \frac{1}{2} \int_{B+1/\sqrt{N}}^1 \frac{d\xi}{\sqrt{(\xi-B)(\xi-A)(1-\xi^2)}} \sim \frac{1}{2} d_1^2 L_2 \qquad (165)$$

while, with $\nu \equiv [4(1-B^2)(B-A)]^{-1/3}$,

$$W_2 = d_2^2 \left[\int_{B-1/\sqrt{N}}^{B} dt + \int_B^{B+1/\sqrt{N}} dt \right] \frac{Ai^2(-N^{2/3}\nu(t-B))}{\sqrt{1-t^2}}$$

$$\sim \frac{d_2^2}{\sqrt{N}} \int_0^1 \frac{Ai^2(N^{1/6}\nu\xi)d\xi}{\sqrt{1-B^2}} + \frac{d_2^2}{\sqrt{N}} \int_0^1 \frac{Ai^2(-N^{1/6}\nu\xi)d\xi}{\sqrt{1-B^2}}.$$

By using the asymptotic forms for $Ai(x)$ (see Ref. 10, 10.4.59, 10.4.60, page 448) one finally finds

$$W_2 \sim \frac{d_2^2}{\pi\sqrt{\nu}\sqrt{1-B^2}} N^{-7/12} = c\frac{d_1^2}{N^{1/4}} = \frac{2c}{L_2}\frac{W_1}{N^{1/4}}$$

where c is independent of N. In a like manner, one finds that all the ratios W_i/W_1, $i = 2,3,\ldots,6$ vanish with increasing N. We omit the details here. Equations (164) and (165) now give $\pi \sim W_1 \sim \frac{1}{2}d_1^2L_2$ so that $d_1 = [2\pi/L_2]^{1/2}$. Equations (146), (151), (156) and (161) now determine all the d's to have the values given in (48).

Recall now that $U_k(N,W;f)$ has k zeros in $-W < f < W$. For the solution we have just constructed, all zeros in $(-W,W)$ are contributed by U_1 of (153). From (154) we see that the number of zeros is given asymptotically by

$$k = \left\lfloor \frac{2}{\pi} \left\{ \frac{N}{2} L_1 - \frac{2}{3} \frac{N^{1/4}}{\sqrt{4(B-A)(1-B^2)}} - \frac{C}{4} L_2 \right\} \right\rfloor \sim \frac{N}{\pi} L_1.$$

Thus if we set $k/N = 2W(1-\epsilon) \sim L_1/\pi$, we must have $L_1 \sim 2\pi W(1-\epsilon) = \pi k/N$ which is (43).

Finally, the two phase continuity requirements (157) and (162) must be met. The first of these is satisfied by the choice of C given in (45). This number lies between zero and $8\pi/L_2$ and hence is $0(1)$ as has been assumed throughout the development. Equation (162) is satisfied by choosing θ as in (46).

Case B. $1 = B > A > -1$ or $k = 0(1)$

If, in the preceding analysis, B is allowed to approach 1, the first turning point approaches $\omega = 0$ and the subinterval of $(-W, W)$ in which U can oscillate becomes vanishingly small. This suggests a separate investigation of (132)–(133) around $\omega = 0$ when $B = 1$.

Substitute

$$\omega = \frac{(2\alpha)^{1/4}}{\sqrt{N}}\, t$$

into (132), where, as before, $\alpha \equiv 1 - A$. One finds

$$\frac{d^2U}{dt^2} + \left[-\frac{C}{4}\sqrt{\frac{2}{\alpha}} - \frac{t^2}{4}\right] U + 0\left(\frac{1}{N}\right) = 0.$$

Asymptotically, then, U is a solution of Weber's equation $\ddot{D} + (\chi - \frac{1}{4}t^2)D = 0$ (see Ref. 9, Vol. II, 8.2, page 116) which has bounded solutions only if $x = k + \frac{1}{2}$ where k is a nonnegative integer. We are thus forced to take

$$C = -4\sqrt{\frac{\alpha}{2}}\left(k + \frac{1}{2}\right), \quad \theta = \frac{1}{4}N^2 - \left(k + \frac{1}{2}\right)\sqrt{\frac{\alpha}{2}}\,N + 0(1). \qquad (166)$$

The corresponding solution is generally denoted by $D_k(t)$ and has exactly k zeros (Ref. 9, Vol. II, 8.6, page 126). Thus we are led to take

$$U_k(\omega) \sim c_1 D_k(t) = c_1 D_k\left(\left(\frac{N^2}{2\alpha}\right)^{1/4} \omega\right) \qquad (167)$$

for fixed t, or $\omega = 0(N^{-1/2})$, as reported in (39). Examination of higher order terms (omitted here) shows (167) to be correct asymptotically even for $\omega = 0(N^{-1/3})$, whence the range of validity shown in (38).

Solutions of (132) near A and in the regions away from the turning points can be obtained in the present case from U_3, U_4, U_5 and U_6 of Section 4.2, Case A, by letting $B = 1$ and $C = -\sqrt{\alpha/2}(k + \frac{1}{2})$ in (138), (143), and (158). The indicated integrals in these last two equations can now be carried out explicitly. Equations (39) result. The constants c_1, $c_2, \ldots, c_5$ are then adjusted so that the solutions match asymptotically at the edges of their regions of validity. Finally, the solution is normalized. Again the oscillatory part near $f = 0$ dominates the asymptotic behavior of $\int_{-1/2}^{1/2} U_k(N,W;f)^2 df$ and we find

$$V_1 \equiv c_1^2 \int_0^{N^{-1/3}} f_1(\omega)^2 d\omega = \left(\frac{2\alpha}{N^2}\right)^{1/4}$$

$$\times c_1^2 \left[\int_0^{\infty} D_k^2(t)dt - \int_{N^{1/6}/(2\alpha)^{1/4}}^{\infty} D_k^2(t)dt \right]$$

$$\sim c_1^2 \left(\frac{2\alpha}{N^2}\right)^{1/4} \left[\sqrt{\frac{\pi}{2}}\, k! - \int_{N^{1/6}/(2\alpha)^{1/4}}^{\infty} t^{2k} e^{-t^2/2} dt \right]$$

$$\sim c_1^2 \frac{(2\alpha)^{1/4}}{\sqrt{N}} \sqrt{\frac{\pi}{2}}\, k! \quad (168)$$

(See Ref. 13, 7.711-1, p. 885 and Ref. 9, Vol. II, 8.4.1, page 122.) This determines c_1 and the values shown in (40) are obtained. We omit the straightforward but tedious details here.

