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Frame properties of shifts of prolate spheroidal wave functions

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ABSTRACT

We provide conditions on a shift parameter and number of basic prolate spheroidal wave functions with a fixed bandwidth and time concentrated to a fixed duration such that the shifts of the basic prolates form a frame or a Riesz basis for the Paley–Wiener space consisting of all square integrable functions with the given bandlimit.

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1. Introduction

A considerable literature has been developed on the subject of principal and finitely generated shift invariant spaces $V(\Psi)=\overline{\operatorname{span}}\{\psi_n(\cdot-k): n=1,\ldots,N;\ k\in\mathbb{Z}\}$, including structural results [5,20,3] and more recent results addressing spaces with further invariance properties [1,11,26]. An equally substantial literature has been developed on the subject of Gabor systems $\mathcal{G}(g_1,\ldots,g_N;\alpha,\beta)$ generated by time–frequency shifts $e^{2\pi i\ell\beta t}\,g_n(t-\alpha k)$ of a single or finite collection of generators, e.g., [7,8,4,10] with more recent applications outlined in [2,19]. A primary question is what properties of the generators g_n and shift parameters α,β are consistent with the Gabor system forming a frame or Riesz basis for $L^2(\mathbb{R})$. We study here very specific shift invariant systems with properties that are in a sense intermediate to structural properties of finitely generated shift invariant spaces on the one hand and of Gabor systems on the other. Specifically, we are interested in frame and Riesz basis properties of systems generated by shifts of certain *prolate spheroidal wave functions* (prolates, for short). These are bandlimited functions that are the most concentrated to a fixed time interval in L^2-norm. They are eigenfunctions of the operator that first truncates to a time interval

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then truncates to a frequency interval, and they are ordered by decreasing magnitude of the eigenvalue. The famous "$2\Omega T$" theorem states that, asymptotically, the number of independent unit norm eigenfunctions φ_n having eigenvalues λ_n close to one is the product of the bandwidth and the duration—the length of the time-concentration interval.

Denote by φ_n the nth prolate in this ordering, $n = 0, 1, 2, \ldots$, bandlimited to $[-\Omega, \Omega]$ and time-concentrated in the interval $[-1, 1]$ with associated eigenvalue λ_n. Fix $\alpha > 0$. We first consider frame properties of families of shifts $\{\sqrt{\lambda_n}\,\varphi_n(\cdot - 2k\alpha) : n = 0, 1, 2, \ldots, k \in \mathbb{Z}\}$ of $\sqrt{\lambda_n}$-normalized prolates. The factor *two* is used here to emphasize that we are shifting by multiples of the duration. The prolates φ_n form an orthonormal basis for the Paley–Wiener space $\mathrm{PW}_{2\Omega}$ of L^2-functions bandlimited to $[-\Omega, \Omega]$ as well as an orthogonal basis for $L^2[-1, 1]$, so it is mildly surprising that a collection of integer shifts of $\sqrt{\lambda_n}$-normalized prolates might form a frame for the Paley–Wiener space. In fact, Theorems 2 and 4 show that they form a tight frame in certain cases and at least a near-tight frame in others.

When considering shifts of the first N prolates $\varphi_0, \ldots, \varphi_{N-1}$, it is possible to obtain frames for the Paley–Wiener space with or without normalizing by $\sqrt{\lambda_n}$. The first main result in this regard, Theorem 10, essentially states that the family forms a frame for $\mathrm{PW}_{2\Omega}$ if $N \geq 2\Omega\alpha$, but that the family is incomplete in $\mathrm{PW}_{2\Omega}$ if $N < 2\Omega\alpha$. However, the frame bounds are less concretely quantified in this case. The proof of Theorem 10 illustrates why the family $\{\varphi_n(\cdot - \alpha k)\}_{n=0,\, k\in\mathbb{Z}}^{N-1}$ is redundant or overcomplete if $N > 2\Omega\alpha$. When $N = 2\Omega\alpha \in \mathbb{N}$ it is possible that the family $\{\varphi_n(\cdot - \alpha k)\}_{n=0,\, k\in\mathbb{Z}}^{N-1}$ actually forms a Riesz basis for $\mathrm{PW}_{2\Omega}$ and the second main result, Theorem 12, shows that this is indeed the case. Notationally we have absorbed the duration into the shift factor α in Theorems 10 and 12, considering shifts $\varphi(\cdot - \alpha k)$ rather than $\varphi(\cdot - 2\alpha k)$ as in the renormalized frames, in order to minimize notational burden in Sections 4 and 5.

When the prolate shifts $\{\varphi_n(t - \alpha k)\}_{n=0,\, k\in\mathbb{Z}}^{N-1}$ form a frame or Riesz basis for $\mathrm{PW}_{2\Omega}$, the time–frequency shifts $\{\mathrm{e}^{4\pi i\ell\Omega t}\varphi_n(t - \alpha k)\}_{n=0,\, k,\ell\in\mathbb{Z}}^{N-1}$ form a corresponding frame or Riesz basis for $L^2(\mathbb{R})$ with the same bounds simply because for $\ell \neq \ell'$, $\langle \mathrm{e}^{4\pi i\ell\Omega t}\varphi_n(t - \alpha k), \mathrm{e}^{4\pi i\ell'\Omega t}\varphi_n(t - \alpha k')\rangle = 0$ since the modulated prolates are frequency-supported on disjoint intervals. This fact does not contradict the Balian–Low theorem, e.g., [8] because $t\varphi_n(t) \notin L^2(\mathbb{R})$, e.g., [22, Eq. 1.10]. Analogues of Gabor frames generated by prolates were studied in the context of joint time–frequency cutoffs by Dörfler and Romero [6].

