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Letter to the Editor: On the Numerical Evaluation of Bandpass Prolates

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Abstract This letter provides a technique for numerical evaluation of certain eigenfunctions of the integral kernel operator corresponding to time truncation of a square-integrable function to a finite interval, followed by frequency limiting to frequencies in an annular band.

Keywords Prolate spheroidal wave function · Bandpass prolate · Time and band limiting · Legendre polynomials

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This note provides a technique for numerical evaluation of certain *bandpass prolate* functions. These functions are eigenfunctions of the integral kernel operator corresponding to time truncation of a function in $L^2(\mathbb{R})$ to a finite interval—$[-1, 1]$ in this work—followed by frequency limiting to frequencies $|\omega| \in [c', c]$. The time- and bandpass-limiting operator was mentioned in the groundbreaking work of Slepian and Pollack [12] and studied more recently for example by Sengupta et al. [11] and by Khare [8]. As in the work of Sengupta et al., our technique utilizes the expansion of such eigenfunctions in *standard prolates* for the full band. The main contribution

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here is an expression for the coefficients of the bandpass functions in terms of certain *partial inner products* of the full-band prolates.

Prolate spheroidal wave functions, or prolates for short, on the interval $[-1,1]$ are eigenfunctions of the prolate differential operator,

$$\mathcal{P}_c\varphi_n(t)=\chi_n\varphi_n(t);\quad \mathcal{P}_c=\frac{\mathrm{d}}{\mathrm{d}t}\big(t^2-1\big)\frac{\mathrm{d}}{\mathrm{d}t}+c^2t^2. \tag{1}$$

Such functions extend to elements of $L^2(\mathbb{R})$. Prolates are also eigenfunctions of the integral operator

$$(F_c f)(t)=\int_{-1}^{1}\mathrm{e}^{\mathrm{i}cst}f(s)\,\mathrm{d}s=\widehat{Qf}(-ct) \tag{2}$$

in which Q is the operation of multiplication by the characteristic function of $[-1,1]$ and $\hat{f}$ denotes the Fourier transform normalized as integration against the kernel $\mathrm{e}^{-\mathrm{i}st}$.

Denote by P_c the operator that sends $f\in L^2(\mathbb{R})$ to $(\hat{f}\mathbb{1}_{[-c,c]})^\vee$ where $\psi^\vee$ denotes the inverse Fourier transform of ψ. Here $\mathbb{1}_{[-c,c]}$ is the characteristic function of the interval $[-c,c]$. Since the differential operator $\mathcal{P}_c$ commutes with the compact integral operator P_cQ, these operators share their eigenfunctions. The eigenvalues of P_cQ are non-degenerate. Let $\lambda_0(c)>\lambda_1(c)>\cdots$ be the eigenvalues of P_cQ arranged in decreasing order and φ_n^c be the eigenfunction corresponding to $\lambda_n(c)$. The fact that φ_n^c is an eigenfunction of the operator F_c of (2) and some other basic properties imply that

$$D_c\widehat{\varphi_n^c}=\frac{\mathrm{i}^n}{\sqrt{\lambda_n}}Q\varphi_n^c \tag{3}$$

where D_c is the unitary dilation $(D_cf)(t)=\sqrt{c}f(ct)$, $c>0$. When $L^2(\mathbb{R})$-normalized, the prolates $\{\varphi_n^c\}$ form an orthonormal basis for PW_c, the closed subspace of $L^2(\mathbb{R})$ of functions bandlimited to $[-c,c]$, as well as a complete, orthogonal set in $L^2[-1,1]$ with $\lambda_n(c)=\int_{-1}^1|\varphi_n^c|^2$. As such, any $f\in\mathrm{PW}_c$ can be expanded in the form $f=\sum_{n=0}^\infty\alpha_n\varphi_n^c$ with $\|f\|^2_{L^2(\mathbb{R})}=\sum\alpha_n^2$ and $\int_{-1}^1|f|^2=\sum\lambda_n\alpha_n^2$. The prolates are real-valued and φ_n^c is even (odd) if n is even (odd).