Case C. $1 > B = A > -1$ or $k = \lfloor 2WN + (b/\pi) \log N \rfloor$

If, in the analysis of Section 4.2, Case A, the parameter B is allowed to approach the value A, the two turning points coincide at $\cos \omega = A$ and a new analysis of U in this neighborhood is now required.

With $\theta = AN^2 + CN + 0(1)$ and

$$\beta \equiv \frac{1}{\sqrt{1 - A^2}} = |\csc 2\pi W|,$$

in (133) substitute

$$\cos \omega - A = i \frac{\xi}{N\beta}, \; U = e^{-1/2\xi} F \quad (169)$$

to obtain

$$\xi F'' + (1 - \xi) F' - \frac{1}{2}(1 - iE\beta)F + 0\left(\frac{1}{N}\right) = 0 \quad (170)$$

where

$$E \equiv \frac{1}{2}(A - C). \quad (171)$$

For large N, then, $F(\xi) \sim \Phi(a,1;\xi)$ where

$$a \equiv \frac{1}{2}(1 - iE\beta)$$

and

$$\Phi(a,c;x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \ldots$$

is the confluent hypergeometric function. (See Ref. 9, Vol. I, 6.1.1, page

248, and 6.2.6, page 250.) Thus for $\cos\omega$ near A, we have

$$U \sim e_2 e^{i(\beta/2)N(\cos\omega - A)} \Phi(a,1;-i\beta N(\cos\omega - A))$$

as reported in (51). This expression is real.

Solutions away from the double turning point $\cos\omega = A$ can be obtained from (153) and (158) by setting $B = A$. The integrations can be done explicitly. The functions h_1 and h_3 of (51) result when C is replaced by E via (171).

There now remains the task of choosing the constants so that h_1, h_2, and h_3 join properly. We indicate a few key steps.

When

$$\cos\omega = A + \frac{u}{N^{2/3}},$$

$$h_1 \sim u^{-1/2} N^{1/3} \cos\left[\frac{N}{2}\arccos A - \frac{1}{2}N^{1/3}\beta u - \frac{E\beta}{2}\log u + E\beta\log\frac{N^{1/3}2^{1/2}}{\beta} - k\frac{\pi}{2}\right]. \quad (172)$$

To develop an asymptotic expression for h_2 at this point, we avail ourselves of the formula (Ref. 9, Vol. 1, 6.13.1.2, p. 278)

$$\Phi(a,c;x) \sim \frac{\Gamma(c)}{\Gamma(c-a)}\left(\frac{e^{i\pi\epsilon}}{x}\right)^a + \frac{\Gamma(c)}{\Gamma(a)} e^x x^{a-c} \quad (173)$$

where $\epsilon = 1$ if $\operatorname{Im} x > 0$ and $\epsilon = -1$ if $\operatorname{Im} x < 0$. One finds

$$h_2\left[\arccos\left(A + \frac{u}{N^{2/3}}\right)\right] \sim \frac{e^{-\pi E\beta/4}}{r(E\beta)\sqrt{2\gamma}} \cos\left[-\gamma - \psi(E\beta) + \frac{\pi}{4} - \frac{1}{2}E\beta\log 2\gamma\right] \quad (174)$$

where $\gamma \equiv \tfrac{1}{2}\beta N^{1/3}u$ and where the real functions r and ψ are defined by

$$\Gamma\left(\frac{1}{2} - \frac{1}{2}i\alpha\right) = r(\alpha)e^{i\psi(\alpha)}.$$

Comparison of (172) and (174) shows that we must have

$$e_2 = \beta^{1/2} r(E\beta) e^{\epsilon\beta\pi/4} N^{1/2} e_1 \quad (175)$$

and

$$\pi WN + \frac{E\beta}{2}\log N - \frac{E\beta}{2}\log\frac{\beta}{2} + \psi(E\beta) - k\frac{\pi}{2} - \frac{\pi}{4} = 0 (\mathrm{mod}\ 2\pi). \quad (176)$$

When $\cos \omega = A - u/N^{2/3}$, from (51) one finds

$$h_3 \sim \frac{N^{1/3}}{\sqrt{u}} \cos\left[\frac{N}{2} \arccos A + \frac{1}{2} N^{1/3}\beta u - \frac{E\beta}{2}\log u + E\beta \log \frac{N^{1/3}2^{1/2}}{\beta} - (k+1)\frac{\pi}{2}\right] \quad (177)$$

while from (51) and (173)

$$h_2\left[\arccos\left(A - \frac{u}{N^{2/3}}\right)\right] \sim \frac{e^{\pi E\beta/4}}{r(E\beta)\sqrt{2\gamma}} \cos\left[\gamma - \psi(E\beta) - \frac{E\beta}{2}\log 2\gamma - \frac{\pi}{4}\right]. \quad (178)$$

Comparison of these last two equations yields

$$e_3 = \beta^{-1/2} r(E\beta)^{-1} e^{E\beta\pi/4} N^{-1/2} e_2 \quad (179)$$

to match the amplitudes, while matching of the cosine arguments gives (176) again.

Now the number of zeros, k_1, of (50) in the interval $(0 \le \omega \le \arccos(A + N^{-2/3}))$ is seen from (172) to be given asymptotically by

$$k_1 \sim \frac{1}{\pi}\left[\pi WN - \frac{1}{2} N^{1/3}\beta + E\beta \log \frac{N^{1/3}2^{1/2}}{\beta}\right]$$

while asymptotically the number, k_2, of zeros of U in $(\arccos(A + N^{-2/3}) \le \omega \le \Omega)$ is obtained from (174) as

$$k_2 \sim \frac{1}{\pi}\left[\frac{1}{2}\beta N^{1/3} + \psi(E\beta) - \frac{\pi}{4} + \frac{E\beta}{2}\log \beta N^{1/3}\right].$$

Thus the number of zeros of $U(f)$ in $(-W, W)$ is given by

$$k = 2(k_1 + k_2) \sim \frac{2}{\pi}\left[\pi WN + \frac{E\beta}{2}\log N - \frac{E\beta}{2}\log\frac{\beta}{2} + \psi(E\beta) - \frac{\pi}{4}\right]. \quad (180)$$

This motivates the choice of E as a root of (53). When this is done, the matching condition (176) is also satisfied.