The remainder of the paper is organized as follows. In Section 2 we review necessary background properties of prolate spheroidal wave functions. Section 3 establishes frame properties of the families $\mathcal{F}_\alpha = \{\sqrt{\lambda_n}\,\varphi_n(\cdot - 2\alpha\ell) : n \geq 0, \ell \in \mathbb{Z}\}$, including Theorems 2 and 4, which provide explicit frame bounds for the Paley–Wiener space $\mathrm{PW}_{2\Omega}$. Section 4 shows that the unrenormalized families $\{\varphi_n(\cdot - \alpha\ell) : n = 0, \ldots, N-1, \ell \in \mathbb{Z}\}$ of shifts of the first N prolates form frames for $\mathrm{PW}_{2\Omega}$ under the condition that there is at least one prolate shift per unit time–bandwidth (Theorem 10), and in Section 5 it is shown that they form a Riesz basis for $\mathrm{PW}_{2\Omega}$ if there is precisely one prolate shift per unit time–bandwidth (Theorem 12).

2. Background on prolate spheroidal wave functions and frames

Much of the mathematical foundation of time and band limiting was laid out in a series of papers written by combinations of Landau, Slepian and Pollak [24,14,15,21,23] appearing in the *Bell System Technical Journal* in the early 1960s. For $T > 0$, the time-limiting operator Q_T corresponds to multiplying $f \in L^2(\mathbb{R})$ by $\mathbb{1}_{[-T,T]}$, the characteristic function of the interval $[-T, T]$. Let P_Ω denote the bandlimiting operator $P_\Omega = \mathcal{F}^{-1}Q_{\Omega/2}\mathcal{F}$ with $(\mathcal{F}f)(\xi) = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-2\pi it\xi}\,\mathrm{d}t$ denoting the Fourier transform. The duration–bandwidth product is $2\Omega T$. The operator $P_\Omega Q_T$ is compact and self-adjoint on the Hilbert space $\mathrm{PW}_\Omega = P_\Omega(L^2(\mathbb{R}))$. The operators $Q = Q_1$ and $P_{c/\pi}$ commute with the differential operator

$$\mathcal{P}_c = \frac{\mathrm{d}}{\mathrm{d}t}\big(t^2 - 1\big)\frac{\mathrm{d}}{\mathrm{d}t} + c^2t^2 \tag{1}$$

whose eigenfunctions are called *prolate spheroidal wave functions*. Their properties were studied as early as the nineteenth century (e.g., [18], cf., [25]) in the context of mathematical physics. We call the corresponding eigenfunctions of $P_\Omega Q_T$ (T,Ω)-prolates, or just prolates. They are usually indexed by listing the eigenvalues λ_n, $n=0,1,2,\dots$ of $P_\Omega Q_T$ in decreasing order. The (T,Ω)-prolates are dilates by $1/T$ of $(1,T\Omega)$-prolates which are eigenfunctions of $\mathcal{P}_{\pi\Omega T}$. Thus, up to a unitary dilation, the (T,Ω) prolates just depend on the *duration–bandwidth product* $2T\Omega$. The $(1,\Omega)$-prolates $\varphi_n=\varphi_n^\Omega$ have interesting properties. They form an orthonormal basis for PW_Ω with L^2-inner product. They also form an orthogonal basis for $L^2[-1,1]$ with $\int_{-1}^1|\varphi_n|^2=\lambda_n$ when $\|\varphi_n\|_{L^2(\mathbb{R})}=1$. As such, any $f\in\mathrm{PW}_\Omega$ can be expanded in the form $f=\sum_{n=0}^\infty\alpha_n\varphi_n^\Omega$ with $\|f\|^2_{L^2(\mathbb{R})}=\sum\alpha_n^2$ and $\int_{-1}^1|f|^2=\sum\lambda_n\alpha_n^2$. The prolates are real-valued and φ_n^Ω is even (odd) if n is even (odd). The nth prolate φ_n has n zeros on $[-1,1]$. The φ_n are eigenfunctions of $f\mapsto\sqrt{2/\Omega}\,D_{\Omega/2}\mathcal{F}^{-1}Qf$, where $(D_af)(t)=\sqrt{a}f(at)$, with eigenvalues $\mu_n=\mathrm{i}^n\sqrt{2\lambda_n(\Omega)/\Omega}$. One has the very useful consequence that their Fourier transforms are just cutoffs of dilates of φ_n: for a $(1,\Omega)$-prolate,

$$\widehat{\varphi}_n\left(\frac{\Omega}{2}\xi\right)=(-\mathrm{i})^n\sqrt{\frac{2}{\Omega\lambda_n}}\,Q\varphi_n(\xi). \tag{2}$$

Landau and Widom's $2\Omega T$ theorem [16] states that $P_\Omega Q_T$ has approximately $2\Omega T$ eigenvalues close to one, with a logarithmic plunge from eigenvalues close to one to very small eigenvalues. Specifically, if $N(\Omega,T,\alpha)$ denotes the number of eigenvalues of $P_\Omega Q_T$ larger than $\alpha\in(0,1)$ then

$$N(\Omega,T,\alpha)=2\Omega T+\log(2\Omega T)\log\left(\frac{1-\alpha}{\alpha}\right)+o\big(\log(\Omega T)\big). \tag{3}$$

When $\alpha=1/2$ one also has $\lfloor 2\Omega T\rfloor+1\ge N(\Omega,T,1/2)\ge\lfloor 2\Omega T\rfloor-1$, see [13,12].

In Section 4 we will make use of the *Markov property* of the $(1,\Omega)$-prolates (e.g., [12, Theorem 2.1.16 and Lemma 2.1.2]): For each N, the first N prolates form a Chebyshev system for $[-1,1]$. We will use this fact in the following form.