Given $0<c'<c$ denote by $\mathrm{PW}_{c',c}$ the orthogonal complement of $\mathrm{PW}_{c'}$ inside PW_c, that is, the closed subspace of $L^2(\mathbb{R})$ of functions whose Fourier transforms $\widehat{f}(\xi)$ are supported in $c'\le|\xi|\le c$, and by $P_{c'c}$ the orthogonal projection onto $\mathrm{PW}_{c'c}$. Denote by $R=R(c',c)$ the matrix with entries $R_{jk}=\frac{\mathrm{i}^{k-j}}{\sqrt{\lambda_j\lambda_k}}\int_{-c'/c}^{c'/c}\varphi_k^c(\xi)\varphi_j^c(\xi)\,\mathrm{d}\xi$. The matrix R is real symmetric, a consequence of the parity properties of the φ_n^c. Also let $\Lambda=\Lambda(c)$ be the diagonal matrix with nth diagonal entry $\lambda_n(c)$.

Proposition 1 *If* $\psi=\sum\alpha_n\varphi_n^c\in\mathrm{PW}_c$ *then*

$$P_{c'c}Q\psi=\sum_k\alpha_k\lambda_k\Big(\varphi_k^c-\sum_j R_{jk}\varphi_j^c\Big).$$

In particular, if $\psi = \sum \alpha_n \varphi_n^c$ *is an eigenfunction of* $P_{c'c}Q$ *with eigenvalue* λ *then*

$$\lambda\alpha_n = \lambda_n\alpha_n - \sum_k \lambda_k\alpha_k R_{nk} \quad i.e. \quad \lambda\boldsymbol{\alpha} = (I-R)\Lambda\boldsymbol{\alpha} \quad \big(\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{\infty}\big).$$

The discrete eigenvectors $\boldsymbol{\alpha}$ of the matrix $(I-R)\Lambda$ give rise to eigenfunctions of $P_{c'c}Q$ and the eigenvalue λ measures the concentration of ψ in $[-1,1]$ just as in the case of standard prolates. The proof will be given in Appendix B.

1 Finite Dimensional Approximations of Bandpass Prolates

Our numerical approximations require two main ingredients. The first is a finite dimensional approximation of the matrix R that gives rise to the eigenfunctions of $P_{c'c}Q$. The latter can be expressed as the integral operator

$$\begin{aligned}(P_{c'c}Q)(f)(t) &= \int_{-1}^{1} \frac{\sin c(t-s) - \sin c'(t-s)}{\pi(t-s)} f(s)\,\mathrm{d}s \\ &= \frac{2}{\pi}\int_{-1}^{1} \frac{\sin a(t-s)\cos b(t-s)}{t-s} f(s)\,\mathrm{d}s\end{aligned}$$

where $a = (c-c')/2$, $b = (c+c')/2$ and with P_cQ corresponding to the limiting case $c' = 0$. The other ingredient is a method to approximate the elements of such a finite dimensional matrix. One can estimate the partial inner product entries of a finite rank approximation of R if one has in turn estimates of the corresponding eigenvalues χ_n of $\mathcal{P}_c$ and the values of the prolates φ_n^c. Specifically, let $\chi_n = \chi_n(c)$ be such that $\mathcal{P}_c\varphi_n^c = \chi_n(c)\varphi_n^c$. The following identity is known as Byerly's identity [4, Chap. 5] in the limiting case $c = 0$ (in which case the prolates collapse to the normalized Legendre polynomials: $\phi_n^0 = \sqrt{n+1/2}P_n$). The proof for the prolate case is similar and is given in Appendix A.