The constant e_1, e_2, e_3 must now be determined by (175), (179) and the normalization requirement (9). Routine calculations show that

$$X_1 \equiv e_1^2 \int_0^{\arccos[A+N^{-2/3}]} [h_1(\omega)]^2 d\omega \sim \frac{1}{2} e_1^2 \int_0^{\arccos[A+N^{-2/3}]} \frac{d\omega}{\cos\omega - A} \sim \frac{e_1^2\beta}{3}\log N, \quad (181)$$

$$X_3 \equiv e_3^2 \int_{\arccos\,[A-N^{-2/3}]}^{\pi} [h_3(\omega)]^2 d\omega$$

$$\sim \frac{1}{2} e_3^2 \int_{\arccos\,[A-N^{-2/3}]}^{\pi} \frac{d\omega}{A - \cos\omega} \sim \frac{e_3^2\beta}{3} \log N = \frac{e_1^2 \beta e^{E\beta\pi}}{3} \log N$$

while

$$X_2 \equiv e_2^2 \int_{\arccos\,[A+N^{-2/3}]}^{\arccos\,[A-N^{-2/3}]} [h_2(\omega)]^2 d\omega \sim e_2^2\, 0\left(\frac{1}{N}\right) \sim e_1^2\, 0(1) \quad (182)$$

which is negligibly small compared to the first two integrals. The normalization integral thus gives

$$e_1^2 \frac{\beta}{3} [1 + e^{E\beta\pi}] \log N = \pi.$$

The values (55) then follow where the factor $(-1)^{\lfloor k/2 \rfloor}$ is dictated by (9).

We have assumed throughout this analysis that $E \equiv \frac{1}{2}(A - C) = 0(1)$. It follows then from (53) that we must have $k = 2WN + (E\beta/\pi) \log N + 0(1)$. If then, we write

$$k = \left\lfloor 2WN + \frac{b}{\pi} \log N \right\rfloor$$

as in (49), it is seen that for the root of (53) we have

$$E \sim b/\beta. \quad (183)$$

Consideration of the detailed nature of ψ shows that E must be taken as the root of (53) of smallest absolute value.

4.3 Asymptotics of $\lambda_k(N,W)$ for large N

The values of $\lambda_k(N,W)$ for large N reported in Section 2.5 are obtained from the asymptotic expressions for $U_k(N,W;f)$ given in Section 2.4 by means of the basic relation

$$\lambda_k(N,W) = \int_{-W}^{W} U_k(N,W;f)^2 df \Big/ \int_{-1/2}^{1/2} U_k(N,W;f)^2 df. \quad (184)$$

Let

$$V_i = \int c_i^2 f_i^2(\omega) d\omega \qquad i = 1,2,\ldots,5$$

$$W_i = \int d_i^2 g_i^2(\omega) d\omega \qquad i = 1,2,\ldots,6$$

$$X_i = \int e_i^2 h_i^2(\omega) d\omega \qquad i = 1,2,3, \quad (185)$$

where the ranges of integration are given by the corresponding intervals of validity shown in (38), (42) and (50).

For fixed k and large N, $1 - \lambda_k \sim (V_4 + V_5)/\pi$ since $\Sigma\, V_i = \pi$. Now straightforward developments yield

$$V_4 = c_4^2 \int_{2\pi W}^{\arccos\,[A-N^{-2/3}]} J_0^2\left[\frac{N}{\sqrt{2-\alpha}}\sqrt{A-\cos\omega}\right] d\omega$$

$$\sim \frac{c_3^2}{N^{2/3}\sqrt{1-A^2}} \int_0^1 J_0^2\left[\frac{N^{2/3}}{\sqrt{2-\alpha}}\sqrt{t}\right] dt \sim \frac{\sqrt{2-\alpha}}{2\pi}\frac{c_3^2}{N^{4/3}}, \qquad (186)$$

$$V_5 = c_5^2 \int_{\arccos\,[A-N^{-2/3}]}^{\pi} f_5^2(\omega) d\omega$$

$$\sim \frac{c_5^2}{2}\int_{-1}^{A-N^{-2/3}} \frac{dt}{\sqrt{(A-t)(1-t)(1-t^2)}} \sim \frac{\pi}{2\sqrt{2(1-A)}} c_5^2. \qquad (187)$$

But, from (40), $c_3^2/(N^{4/3}c_5^2) = 0(N^{-1/3})$ so V_4 is negligible compared to V_5 and we have

$$1 - \lambda_k \sim \frac{1}{\pi} V_5 \sim c_5^2/2\sqrt{2(1-A)}$$

which is (58).

The formula (59) is obtained from $1 - \lambda_k \sim (W_5 + W_6)/\pi$ using (42) and (44). One finds $W_5 \sim d_4^2/(2\pi N^{3/2}\sqrt{B-A})$ and $W_6 \sim \frac{1}{2} d_6^2 L_6$. Equations (48) now show W_5 to be negligible and $1 - \lambda_k \sim W_6/\pi \sim L_6 d_6^2/2\pi$. Insertion of the value for d_6 in (48) gives (59).

Formula (60) arises from

$$\lambda = \frac{1}{\pi}\left[X_1 + \int_{\arccos\,[A+N^{-2/3}]}^{2\pi W} e_2^2 h_2^2(\omega) d\omega\right].$$

We have already commented in connection with (182) that the integral here is of smaller order than X_1 so that

$$\lambda = \frac{1}{\pi} X_1 \sim \frac{e_1^2\beta}{3\pi}\log N = [1 + e^{E\beta\pi}]^{-1} \qquad (188)$$

by (181) and (55). Since $E \sim b/\beta$ by (183), (60) results.

Finally, the approximation (61)–(62) arises from (188) and (53) by solving the latter approximately for $E\beta$. From the theory of the Γ function (Ref. 10, 6.1.27, p. 256, and 6.3.3, p. 258) one finds that $\psi(s) = \frac{1}{2}(\gamma + 2\log 2)s + 0(s^2)$. Inserting this in (53) one finds

$$E\beta \approx -\frac{N\pi W - \dfrac{k\pi}{2} - \dfrac{\pi}{4}}{\dfrac{1}{2}\log\dfrac{8N}{\beta} + \dfrac{1}{2}\gamma}.$$

This together with (188) is (61)–(62).