Proposition 1. *Fix $\Omega>0$, $\beta>0$ and $\tau\in\mathbb{R}$. Let φ_n denote the nth $(1,\Omega)$ prolate and let $t_0<\cdots<t_{N-1}$ denote the lattice points in $(\tau+\beta\mathbb{Z})\cap[-1,1)$. Then there exists $c>0$ depending on β but independent of τ such that $|\det\Phi|\ge c$, where Φ is the matrix with (n,k)th entry $\Phi_{nk}=\varphi_n(t_k)$.*

For the sake of completeness we include here the definitions of frames and Riesz bases, e.g., [9]. Let $\mathcal{H}$ be an infinite dimensional, separable Hilbert space. A family $\{f_1,f_2,\dots\}$ is a frame for $\mathcal{H}$ if there exist constants $0<A\le B<\infty$ such that for any $f\in\mathcal{H}$ one has

$$A\|f\|^2_{\mathcal{H}}\le\sum_n\big|\langle f,f_n\rangle\big|^2\le B\|f\|^2_{\mathcal{H}}.$$

A sequence $\{g_n\}$ is a Riesz basis for its closed span in $\mathcal{H}$ if there exist constants $0<c\le C<\infty$ such that for any sequence $c_1,c_2,\dots$ in $\ell^2(\mathbb{N})$ one has

$$c\sum_{n=1}^\infty|c_n|^2\le\Big\|\sum c_ng_n\Big\|_{\mathcal{H}}\le C\sum_{n=1}^\infty|c_n|^2.$$

3. Frames of normalized shifts using all prolates

Throughout what follows we will assume that $\Omega>0$ is fixed and let φ_n be the nth $(1,2\Omega)$-prolate bandlimited to $[-\Omega,\Omega]$ and time-concentrated on $[-1,1]$ with $\|\varphi_n\|_{L^2(\mathbb{R})}=1$. Thus $P_{2\Omega}Q\varphi_n=\lambda_n\varphi_n$

and $\|Q\varphi_n\|^2 = \lambda_n$. We denote by ψ_n the *renormalized* prolate $\psi_n(t) = \sqrt{\lambda_n}\,\varphi_n(t)$, by $\psi_{n,2\alpha\ell}$ the shift $\tau_{2\alpha\ell}\psi_n(t) = \psi_n(t-2\alpha\ell)$ and, for fixed $\alpha > 0$, $\mathcal{F}_\alpha = \{\psi_{n,2\alpha\ell} : \ell \in \mathbb{Z}, n \geq 0\}$. The purpose of this section is to show that if $\alpha \leq 1$ then $\mathcal{F}_\alpha$ forms a frame for $\mathrm{PW}_{2\Omega}$ and, at least for certain values of α, it forms a tight frame.

Theorem 2. *For $\alpha \leq 1$ fixed, $\mathcal{F}_\alpha = \{\psi_{n,2\alpha\ell} : \ell \in \mathbb{Z}, n \geq 0\}$ forms a frame for $\mathrm{PW}_{2\Omega}$ with lower frame bound $A \geq \lfloor 1/\alpha \rfloor$ and upper frame bound $B \leq \lceil 1/\alpha \rceil$.*

Corollary 3. *If $1/\alpha \in \mathbb{N}$ then $\mathcal{F}_\alpha$ forms a tight frame for $\mathrm{PW}_{2\Omega}$ with frame bound $A = B = 1/\alpha$. In particular, $\mathcal{F} = \mathcal{F}_1$ forms a Parseval frame for $\mathrm{PW}_{2\Omega}$.*

Recall that a Parseval frame is one such that $\|f\|^2 = \sum |\langle f, f_n\rangle|^2$.

Proof. Let $f \in \mathrm{PW}_{2\Omega}$. Then with $\tau_s f(t) = f(t-s)$, after a change of variable and using the self-adjointness of $P_{2\Omega}$ and Q one has

$$\begin{aligned}
\sum_n \sum_\ell \left|\langle f, \psi_{n,2\alpha\ell}\rangle\right|^2 &= \sum_n \sum_\ell \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tau_{-2\alpha\ell} f(t) \overline{\tau_{-2\alpha\ell} f(s)} \lambda_n \varphi_n(t) \varphi_n(s)\, \mathrm{d}t\mathrm{d}s \\
&= \sum_\ell \int_{-\infty}^{\infty} \tau_{-2\alpha\ell} f(t) \sum_n \overline{\langle \tau_{-2\alpha\ell} f,\, \varphi_n\rangle}\, P_{2\Omega} Q \varphi_n(t)\, \mathrm{d}t \\
&= \sum_\ell \langle \tau_{-2\alpha\ell} f,\, P_{2\Omega} Q \tau_{-2\alpha\ell} f\rangle = \sum_\ell \|Q\tau_{-2\alpha\ell} f\|^2 \\
&= \sum_\ell \int_{-1}^{1} \left|f(t+2\alpha\ell)\right|^2 \mathrm{d}t = \sum_\ell \int_{2\alpha\ell-1}^{2\alpha\ell+1} \left|f(t)\right|^2 \mathrm{d}t \\
&= \sum_\ell \int_{-\infty}^{\infty} \mathbb{1}_{[-1,1]}(t-2\alpha\ell) \left|f(t)\right|^2 \mathrm{d}t \\
&= \int_{-\infty}^{\infty} \left(\sum_\ell \mathbb{1}_{[-1,1]}(t-2\alpha\ell)\right) \left|f(t)\right|^2 \mathrm{d}t.
\end{aligned}$$

We used the fact that $f \in \mathrm{PW}_{2\Omega}$ in the fourth identity. For any t, the sequence $\{t-2\alpha\ell\}_{\ell\in\mathbb{Z}}$ forms a lattice having at least $\lfloor 1/\alpha \rfloor$ points in $[-1,1)$ and at most $\lceil 1/\alpha \rceil$ points in $[-1,1)$. Thus the last integral is bounded below by $\lfloor 1/\alpha \rfloor \|f\|^2$ and above by $\lceil 1/\alpha \rceil \|f\|^2$ and the result follows. □