Proposition 2 *If* $n \neq m$ *then, with* χ_n *as in* (1) *and* $-1 \le a \le b \le 1$,

$$(\chi_n - \chi_m)\int_a^b \varphi_n(t)\varphi_m(t)\,\mathrm{d}t = \big(t^2-1\big)\left(\frac{\mathrm{d}\varphi_n}{\mathrm{d}t}\varphi_m(t) - \frac{\mathrm{d}\varphi_m}{\mathrm{d}t}\varphi_n(t)\right)\bigg|_a^b.$$

The partial norm squares $\int_a^b \varphi_n^2(t)\,\mathrm{d}t$ can be computed via Legendre polynomial expansions as outlined below.

In order to produce a finite dimensional approximation of the matrix $(I-R)\Lambda$, first one notes that $I-R$ is a bounded operator since R corresponds, in the prolate basis, to the projection from PW_c onto $\mathrm{PW}_{c'}$. The matrix Λ is understood through the famous $2\Omega T$ theorem of Landau and Widom [10] and other work of Landau et al., e.g., [9] shows that there are approximately $N \approx 2c/\pi$ eigenvalues of P_cQ that are close to one and that the other eigenvalues decay to zero rapidly—faster than exponential in n, e.g., [6, p. 21]. The factor $2c/\pi$ is the product of the length of the time-concentration and frequency-concentration intervals when using the unitary

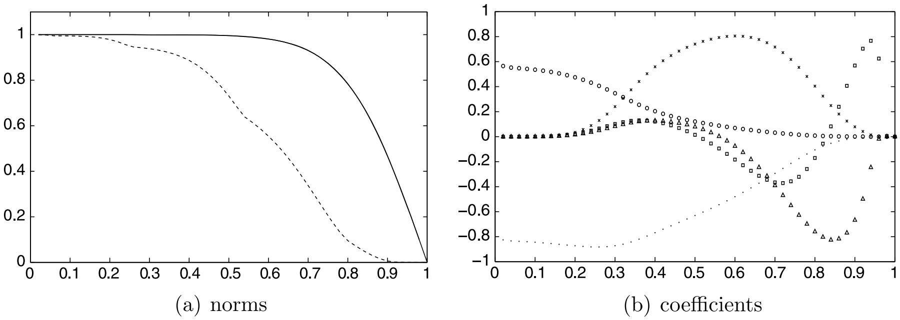

Fig. 1 (**a**) plots the approximate norm of $P_{c'c}Q$ (*dashed*) and the norm of $P_{c-c'}Q$ (*solid*) against c'/c with $c=5\pi/2$. The norm of $P_{c'c}Q$ here is approximated numerically by the norm of the truncated version of the matrix $(I-R)\Lambda$ in Prop. 1. (**b**) plots the coefficients $\langle \psi_0^{c',c}, \varphi_n^c\rangle$ of the symmetric bandpass prolate $\psi_0^{c',c}$ having the largest energy concentration in $[-1,1]$ versus c'/c. The symbols are assigned as follows: $\circ \sim \varphi_0^c$, $\cdot \sim \varphi_2^c$, $\times \sim \varphi_4^c$, $\triangle \sim \varphi_6^c$ and $\square \sim \varphi_8^c$

version of the Fourier transform (integration against $e^{-2\pi i\xi t}$). As a consequence of the $2\Omega T$ theorem, there are at most essentially $2c/\pi + \log(2c/\pi)\log((1-\alpha)/\alpha) + o(\log(c))$ eigenvalues of P_cQ larger than any fixed $\alpha > 0$. If c is large, then the initial eigenvalues of P_cQ are close to 1 and, therefore, close to one another. Eigenfunction estimates based on the integral equation are then inaccurate. In addition, the rapid decay of the eigenvalues also leads to a loss of accuracy in estimating eigenfunctions φ_n^c for large n. Similar limitations apply to the operator $P_{c'c}Q$.