4.4 Asymptotics of $\lambda_k(N,W)$ for small W

Consider the matrix eigenvalue problem

$$\sum_{j=0}^{N-1} K(cx_i,cx_j)w_j\psi_j = \mu\psi_i, \quad i = 0,1,\ldots,N-1 \tag{189}$$

which has solutions only for those values of $\nu \equiv 1/\mu$ for which the determinant $\mathcal{D}(\nu) \equiv |\delta_{ij} - \nu K(cx_i,cx_j)|_{N-1}$ vanishes. Here, as in the rest of this section, we denote by $|f(i,j)|_{N-1}$ the determinant of the $N \times N$ matrix whose element in the ith row and jth column is $f(i,j)$, $i,j = 0,1,\ldots,N-1$. In (189) we consider the function $K(\cdot\,,\cdot)$, the weights w_j and the points x_j, $j = 0,1,\ldots,N-1$ as given. The number c is a parameter. For the determinant we have the development in powers of ν

$$\mathcal{D}(\nu) = 1 + \sum_{n=1}^{N} (-1)^n d_n \nu^n,$$

where

$$d_n = \frac{1}{n!}\sum_{\ell_0=0}^{N-1}\cdots\sum_{\ell_{n-1}=0}^{N-1} |K(cx_{\ell_i},cx_{\ell_j})w_{\ell_j}|_{n-1}.$$

If now

$$K(x,y) = \sum_0^\infty a_{ij}x^iy^j, \quad a_{00} = 1, \tag{190}$$

the development in the appendix of Ref. 21 from equation (A4) to (A9) can be repeated step by step with all integrals replaced by sums to show that for small c we have for the eigenvalues of (189)

$$\mu_n = c^{2n}\chi_0(n)[1 + 0(c)] \tag{191}$$

$$n = 0,1,\ldots,N-1$$

where

$$\chi_0(n) = \frac{|a_{ij}|_n|h_{i+j}|_n}{|a_{ij}|_{n-1}|h_{i+j}|_{n-1}}. \tag{192}$$

Here

$$h_\gamma = \sum_{i=0}^{N-1} x_i^\gamma w_i. \tag{193}$$

To use this result to obtain asymptotics of $\lambda_k(N,W)$ for small W, divide (18) by $2W$, and write

$$c \equiv 2\pi W,\ \mu = \frac{\lambda}{2W},\ x_j = j,\ j = 0,1,\ldots,N-1. \tag{194}$$

Equation (18) then becomes (189) with $w_j = 1$ and

$$K(x,y) = \frac{\sin(x-y)}{x-y} = \sum_0^\infty \frac{(-1)^n(x-y)^{2n}}{(2n+1)!}$$

$$= \sum_{n,j} \frac{(-1)^n \binom{2n}{j} x^j(-y)^{2n-j}}{(2n+1)!}. \tag{195}$$

For evaluation of (192) we thus have

$$a_{ij} = \begin{cases} 0, & i+j \text{ odd} \\ \dfrac{(-1)^{j+(i+j)/2}}{i!j!(i+j+1)}, & i+j \text{ even} \end{cases} \tag{196}$$

and

$$h_\gamma = \sum_{\ell=0}^{N-1} \ell^\gamma. \tag{197}$$

To evaluate (192) we first note that the equations

$$\sum_{j=0}^{n} a_{ij} Y_j = \delta_{in}, \quad i = 0,1,\ldots,n \tag{198}$$

yield

$$Y_n = \frac{|a_{ij}|_{n-1}}{|a_{ij}|_n}, \tag{199}$$

the reciprocal of one factor of (192). Now, from (196) we see that we can also write

$$a_{ij} = \frac{(-1)^{j+(i+j)/2}}{i!j!} \frac{1}{2} \int_{-1}^{1} t^{i+j} dt. \tag{200}$$

Insert this expression for a_{ij} into (198) and define

$$F(t) \equiv \sum_{j=0}^{n} (-1)^{3j/2} \frac{t^j}{j!} Y_j. \tag{201}$$

Equation (198) then reads

$$\int_{-1}^{1} F(t) t^i dt = \frac{2n!}{(-1)^{n/2}} \delta_{in} \tag{202}$$

$$i = 0,1,\ldots,n.$$

But $F(t)$ is a polynomial of degree n orthogonal on $(-1,1)$ to t^i, $i = 0,\ldots,n-1$. We can write therefore $F(t) = kP_n(t)$ with $P_n(t)$ the usual Legendre polynomial. Now

$$\int_{-1}^{1} P_n(t)t^n dt = \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

(see Ref. 10, p. 786, 22.13.8-.9) so the last of equations (202) shows that $k = (2n+1)!/2^n n!(-1)^{n/2}$. We thus have

$$F(t) = kP_n(t) = \frac{(2n+1)!}{2^n n!(-1)^{n/2}} \cdot \frac{(2n)!}{2^n (n!)^2}\left[t^n - \frac{n(n-1)}{2(2n-1)} t^{n-2} + \ldots\right]$$

$$= \frac{(-1)^{3n/2}}{n!} Y_n [t^n + \ldots]$$

on using an explicit form for $P_n(t)$ (see Ref. 10, p. 775, 22.3.8) and on recalling the definition (201). Comparing coefficients of t^n we have now established that

$$\frac{1}{Y_n} = \frac{|a_{ij}|_n}{|a_{ij}|_{n-1}} = \frac{2^{2n}(n!)^2}{(2n+1)!(2n)!}. \tag{203}$$

It is not difficult to obtain this result by direct evaluation of $|a_{ij}|_n$ which is a product of Cauchy determinants.

We use a similar technique to evaluate the second factor in (192). The equations

$$\sum_{j=0}^{n} h_{i+j} Z_j = \delta_{in}, \quad i = 0,1,\ldots,n \tag{204}$$

yield

$$Z_n = \frac{|h_{i+j}|_{n-1}}{|h_{i+j}|_n}. \tag{205}$$

Using the definition (197) of h_{i+j}, (205) becomes

$$\sum_{x=0}^{N-1} x^i G(x) = \delta_{in}, \quad i = 0,1,\ldots,n \tag{206}$$

where we have written

$$G(x) = \sum_{j=0}^{n} Z_j x^j. \tag{207}$$

Thus we seek an nth degree polynomial $G(x)$ satisfying (206). The coefficient of x^n will give the desired ratio (205).

The Tchebyshev polynomial $t_n(x)$ (see Ref. 9, vol. 2, pp. 221–223) has just the properties sought. It satisfies

$$\sum_{x=0}^{N-1} x^i t_n(x) = 0, \quad i = 0,1,\ldots,n-1. \tag{208}$$

An explicit formula for the polynomial is

$$t_0(x) \equiv 1$$
$$t_n(x) \equiv n!\Delta^n[(x)_n(x-N)_n], \quad n = 1, \ldots, N-1 \tag{209}$$

where we write $(x)_n \equiv x(x-1)(x-2)\ldots(x-n+1)$ and define the forward difference operator Δ by $\Delta f(x) = f(x+1) - f(x)$ and $\Delta^n f(x) = \Delta[\Delta^{n-1} f(x)]$. The polynomials satisfy the recurrence

$$(n+1)t_{n+1}(x) - (2n+1)(2x-N+1)t_n(x) + n(N^2-n^2)t_{n-1}(x) = 0 \tag{210}$$

$$n = 1,2,\ldots, N-2$$

(Ref. 9, vol. 2, p. 223, (6)).