The normalization of the functions ψ_n is not intuitive. The functions φ_n form an orthonormal basis for $\mathrm{PW}_{2\Omega}$ and $\varphi_n/\sqrt{\lambda_n}$ form an orthonormal basis for $L^2[-1,1]$. Giving ψ_n $L^2(\mathbb{R})$-norm $\sqrt{\lambda_n}$ insures that high order terms do not add too much energy to the coefficient sum when f is shifted into a region in which φ_n is more concentrated. The inequalities

$$\lfloor 1/\alpha \rfloor \|f\|^2 \leq \sum_n \sum_\ell \left|\langle f,\, \psi_{n,2\alpha\ell}\rangle\right|^2 \leq \lceil 1/\alpha \rceil \|f\|^2$$

do not preclude the possibility of the functions $\{\psi_{n,2\alpha\ell}\}$ forming a tight frame for some cases when $1/\alpha \notin \mathbb{N}$. In the remainder of this section we study frames of the form $\mathcal{F}_\alpha$ or $\mathcal{F}_{\alpha,N} = \{\psi_{n,2\alpha\ell} : n = 0, \ldots, N-1, \ell \in \mathbb{Z}\}$ when α is such that $1/(4\Omega\alpha) \in \mathbb{N}$. Theorem 4 shows that $\mathcal{F}_\alpha$ forms a tight frame in this case, just as it does when $1/\alpha \in \mathbb{N}$; however, the proof when $1/(4\Omega\alpha) \in \mathbb{N}$ (and $1/\alpha \notin \mathbb{N}$) requires fundamentally different techniques.

Theorem 4. *If $1/\alpha = 4m\Omega$ for some $m \in \mathbb{N}$ then $\mathcal{F}_\alpha = \{\psi_{n,2\alpha\ell}\}$ forms a tight frame for $\mathrm{PW}_{2\Omega}$ with frame bound $A = B = 4m\Omega$.*

The theorem requires two auxiliary facts.

Proposition 5. *For $\alpha = 1/4\Omega$ and any $N \in \mathbb{N}$ one has*

$$\sum_{\ell}\sum_{n=0}^{N-1}\left|\langle f,\,\psi_{n,2\alpha\ell}\rangle\right|^2 = 2\int_{-\Omega}^{\Omega}\left|\widehat{f}(\xi)\right|^2\sum_{n=0}^{N-1}\frac{1}{\lambda_n}\left|\psi_n\left(\frac{\xi}{\Omega}\right)\right|^2 \mathrm{d}\xi.$$

Observe that $\sum_{n=0}^{N-1}|\psi_n(\xi/\Omega)|^2/\lambda_n = \sum_{n=0}^{N-1}|\varphi_n(\xi/\Omega)|^2$. The tight frame bound in Theorem 4 relies on the fact that the latter sum tends to a constant on $[-\Omega,\Omega]$ when $N \to \infty$.

Lemma 6. *For the $(1, 2\Omega)$-prolates φ_n, on $[-1, 1]$ one has $\sum_{n=0}^{\infty}|\varphi_n(t)|^2 = 2\Omega$ in the sense of both L^2 and uniform convergence on $[-1, 1]$.*

When $m = 1$ the frame bound in Theorem 4 is 4Ω, which is equal to the duration–bandwidth product for the $(1, 2\Omega)$-prolates φ_n. The uniformity of the limit in Lemma 6 suggests that shifts of $\psi_0, \ldots, \psi_{N-1}$ also can form a frame, albeit no longer tight, for $\mathrm{PW}_{2\Omega}$.

Corollary 7. *For $\alpha = 1/(2\Omega)$, $\mathcal{F}_{\alpha,N} = \{\psi_{n,2\alpha\ell} : n = 0, \ldots, N-1,\ \ell \in \mathbb{Z}\}$ forms a frame for $\mathrm{PW}_{2\Omega}$ whose lower and upper frame bounds A_N and B_N satisfy*

$$2\inf_{|\xi|\le 1}\sum_{n=0}^{N-1}\frac{1}{\lambda_n}\left|\psi_n(\xi)\right|^2 \le A_N; \qquad B_N \le 2\sup_{|\xi|\le 1}\sum_{n=0}^{N-1}\frac{1}{\lambda_n}\left|\psi_n(\xi)\right|^2.$$

When $N = 1$ the lower frame bound holds because φ_0 is nonvanishing on $[-1, 1]$, though small near ± 1. One can use exactly the same argument used in proving Proposition 5 to establish corresponding frame bounds generated by the first N (unrenormalized) shifted prolates $\varphi_n(\cdot - \ell/(2\Omega))$.

Corollary 8. *For N fixed the $(1, 2\Omega)$ shifted prolates $\{\varphi_n(\cdot - \ell/(2\Omega)) : n = 0, \ldots, N-1,\ \ell \in \mathbb{Z}\}$ form a frame for $\mathrm{PW}_{2\Omega}$ whose lower and upper frame bounds A'_N and B'_N satisfy*

$$2\inf_{|\xi|\le 1}\sum_{n=0}^{N-1}\frac{1}{\lambda_n}\left|\varphi_n(\xi)\right|^2 \le A'_N; \qquad B'_N \le 2\sup_{|\xi|\le 1}\sum_{n=0}^{N-1}\frac{1}{\lambda_n}\left|\varphi_n(\xi)\right|^2.$$

Recovery from frame coefficients is most effective when the frame is tight or, at least, *snug*, that is, $(B-A)/(B+A)$ is small. Since $\int_{-1}^{1}\sum_{n=0}^{N-1}\frac{1}{\lambda_n}|\varphi_n(\xi)|^2 = N$, the average value of $\sum_{n=0}^{N-1}\frac{1}{\lambda_n}|\varphi_n(\xi)|^2$ is $N/2$. The function $\sum_{n=0}^{N-1}\frac{1}{\lambda_n}|\varphi_n(\xi)|^2$ is plotted for a fixed Ω and different values of N in Fig. 1.

We proceed now in proving Theorem 4 assuming Proposition 5 and Lemma 6, which will be proved subsequently.