For small values of c, the prolates φ_n^c can be approximated on $[-1,1]$ by sums of Legendre polynomials up to an appropriate order. The method for doing so relies on the prolate differential equation. In the case $c=0$ the solutions of (1) are the Legendre polynomials on $[-1,1]$. The coefficients β_{nk} of the prolate expansion $\varphi_n^c = \sum_{n=0}^{\infty}\beta_{nk}P_k$ in which P_k is the kth Legendre polynomial can be *solved* efficiently by finding the eigenvector–eigenvalue decomposition of a certain *tridiagonal matrix*. This variant of Bouwkamp's method [1] was considered independently by Xiao et al., [13] and Boyd [2, 3] who also considered what truncation dimension would provide accurate expansions, and by Sengupta et al. [11] in obtaining eigenvalue estimates for the bandpass problem. In particular, Boyd [3] suggested that if c is not too large ($c \le c^*(N) \approx \pi(N+1/2)/2$) then the prolates φ_n^c and their eigenvalues χ_n are approximated up to order N with negligible error by truncating this tridiagonal matrix to the first $N_{\rm tr}$ rows and columns, taking $N_{\rm tr} \approx 2N+30$. We use the same truncation dimension to estimate the matrix R, as justified by the $2\Omega T$ theorem.

In the remainder of this note we provide a few figures to illustrate the approximations just mentioned. Figure 1(a) compares the norm of the time- and band-limiting operator $P_{c-c'}Q$ with the norm of $P_{c'c}Q = (P_c - P_{c'})Q$. Here $c = 5\pi/2$. It was proved by Donoho and Stark [5] that if P_Σ denotes the operator of frequency limiting to a set Σ of finite measure then, provided the measure is sufficiently small (less than 0.88) one has the operator norm inequality $\|P_\Sigma Q\| \le \|P_{|\Sigma|}Q\|$ with $|\Sigma|$ the measure of Σ. Figure 1(a) quantifies the difference between the two norms when $\Sigma = [-c,c]\setminus[-c',c']$.

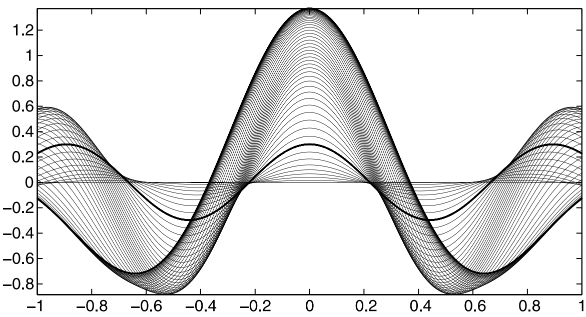

Fig. 2 Plots of bandpass prolates $\psi_0^{c'c}$ for $c = 5\pi/2$. These plots show the symmetric bandpass prolates having largest energy concentration to $[-1, 1]$. Each curve corresponds to a fixed value of the parameter c'/c ranging from 0.02 to 0.98 in increments of 0.02. The *thick curves* correspond to the starting value $c'/c = 0.02$—the curve with the highest peak—and the value $c'/c = 0.8$. The amplitudes tend to zero as $c'/c \to 1$, see Fig. 1(a)

When restricted to $[-1, 1]$, the bandpass prolate $\psi^{c'c}$ having the largest eigenvalue is essentially a sum of low order prolates φ_n^c for small c'/c. The coefficients in these linear combinations vary continuously with c'/c, with the general trend that energy moves from coefficients φ_n^c for smaller n to coefficients for larger $n \le 4c/\pi$ as $c' \uparrow c$. In Fig. 1(b) we take $c = 5\pi/2$ so that, as indicated by the $2\Omega T$ theorem, φ_n^c has negligible energy in $[-1, 1]$ for $n \ge 10$. The coefficients in Fig. 1(b) correspond to eigenvectors $\boldsymbol{\alpha}$ of an approximate truncation of $(I - R)\Lambda$. The prolates used in approximating R were obtained by the method outlined in Boyd [2] with $N = 5 = 2c/\pi$ and $N_{\text{tr}} = 2N + 30 = 40$. Likewise, R was truncated to size N_{tr}. As is well known, the nth prolate φ_n^c has n zeros in $[-1, 1]$ (e.g., [6, p. 18]) so, roughly speaking, the center frequency of φ_n^c increases with n. Thus the coefficient trend is consistent with the observation that the energy is localized in higher frequencies as $c' \uparrow c$, since $P_{c'c}$ localizes the Fourier transform to $[-c, c] \setminus [-c', c']$. This behavior is illustrated in Fig. 2, where we plot the bandpass prolate, which we denote by $\psi_0^{c'c}$, having the largest fraction of its energy in $[-1, 1]$ among all *symmetric* bandpass prolates. Only for very small values of c' does the φ_0^c term become dominant and thus $\psi_0^{c'c} \approx \varphi_0^c$. In our case, this happens for $c'/c < 0.0005$ (not shown).