From (210) by using (208) we can easily calculate $\Sigma_{x=0}^{N-1} x^n t_n(x) \equiv S_n$. To do so, multiply (210) by x^{n-1} and sum. Recalling (208), one finds $2(2n+1)S_n = n(N^2-n^2)S_{n-1}$. Since $S_0 = N$, we have

$$S_n = \frac{Nn! \prod_{k=1}^{n} (N^2-n^2)}{2^n 1 \cdot 3 \cdot 5 \ldots \cdot (2n+1)}. \tag{211}$$

It follows then that

$$G(x) = \frac{1}{S_n} t_n(x) \tag{212}$$

is the nth degree polynomial satisfying (206).

We now seek the coefficient of x^n in $G(x)$. The coefficient of x^n in $t_n(x)$ is not evident from the definition (209). However, it is easy to show that $\Delta(x)_n = n(x)_{n-1}$ and that

$$\Delta^n[f(x)g(x)] = \sum_{j=0}^{n} \binom{n}{j} \Delta^j f(x) \Delta^{n-j} g(x+j).$$

Applying these rules to (209) we obtain the alternative expression

$$t_n(x) = \sum_{j=0}^{n} \binom{n}{j}^2 (x)_{n-j}(x-N+j)_j. \tag{213}$$

It follows then that the coefficient of x^n in $t_n(x)$ is

$$k \equiv \sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n} \tag{214}$$

(see Ref. 13, p. 4, 0.157-1). From (207), (212) and (213) it follows that Z_n

$= k/S_n$. From (205), (211) and (214), then one has

$$\frac{|h_{i+j}|_n}{|h_{i+j}|_{n-1}} = \frac{1}{Z_n} = \frac{\prod_{j=-n}^{n} (N-j)}{(2n+1)\binom{2n}{n}^2}. \tag{215}$$

This result combined with (203), (191), (192) and (194) yields (64)–(66).

Since $|h_{i+j}|_0 = N$, (215) yields

$$|h_{i+j}|_n = N^{n+1} \prod_{j=1}^{n} \frac{(j!)^4(N^2-j^2)^{n+1-j}}{(2j+1)[(2j)!]^2}, \tag{216}$$

a formula that seems difficult to arrive at by direct manipulation of the determinant.

APPENDIX A

Asymptotic Behavior of η_0

We here investigate the product (93) for large N. We adopt the abbreviation $W' = W_0 T_0$.

Suppose first that $0 < W' < 1/2$. We write (93) in the form

$$\frac{1}{N} \log \eta_0 = P_1 + P_2 + P_3 \tag{217}$$

where

$$\begin{aligned} P_1 &= \frac{1}{N} \log \left(\frac{\sigma^2}{2W'} \lambda_0(N+1,W') \right) \\ P_2 &= \frac{1}{N} \sum_{k=0}^{2W'N-2} \log \frac{\lambda_{k+1}(N+1,W')}{\lambda_k(N,W')} \\ P_3 &= \frac{1}{N} \sum_{k=2W'N-1}^{N-1} \log \frac{\lambda_{k+1}(N+1,W')}{\lambda_k(N,W')}. \end{aligned} \tag{218}$$

In this last sum set $k = N - 1 - \ell$ and use (13) to obtain

$$\begin{aligned} P_3 &= \frac{1}{N} \sum_{\ell=0}^{N(1-2W')} \log \frac{\lambda_{N-\ell}(N+1,W')}{\lambda_{N-1-\ell}(N,W')} \\ &= \frac{1}{N} \sum_{\ell=0}^{2N\overline{W}} \log \frac{1-\lambda_\ell(N+1,\overline{W})}{1-\lambda_\ell(N,\overline{W})} \end{aligned} \tag{219}$$

where we have written $\overline{W} = 1/2 - W'$. Now from (59), if $\ell = sN$, with s fixed and $0 < s < 2\overline{W}$, $1 - \lambda_\ell(N,\overline{W}) \sim \exp[-1/2 C(B,N)L_4(B) - NL_3(B)]$ where C, L_4 and L_3 are given by (45) and (47) and B is determined as a function of s by (43), namely

$$\frac{1}{\pi}\int_{B(s)}^{1}\sqrt{\frac{\xi - B(s)}{(\xi - A)(1 - \xi^2)}}\,d\xi = s. \tag{220}$$

In these formulas we now have $A = \cos 2\pi\overline{W}$.

For large N, then, and fixed s, a term in (219) takes the value

$$J \equiv \log\frac{1 - \lambda_\ell(N + 1, \overline{W})}{1 - \lambda_\ell(N, \overline{W})} \sim -\frac{1}{2}C(\hat{B}, N + 1)L_4(\hat{B}) - (N + 1)L_3(\hat{B}) - \left[-\frac{1}{2}C(B,N)L_4(B) - NL_3(B)\right] \tag{221}$$

where

$$\frac{1}{\pi}\int_{\hat{B}}^{1}\sqrt{\frac{\xi - \hat{B}}{(\xi - A)(1 - \xi^2)}}\,d\xi = \frac{k}{N + 1} = \frac{k}{N} - \frac{k}{N(N + 1)} = s - \frac{s}{N + 1}\cdot \tag{222}$$

Thus

$$\hat{B} = B\left(s - \frac{s}{N + 1}\right) = B(s) - sB'(s)\frac{1}{N} + 0\left(\frac{1}{N^2}\right)\cdot$$

Straightforward Taylor expansions of quantities on the right of (221) now give

$$J = -\left[L_3(B(s)) - s\frac{d}{ds}L_3(B(s))\right] + o(1)$$

Returning to (219), we have

$$P_3 \sim -\frac{1}{N}\sum_{\ell=0}^{2N\overline{W}}\left[L_3\left(B\left(\frac{\ell}{N}\right)\right) - \frac{\ell}{N}\frac{d}{ds}L_3(B(s))|_{s=\ell/N}\right] + o(1)$$

so that

$$\begin{aligned}\lim_{N\to\infty} P_3 &= -\int_0^{2\overline{W}}\left[L_3(B(s)) - s\frac{d}{ds}L_3(B(s))\right]ds \\ &= -\int_0^{2\overline{W}} L_3 ds + sL_3\Big|_{s=0}^{2\overline{W}} - \int_0^{2\overline{W}} L_3 ds \\ &= -2\int_0^{2\overline{W}} L_3(B(s))ds.\end{aligned} \tag{223}$$