Proof of Theorem 4. By Lemma 6 and Plancherel's theorem, the quantity $\int_{-\Omega}^{\Omega}|\widehat{f}(\xi)|^2\sum_{n=0}^{N-1}|\varphi_n(\frac{\xi}{\Omega})|^2\,\mathrm{d}\xi$ converges to $2\Omega\|f\|^2$ as $N \to \infty$ so that, by Proposition 5 it also follows that $\sum_{n=0}^{N-1}\sum_{\ell}|\langle f,\,\psi_{n,2\alpha\ell}\rangle|^2$

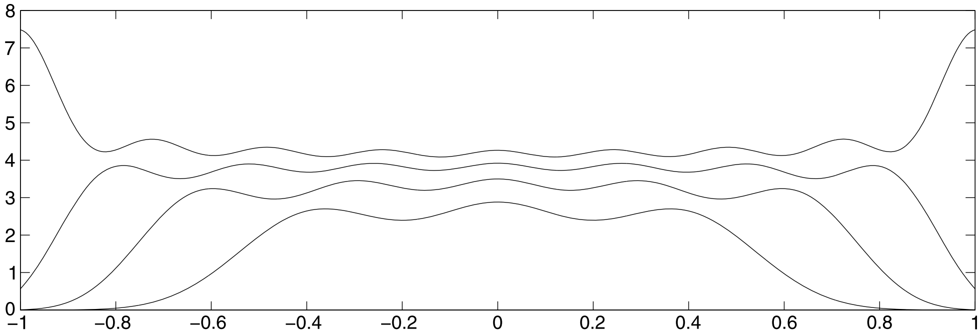

Fig. 1. Plot of $\sum_{n=0}^{N-1} |\varphi_n(\xi)|^2/\lambda_n$ for $N = 3, 5, 7, 9$ and $\Omega = 2$, so there about 8 eigenvalues larger than 1/2. The tightest frame bounds correspond to $N = 7$, 9. The lower frame bound is on the order 10^{-5} when $N = 3$ and 6×10^{-3} when $N = 5$.

converges to $4\Omega\|f\|^2$ as $N \to \infty$ when $\alpha = 1/(4\Omega)$. The case $m = 1$ then follows once the proposition and lemma are proved. For the general case $m \geq 1$, with $\alpha = 1/(4m\Omega)$ one can write

$$\begin{aligned}
&\sum_{n=0}^{N-1} \sum_{\ell} \left|\langle f, \psi_{n,2\alpha\ell}\rangle\right|^2 \\
&\quad = \sum_{k=0}^{m-1} \sum_{n=0}^{N-1} \sum_{\ell} \left|\left\langle f, \sqrt{\lambda_n}\varphi_n\left(\cdot - \frac{\ell}{2\Omega} - \frac{k}{2m\Omega}\right)\right\rangle\right|^2 \\
&\quad = \sum_{k=0}^{m-1} \sum_{n=0}^{N-1} \sum_{\ell} \left|\left\langle f\left(\cdot + \frac{k}{2m\Omega}\right), \sqrt{\lambda_n}\varphi_n\left(\cdot - \frac{\ell}{2\Omega}\right)\right\rangle\right|^2 \\
&\quad \to \sum_{k=0}^{m-1} 4\Omega \left\| f\left(\cdot + \frac{k}{2m\Omega}\right)\right\|^2 = 4m\Omega\|f\|^2. \qquad \square
\end{aligned}$$

As a side note, the tight frame bound $4m\Omega$ extends to the case in which the shifts of the ψ_n by multiples of $2\alpha = 1/(2m\Omega)$ are replaced by their shifts by a union of m arbitrary but fixed translates of the lattice $\mathbb{Z}/(2\Omega)$.

Proof of Proposition 5. As before we assume that the prolates φ_n are $(1, 2\Omega)$ prolates, that is, they are bandlimited to $[-\Omega, \Omega]$. Using the Fourier property (2) of the prolates and Plancherel's theorem we have

$$\begin{aligned}
\sum_{\ell} \sum_{n=0}^{N-1} \left|\left\langle f, \sqrt{\lambda_n}\varphi_n\left(\cdot - \frac{\ell}{2\Omega}\right)\right\rangle\right|^2 &= \sum_{\ell} \sum_{n=0}^{N-1} \lambda_n \left|\langle \widehat{f}, e^{\pi i\ell \cdot/\Omega}\widehat{\varphi}_n(\cdot)\rangle\right|^2 \\
&= \sum_{\ell} \sum_{n=0}^{N-1} \lambda_n \left|\int_{-\Omega}^{\Omega} \widehat{f}(\xi)\, e^{-\pi i\ell\xi/\Omega}\, \overline{\widehat{\varphi}_n(\xi)}\, d\xi\right|^2 \\
&= 2\Omega \sum_{n=0}^{N-1} \lambda_n \int_{-\Omega}^{\Omega} \left|\widehat{f}(\xi)\, \overline{\widehat{\varphi}_n(\xi)}\right|^2 d\xi \\
&= 2\Omega \int_{-\Omega}^{\Omega} \left|\widehat{f}(\xi)\right|^2 \sum_{n=0}^{N-1} \lambda_n \left|\widehat{\varphi}_n(\xi)\right|^2 d\xi
\end{aligned}$$

$$= 2\Omega \int_{-\Omega}^{\Omega} \left|\widehat{f}(\xi)\right|^2 \sum_{n=0}^{N-1} \lambda_n \left|\frac{1}{\sqrt{\Omega\lambda_n}} \varphi_n\left(\frac{\xi}{\Omega}\right)\right|^2 \mathrm{d}\xi$$

$$= 2 \int_{-\Omega}^{\Omega} \left|\widehat{f}(\xi)\right|^2 \sum_{n=0}^{N-1} \left|\varphi_n\left(\frac{\xi}{\Omega}\right)\right|^2 \mathrm{d}\xi$$

$$= 2 \int_{-\Omega}^{\Omega} \left|\widehat{f}(\xi)\right|^2 \sum_{n=0}^{N-1} \frac{1}{\lambda_n} \left|\psi_n\left(\frac{\xi}{\Omega}\right)\right|^2 \mathrm{d}\xi. \qquad \square$$

It remains to prove Lemma 6.