It should be mentioned that $\psi_0^{c'c}$ is not always the most concentrated among all bandpass prolates: for some values of c', the most concentrated *odd* bandpass prolate (not shown) appears to have slightly more energy in $[-1, 1]$ than $\psi_0^{c'c}$ does—consistent with Slepian and Pollak's observation [12] of degeneracies in $P_{c'c}Q$ for certain values of c' and c. The notation $\psi_0^{c'c}$ then does not necessarily indicate the most concentrated bandpass prolate as it does in the case $c' = 0$, but rather it denotes a continuous deformation of the full-band prolate φ_0^c in the parameter c'.

Khare [8] also considered the problem of numerical evaluation of bandpass prolates, focusing instead on the role of the interpolating function (sinc multiplied by a suitably dilated cosine) and establishing that the bandpass prolate samples form a discrete eigenvector of the matrix of partial integrals on $[-1, 1]$ of shifts of the interpolating kernel, cf. also Hogan et al., [7]. Khare did not investigate dependence on c'/c.

Appendix A: Proof of Proposition 2

One has

$$\chi_n \int_x^1 \varphi_n(t)\varphi_m(t)\,\mathrm{d}t = \int_x^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(t^2-1\right)\frac{\mathrm{d}}{\mathrm{d}t} + c^2t^2\right)\varphi_n(t)\varphi_m(t)\,\mathrm{d}t$$
$$= \left(t^2-1\right)\frac{\mathrm{d}\varphi_n}{\mathrm{d}t}\varphi_m(t)\Big|_x^1 + c^2\int_x^1 t^2\varphi_n(t)\varphi_m(t)\,\mathrm{d}t$$
$$- \int_x^1 \left(t^2-1\right)\frac{\mathrm{d}\varphi_n}{\mathrm{d}t}\frac{\mathrm{d}\varphi_m}{\mathrm{d}t}\,\mathrm{d}t.$$

Interchanging the roles of n and m one has

$$\chi_m \int_x^1 \varphi_n(t)\varphi_m(t)\,\mathrm{d}t = \int_x^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(t^2-1\right)\frac{\mathrm{d}}{\mathrm{d}t} + c^2t^2\right)\varphi_m(t)\varphi_n(t)\,\mathrm{d}t$$
$$= \left(t^2-1\right)\frac{\mathrm{d}\varphi_m}{\mathrm{d}t}\varphi_n(t)\Big|_x^1 + c^2\int_x^1 t^2\varphi_n(t)\varphi_m(t)\,\mathrm{d}t$$
$$- \int_x^1 \left(t^2-1\right)\frac{\mathrm{d}\varphi_m}{\mathrm{d}t}\frac{\mathrm{d}\varphi_n}{\mathrm{d}t}\,\mathrm{d}t.$$

Subtracting the two results in the identity

$$(\chi_n - \chi_m)\int_x^1 \varphi_n(t)\varphi_m(t)\,\mathrm{d}t = \left(t^2-1\right)\left(\frac{\mathrm{d}\varphi_n}{\mathrm{d}t}\varphi_m(t) - \frac{\mathrm{d}\varphi_m}{\mathrm{d}t}\varphi_n(t)\right)\Big|_x^1.$$