Now recalling (220) and the definition (47) of L_3, we have

$$\lim_{N\to\infty} P_3 = -2 \int_0^{2\overline{W}} ds \int_A^{B(s)} \sqrt{\frac{B-\xi}{(\xi - A)(1-\xi^2)}}\, d\xi$$
$$= -\frac{1}{\pi} \int_A^1 dB \int_A^B d\xi \sqrt{\frac{B-\xi}{(\xi - A)(1-\xi^2)}}$$
$$\int_B^1 \frac{dt}{\sqrt{(t-B)(t-A)(1-t^2)}} = 2 \log (\sin \pi W_0 T_0). \quad (224)$$

Details of the evaluation of the triple integral to obtain the last line here are given in Appendix B.

It is not difficult to see that both P_1 and P_2 approach zero as N increases. We omit the demonstration here. Our result thus far reads

$$\lim_{N\to\infty} \frac{\log \eta_0}{N} = 2 \log (\sin \pi W_0 T_0), \quad 0 < W_0 T_0 < 1/2$$

which is (94).

To study η_0 when $W' = W_0 T_0 > 1/2$, we return to (18) and consider it now for arbitrary values of W. Since

$$\frac{\sin 2\pi \left(W + \frac{1}{2}\right)(n-m)}{\pi(n-m)} = \left[\frac{\sin 2\pi W(n-m)}{\pi(n-m)} + \delta_{nm}\right](-1)^{n-m}, \quad (225)$$

the eigenvalue equation

$$\sum_{m=0}^{N-1} \frac{\sin 2\pi \left(W + \frac{1}{2}\right)(n-m)}{\pi(n-m)} v_m^{(k)}\left(N, W + \frac{1}{2}\right) = \lambda_k \left(N, W + \frac{1}{2}\right) v_n^{(k)}\left(N, W + \frac{1}{2}\right)$$

can be rewritten on direct substitution of (225) as

$$\sum_{m=0}^{N-1} \frac{\sin 2\pi W(n-m)}{\pi(n-m)} \left[(-1)^m v_m^{(k)}\left(N, W + \frac{1}{2}\right)\right] = \left[-1 + \lambda_k\left(N, W + \frac{1}{2}\right)\right]\left[(-1)^n v_n^{(k)}\left(N, W + \frac{1}{2}\right)\right].$$

Comparison with (18) now shows that

$$\lambda_k\left(N, W + \frac{1}{2}\right) = 1 + \lambda_k(N, W)$$
$$v_m^{(k)}\left(N, W + \frac{1}{2}\right) = C(-1)^m v_m^{(k)}(N, W). \quad (226)$$

Suppose now that

$$\frac{n}{2} < W' < \frac{n+1}{2} \tag{227}$$

where n is a positive integer. Then in virtue of (226) Eq. (93) becomes

$$\eta_0 = \frac{\sigma^2}{2W'} [n + \lambda_0(N+1, W'')] \prod_0^{N-1} \frac{n + \lambda_{k+1}(N+1, W'')}{n + \lambda_k(N, W'')} \tag{228}$$

$$W'' \equiv W' - \frac{n}{2}$$

$$0 < W'' < \frac{1}{2} \cdot \tag{229}$$

Then

$$\log \eta_0 = Q_1 + Q_2 \tag{230}$$

where

$$Q_1 \equiv \log \frac{\sigma^2[n + \lambda_0(N+1, W'')]}{2W'} \sim \log \frac{\sigma^2(n+1)}{2W'} \tag{231}$$

and

$$Q_2 \equiv \sum_0^{N-1} \log \frac{n + \lambda_{k+1}(N+1, W'')}{n + \lambda_k(N, W'')} \cdot \tag{232}$$

When N is large only the terms for k near $2W''N$ contribute significantly to Q_2, for if $k = 2W''N(1-\epsilon)$, λ_k approaches 1 exponentially, while if $k = 2W''N(1+\epsilon)$, λ_k approaches zero exponentially. In either event, the summand in (232) approaches zero exponentially while the number of term grows linearly with N.

Consider now

$$H(\overline{\alpha}, N) = \sum_{k=2WN-(\overline{\alpha}/\pi)\log N}^{2WN+(\overline{\alpha}/\pi)\log N} \log \frac{n + \lambda_{k+1}(N+1, W'')}{n + \lambda_k(N, W'')} \tag{233}$$

where $\overline{\alpha}$ is an arbitrary positive real number. We change from the variables N, k to new variables Δ and b via the transformation

$$\Delta = \frac{\pi}{\log N} \qquad N = e^{\pi/\Delta}$$

$$b = (k - 2W''N)\Delta \qquad k = \frac{b}{\Delta} + 2W''e^{\pi/\Delta} \tag{234}$$

and write

$$\lambda_k(N, W'') \equiv g(\Delta, b). \tag{235}$$

Then $\lambda_{k+1}(N+1,W'') = g(\Delta',b')$ where

$$\Delta' = \frac{\pi}{\log (N+1)} = \Delta + 0\left(\frac{\Delta^2}{N}\right)$$

and

$$b' = [k+1-2W''(N+1)]\Delta' = b + (1-2W'')\Delta + 0\left(\frac{\Delta}{N}\right) \cdot$$

Thus

$$g(\Delta',b') = g(\Delta,b) + (1-2W'')\Delta \frac{\partial g(\Delta,b)}{\partial b} + 0\left(\frac{\Delta}{N}\right)$$

and the summand of (233) becomes

$$\log \frac{n+\lambda_{k+1}(N+1,W'')}{n+\lambda_k(N,W'')} = \log \frac{n+g(\Delta,b)+(1-2W'')\Delta\dfrac{\partial g}{\partial b} + 0\left(\dfrac{\Delta}{N}\right)}{n+g(\Delta,b)}$$

$$= (1-2W'')\Delta L(\Delta,b) + 0\left(\frac{\Delta}{N}\right)$$

where

$$L(\Delta,b) \equiv \frac{d}{db} \log\,[n+g(\Delta,b)]. \tag{236}$$

Now write $j = k - 2W''N$. Equation (233) becomes

$$H(\overline{\alpha},N) = (1-2W'') \sum_{j=-\overline{\alpha}/\Delta}^{\overline{\alpha}/\Delta} \left[L(\Delta,j\Delta)\Delta + 0\left(\frac{\Delta}{N}\right)\right] \cdot$$