Proof of Lemma 6. The eigenvalue property of the $(1, 2\Omega)$ prolates is

$$\lambda_n \varphi_n(t) = \int_{-1}^{1} \frac{\sin(2\pi\Omega(t-s))}{\pi(t-s)} \varphi_n(s)\,\mathrm{d}s.$$

Since the functions $\varphi_n/\sqrt{\lambda_n}$ form an orthonormal basis for $L^2[-1,1]$, one has

$$\sum_{n=0}^{\infty} \left|\varphi_n(t)\right|^2 = \sum_n \frac{1}{\lambda_n} \left(\int_{-1}^{1} \frac{\sin(2\pi\Omega(t-s))}{\pi(t-s)} \varphi_n(s)\,\mathrm{d}s \right) \varphi_n(t)$$

$$= \sum_n \left\langle \frac{\sin(2\pi\Omega(t-\cdot))}{\pi(t-\cdot)}, \frac{Q\varphi_n}{\sqrt{\lambda_n}} \right\rangle \frac{Q\varphi_n}{\sqrt{\lambda_n}} = \left.\frac{\sin(2\pi\Omega(t-s))}{\pi(t-s)}\right|_{t=s} = 2\Omega$$

where the series converges in $L^2([-1,1])$. In addition, the functions $\varphi_n(t)$ are continuous on $[-1,1]$ so the partial sums $\sum_{n=0}^{N-1} |\varphi_n(t)|^2$ form a sequence of continuous functions monotone increasing in N. By [12, (2.22), p. 59] one has $\varphi_n^2(t) \le C\,n\lambda_n$ and then by [12, Lemma 1.2.11], $\lambda_n \le C\mathrm{e}^{-n}$ so $\sum_{n=N}^{\infty} |\varphi_n(t)|^2 \to 0$ uniformly as $N \to \infty$. Hence the sum converges uniformly on $[-1,1]$ to the constant limit. $\square$

4. Prolate shift frames with low redundancy

The normalized prolate shift families $\mathcal{F}_\alpha$ above are highly redundant systems. In this section we consider families of translates of the first N prolates $\varphi_0, \dots, \varphi_{N-1}$ having $L^2(\mathbb{R})$-norm one by integer multiples of a factor $\alpha > 0$ such that neither $2/\alpha$ nor $1/(2\Omega\alpha)$ is necessarily an integer. We seek conditions such that the family of shifted prolates $\{\varphi_n(t-\alpha k)\}_{n=0,\,k\in\mathbb{Z}}^{N-1}$ forms a frame for $\mathrm{PW}_{2\Omega}$. For notational convenience we have absorbed the duration factor two into the shift factor α here. As before we assume that the prolates are bandlimited to $[-\Omega, \Omega]$ and essentially time-limited to $[-1,1]$ so that the duration–bandwidth product is 4Ω. We will see that it is sufficient, in general, that $N \ge 2\Omega\alpha$, meaning that there is at least one prolate shift per unit time–duration–bandwidth. In contrast to the high redundancy case, the frame bounds here depend on the norm of a certain positive definite matrix. Estimated bounds are not as concrete as in the high redundancy case.

To get quickly to the role of this matrix, the proof of the following lemma, which follows the lines of that of Proposition 5, is left to the end of this section.

Lemma 9. *Given* $\alpha > 0$ *and the* $(1, 2\Omega)$*-prolates* $\varphi_0, \dots, \varphi_{N-1}$*, for* $f \in \mathrm{PW}_{2\Omega}$*,*

$$\sum_{n=0}^{N-1}\sum_{k\in\mathbb{Z}}\big|\langle f,\,\varphi_n(\cdot-\alpha k)\rangle\big|^2$$
$$=\sum_{n=0}^{N-1}\frac{1}{\lambda_n\alpha\Omega}\int_0^{1/\alpha}\left|\sum_{\ell=0}^{\lceil 2\Omega\alpha\rceil-1}\widehat{f}(\xi+\ell/\alpha-\Omega)\,\varphi_n\left(\frac{\xi+\ell/\alpha}{\Omega}-1\right)\right|^2\mathrm{d}\xi. \tag{4}$$

The vector $\{\widehat{f}(\xi+\ell/\alpha-\Omega)\}_{\ell=0}^{D-1}$ with $D=\lceil 2\Omega\alpha\rceil$ can be regarded as an arbitrary $\mathbb{C}^D$-valued function $\vec{s}(\xi)=\{s_\ell(\xi)\}_{\ell=0}^{D-1}$ with components in $L^2[0,1/\alpha)$, with the caveat that the last component of $\vec{s}(\xi)$ is zero if $\xi>(1-\gamma)/\alpha$. We seek uniform bounds

$$A\big\|\vec{s}(\xi)\big\|_{\mathbb{C}^D}^2\le\sum_{n=0}^{N-1}\frac{1}{\lambda_n\alpha\Omega}\left|\sum_{\ell=0}^{D-1}s_\ell(\xi)\,\varphi_n\left(\frac{\xi+\ell/\alpha}{\Omega}-1\right)\right|^2\le B\big\|\vec{s}(\xi)\big\|_{\mathbb{C}^D}^2, \tag{5}$$

that is, with A and B independent of ξ. This is equivalent to the matrices

$$M_{k,\ell}(\xi)=\sum_{n=0}^{N-1}\frac{1}{\lambda_n\alpha\Omega}\varphi_n\left(\frac{\xi+k/\alpha}{\Omega}-1\right)\varphi_n\left(\frac{\xi+\ell/\alpha}{\Omega}-1\right),\quad 0\le k,\ell<D=\lceil 2\Omega\alpha\rceil,0\le\xi<1/\alpha \tag{6}$$

having spectra bounded below uniformly by $A>0$.