Appendix B: Proof of Proposition 1

Suppose that $\psi = \sum \alpha_n \varphi_n^c$ with φ_n^c $L^2(\mathbb{R})$-normalized. Then

$$P_{c'c}Q\psi = \sum_k \alpha_k (P_c - P_{c'})Q\varphi_k^c = \sum_k \alpha_k\lambda_k\varphi_k^c - \sum_k \alpha_k P_{c'}Q\varphi_k^c$$
$$= \sum_k \alpha_k\lambda_k\varphi_k^c - \sum_k \alpha_k \sum_j \left\langle P_{c'}Q\varphi_k^c, \varphi_j^c\right\rangle\varphi_j^c$$

where in the last line we use the fact that $\{\varphi_j^c\}$ is complete in PW_c and $\mathrm{PW}_{c'} \subset \mathrm{PW}_c$. Use of the Plancherel identity, Eq. (3), the eigenfunction property of the prolates and the property $P_cP_{c'} = P_{c'}P_c = P_{c'}$, shows that the quantity $\langle P_{c'}Q\varphi_k^c, \varphi_j^c\rangle$ in the last line of the formula above may be written as

$$\left\langle P_{c'}Q\varphi_k^c, \varphi_j^c\right\rangle = \left\langle P_c'Q\varphi_k^c, P_c\varphi_j^c\right\rangle = \left\langle P_cP_{c'}Q\varphi_k^c, \varphi_j^c\right\rangle$$
$$= \left\langle P_{c'}P_cQ\varphi_k^c, \varphi_j^c\right\rangle = \lambda_k\left\langle P_{c'}\varphi_k^c, \varphi_j^c\right\rangle$$

$$= \lambda_k \int_{-c'}^{c'} \widehat{\varphi_k^c}(\xi)\overline{\widehat{\varphi_j^c}}(\xi)\,\mathrm{d}\xi$$

$$= \lambda_k c \int_{-c'/c}^{c'/c} \widehat{\varphi_k^c}(c\xi)\overline{\widehat{\varphi_j^c}}(c\xi)\,\mathrm{d}\xi$$

$$= \lambda_k \frac{i^{k-j}}{\sqrt{\lambda_k\lambda_j}} \int_{-c'/c}^{c'/c} \varphi_k^c(t)\varphi_j^c(t)\,\mathrm{d}t = \lambda_k R_{jk}.$$

That is, for $\psi = \sum_k \alpha_k \varphi_k^c$ one has

$$P_{c'c}Q\psi = \sum_k \alpha_k\lambda_k\Big(\varphi_k^c - \sum_j R_{jk}\varphi_j^c\Big).$$

Thus if $P_{c'c}Q\psi = \lambda\psi$ then

$$\lambda\alpha_j = \lambda_j\alpha_j - \sum_k \lambda_k\alpha_k R_{jk}$$

or, equivalently, $\lambda\boldsymbol{\alpha} = (I - R)\Lambda\boldsymbol{\alpha}$ with $\boldsymbol{\alpha} = \{\alpha_k\}$.

Notice that if $\boldsymbol{\alpha} \in \ell^2(\mathbb{Z}_+)$ and $f_\alpha \in L^2[-1,1]$ is defined by $f_\alpha(t) = \sum_k \frac{\mathrm{i}^k}{\sqrt{\lambda_k}}\overline{\alpha_k}\varphi_k^c$ then

$$\langle R\alpha, \alpha\rangle = \int_{-c'/c}^{c'/c} \big|f_\alpha(t)\big|^2\,\mathrm{d}t$$

and, since R is self adjoint,

$$\|I - R\| = \sup_{\boldsymbol{\alpha}:\|\boldsymbol{\alpha}\|=1} \big\langle (I-R)\boldsymbol{\alpha}, \boldsymbol{\alpha}\big\rangle = \sup_{\boldsymbol{\alpha}} \int_{c'/c\le|t|\le1} \big|f_\alpha(t)\big|^2\,\mathrm{d}t.$$

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