In the limit of large N, we have

$$H(\overline{\alpha},N) \sim (1-2W'') \int_{-\overline{\alpha}}^{\overline{\alpha}} L(0,x)dx$$

$$= (1-2W) \log\,[n+g(0,x)]\Big|_{x=-\overline{\alpha}}^{\overline{\alpha}} \tag{237}$$

from (236). But (60) and (235) show that

$$g(0,b) = \frac{1}{1+e^{\pi b}}$$

and so

$$H(\overline{\alpha},N) \to (1-2W'') \log \frac{n+[1+e^{\pi\overline{\alpha}}]^{-1}}{n+[1+e^{-\pi\overline{\alpha}}]^{-1}}.$$

Finally, since $\overline{\alpha}$ can be chosen arbitrarily large,

$$H(\overline{\alpha},N) \to (1-2W'') \log \frac{n}{n+1} = -\log\left(1+\frac{1}{n}\right)^{(1-2W'')}. \tag{238}$$

Thus if

$$\lim_{N\to\infty} Q_2 = \lim_{N\to\infty} H(\overline{\alpha},N),$$

we can write

$$\eta_0 \sim \eta_\infty = \frac{\sigma^2(n+1)}{2W'} \frac{1}{\left(1+\frac{1}{n}\right)^{1-2W''}} \tag{239}$$

in virtue of (229)–(231). This result is (95).

APPENDIX B

Evaluation of Integral (224)

$$\begin{aligned} J \equiv \lim_{N\to\infty} P_3 &= -\frac{1}{\pi}\int_A^1 dB \int_A^B d\xi \sqrt{\frac{B-\xi}{(\xi-A)(1-\xi^2)}} \\ &\times \int_B^1 \frac{dt}{\sqrt{(t-B)(t-A)(1-t^2)}} = -\frac{1}{\pi}\int_A^1 \frac{dt}{\sqrt{(t-A)(1-t^2)}} \\ &\times \int_A^t \frac{d\xi}{\sqrt{(\xi-A)(1-\xi^2)}} \int_\xi^t dB \sqrt{\frac{B-\xi}{t-B}} \\ &= -\frac{1}{2}\int_A^1 dt \int_A^t d\xi \frac{t-\xi}{\sqrt{(1-\xi^2)(1-t^2)(A-t)(A-\xi)}} \end{aligned} \tag{240}$$

since under the substitution $u^2 = (B-\xi)/(t-B)$

$$\begin{aligned} \int_\xi^t dB \sqrt{\frac{B-\xi}{t-B}} &= (t-\xi)\int_0^\infty u \frac{2u}{(1+u^2)^2} du \\ &= (t-\xi)\left[-u\frac{1}{1+u^2}\Big|_0^\infty + \int_0^\infty \frac{du}{1+u^2}\right] = (t-\xi)\frac{\pi}{2}. \end{aligned}$$

Now change from the variables of integration ξ,t to α,β via

$$\xi = \frac{1+A}{2} + \frac{1-A}{2}\sin\alpha$$

$$t = \frac{1+A}{2} + \frac{1-A}{2}\sin\beta$$

and obtain

$$J = -\frac{1}{2}\int_{-\pi/2}^{\pi/2} d\beta \int_{-\pi/2}^{\beta} d\alpha \frac{\sin\beta - \sin\alpha}{\sqrt{(x+\sin\alpha)(x+\sin\beta)}}$$

where we write

$$x = \frac{3+A}{1-A}, \; 1 < x < \infty.$$

Changing now to variables ϕ,ψ through the 90° rotation $\frac{1}{2}(\alpha + \beta) = \phi$, $\frac{1}{2}(\alpha - \beta) = \psi$, we find that

$$J = 2 \int_{-\pi/2}^{0} d\psi \int_{-\psi-\pi/2}^{\psi+\pi/2} d\phi \frac{\cos\phi \sin\psi}{\sqrt{x^2 - \sin^2\psi + 2x\cos\psi\sin\phi + \sin^2\phi}}$$

$$= 2 \int_{-\pi/2}^{0} d\psi \sin\psi \,[\log |x \cos\psi$$

$$+ \sin\phi \sqrt{x^2 - \sin^2\psi + 2x\cos\psi\sin\phi + \sin^2\phi}|]_{\phi=-(\psi+\pi/2)}^{\psi+\pi/2}$$

$$= 2 \int_{-\pi/2}^{0} d\psi \sin\psi$$

$$\times \log \left| \frac{x\cos\psi + \cos\psi + \sqrt{x^2 - \sin^2\psi + 2x\cos^2\psi + \cos^2\psi}}{x\cos\psi - \cos\psi + \sqrt{x^2 - \sin^2\psi - 2x\cos^2\psi + \cos^2\psi}} \right|$$

$$= -2 \int_0^1 du \log \frac{a[au + \sqrt{b^2 + u^2}]}{b[bu + \sqrt{a^2 - u^2}]}$$

where we have set $\cos\psi = u$ and

$$a = \sqrt{\frac{x+1}{2}}, \quad b = \sqrt{\frac{x-1}{2}}, \quad a^2 - b^2 = 1. \tag{241}$$

Thus

$$J = -2 \log \frac{a}{b} - 2 \int_0^1 du \log [au + \sqrt{b^2 + u^2}]$$

$$+ 2 \int_0^1 du \log [bu + \sqrt{a^2 - u^2}]. \tag{242}$$

Now it can be verified by direct differentiation that when (241) holds

$$\int du \log [au + \sqrt{b^2 + u^2}]$$
$$= (u + 1) \log [au + \sqrt{b^2 + u^2}] - u - \log [u + b^2 + a\sqrt{b^2 + u^2}]$$

and

$$\int du \log [bu + \sqrt{a^2 - u^2}]$$
$$= (u + 1) \log [bu + \sqrt{a^2 - u^2}] - u - \log [a^2 - u + b\sqrt{a^2 - u^2}].$$

It then follows readily that the last two terms of (242) both have magnitude $\log 2(a + b) - 1$, and that equation becomes simply

$$J = -2 \log \frac{a}{b} = -2 \log \sqrt{\frac{x+1}{x-1}} = 2 \log \sqrt{\frac{1+A}{2}}. \tag{243}$$

When $A = \cos 2\pi \overline{W} = \cos 2\pi(\frac{1}{2} - W_0T) = -\cos 2\pi W_0 T$, (243) yields (224).