The matrix $M(\xi)$ is positive semidefinite since $M=\sum_n M_n$ with $M_n=\frac{1}{\lambda_n\alpha\Omega}\Phi_n(\xi)\Phi_n^T(\xi)$ where Φ_n is the vector with kth coordinate $\varphi_n(\frac{\xi+k/\alpha}{\Omega}-1)$, $(0\le k<D)$. That $\vec{x}^TM(\xi)\vec{x}=0$ then requires that $\vec{x}^TM_n\vec{x}=0$ for each n. This is equivalent to $\Phi_n^T(\xi)\vec{x}=0$ for each $n=0,\dots,N-1$. That is, $\vec{x}$ must be orthogonal to each of the vectors $\Phi_n^T(\xi)$. Thus $\vec{x}$ must be in the kernel of the matrix whose nth row is Φ_n^T, $0\le n<N$.

Consider a matrix $W(\xi)$ whose rows are $\Phi_n^T(\xi)$. That is, the (n,ℓ)th entry of W is $\varphi_n(\frac{\xi+\ell/\alpha}{\Omega}-1)$, $0\le\ell<D$. The matrix has D columns. Its last column is zero if $\xi>(1-\gamma)/\alpha$.

If W has fewer than D rows then there is a unit vector $\vec{s}(\xi)$ in $\mathbb{C}^D$ that is orthogonal to each of the rows of W. In this case the left hand inequality in (5) fails. Thus we need to use $N\ge D$ prolates. On the other hand, if W has D rows then the Chebyshev condition (see Proposition 1) implies that W is nonsingular for $\xi\le(1-\gamma)/\alpha$: the Chebyshev condition applies to the arguments of φ_n lying in the full interval $[-1,1]$ and $\xi=(1-\gamma)/\alpha$ corresponds to the argument of φ_n equal to one. By continuity of the determinant and compactness of $[0,(1-\gamma)/\alpha]$, the determinant is bounded away from zero for $\xi\in[0,(1-\gamma)/\alpha]$. On the other hand, if $1-\gamma\le\alpha\xi\le 1$ then the minor consisting of the first $D-1$ rows and columns of W is nonsingular and the determinant of this minor is likewise bounded away from zero on $[(1-\gamma)/\alpha,\,1/\alpha]$. Since the matrix $M(\xi)$ in (6) varies continuously with ξ we can conclude that its spectrum is bounded below uniformly on $[0,\,(1-\gamma)/\alpha]$ while the spectrum of the submatrix corresponding to $0\le k,\ell<D-1$ is bounded below uniformly on $[(1-\gamma)/\alpha,\,1/\alpha]$. Together this implies that the left hand inequality in (5) holds uniformly in ξ with the caveat that the last coordinate of $\vec{s}$ is zero when $\xi>(1-\gamma)/\alpha$. The right hand inequality is a simple consequence of the continuity of the prolate functions. These facts show that the functions $\{\varphi_n(\cdot-\alpha k)\}_{n=0,k\in\mathbb{Z}}^{N-1}$ form a frame for $\mathrm{PW}_{2\Omega}$ provided that $N\ge\lceil 2\Omega\alpha\rceil$.

Theorem 10. *Let φ_n denote the nth prolate bandlimited to $[-\Omega,\,\Omega]$ and time-concentrated in $[-1,1]$. If $N\ge\lceil 2\Omega\alpha\rceil$ then the functions $\{\varphi_n(\cdot-\alpha k)\}_{n=0,k\in\mathbb{Z}}^{N-1}$ form a frame for $\mathrm{PW}_{2\Omega}$. That is, there exist constants $0<A\le B<\infty$ such that for any $f\in\mathrm{PW}_{2\Omega}$ one has*

$$A\|f\|_{L^2}^2\le\sum_{n=0}^{N-1}\sum_{k\in\mathbb{Z}}\big|\langle f,\,\varphi_n(\cdot-\alpha k)\rangle\big|^2\le B\|f\|_{L^2}^2.$$

Conversely, if $N<\lceil 2\Omega\alpha\rceil$ then the lower frame bound fails.

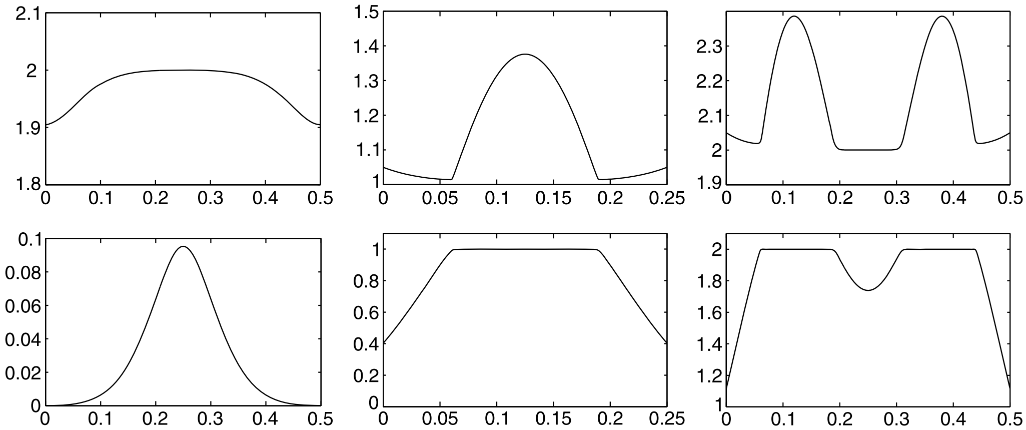

Fig. 2. *Plots of matrix norm* $\|M(\xi)\|$ (*top*) and $1/\|M^{-1}(\xi)\|$ (*bottom*) in (6). In each figure, $\Omega = 2$ so the duration–bandwidth product is 8, and the horizontal axis is indexed by the subset $[0, 1/\alpha\Omega]$ of $[-1, 1]$ corresponding to the argument of the prolate. In the leftmost figure we take $\alpha = 1$ and $N = 4$ so that $D = 2\Omega\alpha = 8$ as well, and the corresponding prolate shifts form a Riesz basis. In this case, the numerical lower bound for $1/\|M^{-1}\|$ is on the order of 10^{-4}. In the middle figure, $\alpha = 2$ and $N = 8$ and, again, the prolate shifts form a Riesz basis but in this case the frame bounds are much closer. In the right figure, $\alpha = 1$ and $N = 8$ so the prolate shifts form a frame with redundancy two.