The simple result (243) for the integral given by the last member of

(240) was first obtained by J. A. Morrison by a route involving elliptic functions and identities among them.

APPENDIX C

Commutation of L and M

Let operators

$$L \equiv \int_a^b df' K(f,f')$$

and

$$M = \frac{d}{df} p(f) \frac{d}{df} + q(f)$$

be given. Then

$$MLg = \int_a^b df' M_f K(f,f') g(f')$$

while

$$LMg = \int_a^b df' K(f,f') M_{f'} g(f')$$

$$= \left[p(f') \left\{ K \frac{dg(f')}{df'} - \frac{\partial K(f,f')}{\partial f'} g(f') \right\} \right]_{f'=a}^{b} + \int_a^b df' g(f') M_{f'} K(f,f').$$

Now if

$$p(a) = p(b) = 0, \tag{244}$$

we have

$$MLg - LMg = \int_a^b df' g(f') [M_f - M_{f'}] K(f,f')$$

and so, if in addition

$$M_f K(f,f') \equiv M_{f'} K(f,f'), \tag{245}$$

the operators commute. In the special case when $K(f,f') = K(|f - f'|)$, (245) becomes

$$[p(f) - p(f')] \frac{\partial^2 K(|f - f'|)}{\partial f^2} + [p'(f) + p'(f')] \frac{\partial K(|f - f'|)}{\partial f} + [q(f) - q(f')] K(|f - f'|) \equiv 0. \tag{246}$$

Applying this to the operators of (110) and (111) we have $p(f) = (1/4\pi^2)[\cos 2\pi f - A]$, $q(f) = 1/4(N^2 - 1) \cos 2\pi f$ and $a = -b = W$. It is seen that (244) is satisfied. To verify (246) observe that

$$p(f) - p(f') = -\frac{1}{2\pi^2} \sin \pi(f + f') \sin \pi(f - f')$$

$$p'(f) + p'(f') = -\frac{1}{\pi} \sin \pi(f + f') \cos \pi(f - f')$$

$$q(f) - q(f') = \frac{1}{2}(N^2 - 1) \sin \pi(f + f') \sin \pi(f - f').$$

Thus every term of (246) in the present case contains a factor $\sin \pi(f + f')$. This equation then is equivalent to

$$\sin t \frac{d^2}{dt^2} \frac{\sin Nt}{\sin t} + 2 \cos t \frac{d}{dt} \frac{\sin Nt}{\sin t} + (N^2 - 1) \sin t \frac{\sin Nt}{\sin t} = 0 \quad (247)$$

where we have written $t \equiv \pi(f - f')$. The reader can readily verify that (247) holds identically in t.

REFERENCES

1. D. Slepian and H. O. Pollak, "Prolate Spheroidal Wave Functions, Fourier Analysis, and Uncertainty—I," B.S.T.J., *40,* No. 1 (January 1961), pp. 43–64.
2. H. J. Landau and H. O. Pollak, "Prolate Spheroidal Wave Functions, Fourier Analysis, and Uncertainty—II," B.S.T.J., *40,* No. 1 (January 1961), pp. 65–84.
3. H. J. Landau and H. O. Pollak, "Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty—III," B.S.T.J., *41,* No. 4 (July 1962), pp. 1295–1336.
4. D. Slepian, "Prolate Spheroidal Wave Functions, Fourier Analysis, and Uncertainty—IV," B.S.T.J., *43,* No. 6 (November 1964), pp. 3009–3058.
5. D. Slepian, "Some Asymptotic Expansions for Prolate Spheroidal Wave Functions," J. Math. and Phys., *44,* No. 2 (June 1965), pp. 99–140.
6. D. Slepian and E. Sonnenblick, "Eigenvalues Associated with Prolate Spheroidal Wave Functions of Zero Order," B.S.T.J., 44, No. 8 (October 1965), pp. 1745–1760.
7. D. Slepian, "On Bandwidth," Proc. IEEE, 64, No. 3 (March 1976), pp. 292–300.
8. E. N. Gilbert and D. Slepian, "Doubly Orthogonal Concentrated Polynomials," SIAM J. Appl. Math., *8,* No. 2 (April 1977), pp. 290–319.
9. A. Erdélyi, *Higher Transcendental Functions,* New York: McGraw-Hill, 1953.
10. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions,* New York: Dover Publications, 1965.
11. Harald Cramér, *Mathematical Methods of Statistics,* Princeton, New Jersey: Princeton University Press, 1946.
12. A. Papoulis and M. S. Bertran, "Digital Filtering and Prolate Functions," IEEE Trans. Circuit Theory, *CT-19,* No. 6 (November 1972), pp. 674–681.
13. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products,* New York: Academic Press, 1965.
14. P. M. Morse and H. Feshback, "Methods of Theoretical Physics," New York: McGraw-Hill, 1953.
15. J. Von Neumann, *Mathematical Foundations of Quantum Mechanics,* Princeton, New Jersey: Princeton University Press, 1955, pp. 170–178.
16. D. W. Tufts and J. T. Francis, "Designing Digital Low-Pass Filters—Comparison of Some Methods and Criteria," IEEE Trans. Audio and Electroacoust., *AU-18,* No. 4 (Dec., 1970), pp. 487–494.
17. A. Eberhard, "An Optimum Discrete Window for the Calculation of Power Spectra," IEEE Trans. Audio and Electroacoust., *AU-21* (February 1973), pp. 37–43.

18. D. W. Tufts, "Comments on 'FIR Digital Filter Design Techniques Using Weighted Chebyshev Approximation'," Proc. IEEE, *63* (November 1975), p. 1618.
19. A. N. Balakrishnan, "Essentially Band-Limited Stochastic Processes," IEEE Trans. Infor. Theory, *IT-11* No. 1 (January 1965), pp. 154–156.
20. A. Viterbi, "On the Minimum Mean Square Error Resulting from Linear Filtering of Stationary Signals in White Noise," IEEE Trans. Inform. Theory, *IT-11,* No. 4 (October 1965), pp. 594–595.
21. D. Slepian, "A Numerical Method for Determining the Eigenvalues and Eigenfunctions of Analytic Kernels," SIAM J. Numerical Analysis, *5*, No. 3 (September 1968), pp. 586–600.
22. H. Widom, "The Strong Szegö Limit Theorem for Circular Arcs," Indiana Univ. Math. J., *21,* No. 3 (1971), pp. 277–283.
23. F. Gori and G. Guattari, "Degrees of Freedom of Images from Point-Like-Element Pupils," J. Optical Soc. Am., *64,* No. 4 (April 1974), pp. 453–458.