Several remarks are in order. First, the proof relies crucially on the Chebyshev property of the prolates and, in particular, on the ordering of the prolates by magnitude of their eigenvalues. Secondly, using more than the minimum number $N = \lceil 2\Omega\alpha \rceil$ of prolates still results in a frame since the quantity $\vec{s}^T(\xi)M(\xi)\vec{s}(\xi)$ in (5) only increases when we add more terms to the sum over n. Third, when $2\Omega\alpha \in \mathbb{N}$ it is possible that the α-shifts of the first $2\Omega\alpha$ prolates can form a Riesz basis for $\mathrm{PW}_{2\Omega}$. This will be investigated in the next section. Finally, the lower frame bound in Theorem 10 is determined by the lower bound of the spectra of the matrices $M(\xi)$ in (6). This bound is computable numerically, see Fig. 2. However, there is no known straightforward method to obtain effective analytical bounds. In particular, it is not obvious which combinations of N and α yield the snuggest possible frame bounds when N/α is fixed.

To complete the proof of Theorem 10 it remains to prove Lemma 9.

Proof of Lemma 9. By the Parseval identity, properties of Fourier transforms, and (2),

$$\begin{aligned}
\sum_{n=0}^{N-1}\sum_{k\in\mathbb{Z}}\bigl|\langle f, \varphi_n(\cdot-\alpha k)\rangle\bigr|^2 &= \sum_{n=0}^{N-1}\sum_{k\in\mathbb{Z}}\bigl|\langle \widehat{f}, \mathrm{e}^{2\pi\mathrm{i}\alpha k\xi}\widehat{\varphi}_n(\xi)\rangle\bigr|^2 \\
&= \sum_{n=0}^{N-1}\sum_{k\in\mathbb{Z}}\left|\int_{-\Omega}^{\Omega}\widehat{f}(\xi)\,\mathrm{e}^{-2\pi\mathrm{i}\alpha k\xi}\,\overline{\widehat{\varphi}_n}(\xi)\,\mathrm{d}\xi\right|^2 \\
&= \sum_{n=0}^{N-1}\sum_{k\in\mathbb{Z}}\left|\int_{-\Omega}^{\Omega}\widehat{f}(\xi)\,\mathrm{e}^{-2\pi\mathrm{i}\alpha k\xi}(-\mathrm{i})^n\sqrt{\frac{1}{\lambda_n\Omega}}\varphi_n\left(\frac{\xi}{\Omega}\right)\mathrm{d}\xi\right|^2 \\
&= \sum_{n=0}^{N-1}\frac{1}{\lambda_n\Omega}\sum_{k\in\mathbb{Z}}\left|\int_{-\Omega}^{\Omega}\widehat{f}(\xi)\,\mathrm{e}^{-2\pi\mathrm{i}\alpha k\xi}\varphi_n\left(\frac{\xi}{\Omega}\right)\mathrm{d}\xi\right|^2 \\
&= \sum_{n=0}^{N-1}\frac{1}{\lambda_n\Omega}\sum_{k\in\mathbb{Z}}\left|\sum_{\ell=0}^{\lceil 2\Omega\alpha\rceil-1}\int_{-\Omega+\ell/\alpha}^{-\Omega+(\ell+1)/\alpha}\widehat{f}(\xi)\,\mathrm{e}^{-2\pi\mathrm{i}\alpha k\xi}\varphi_n\left(\frac{\xi}{\Omega}\right)\mathrm{d}\xi\right|^2 \\
&= \sum_{n=0}^{N-1}\frac{1}{\lambda_n\Omega}\sum_{k\in\mathbb{Z}}\left|\sum_{\ell=0}^{\lceil 2\Omega\alpha\rceil-1}\int_{0}^{1/\alpha}\widehat{f}(\xi+\ell/\alpha-\Omega)\,\mathrm{e}^{-2\pi\mathrm{i}\alpha k\xi}\varphi_n\left(\frac{\xi+\ell/\alpha}{\Omega}-1\right)\mathrm{d}\xi\right|^2.
\end{aligned}$$

When $2\Omega\alpha \in \mathbb{N}$ the sum over ℓ covers all values of φ_n running from -1 to 1. When $2\Omega\alpha \notin \mathbb{N}$ the integral corresponding to $\ell = \lceil 2\Omega\alpha \rceil - 1$ has to be corrected. Setting $\gamma = \lceil 2\Omega\alpha \rceil - 2\Omega\alpha$,

$$\xi + \frac{\ell}{\alpha} - \Omega = \xi + \frac{\lceil 2\Omega\alpha \rceil - 1}{\alpha} - \Omega = \xi + \frac{\gamma - 1}{\alpha} + \Omega > \Omega$$

if $\xi > (1-\gamma)/\alpha$ when $\ell = \lceil 2\Omega\alpha \rceil - 1$. With this caveat, the sum

$$\sum_{\ell=0}^{\lceil 2\Omega\alpha \rceil - 1} \widehat{f}(\xi + \ell/\alpha - \Omega)\, \mathrm{e}^{-2\pi \mathrm{i}\alpha k\xi} \varphi_n\left(\frac{\xi + \ell/\alpha}{\Omega} - 1\right)$$

can be regarded as a function on $[0, 1/\alpha)$. For $\xi \in [0, 1/\alpha)$, the arguments of the terms in the sum run over disjoint subintervals of $[-\Omega, \Omega]$. When $\xi \in [0, (1-\gamma)/\alpha)$ each term lies in the support $[-\Omega, \Omega]$ of $\widehat{f}$ and there are potentially $D = \lceil 2\Omega\alpha \rceil$ nonzero terms in the sum. When $\xi \in ((1-\gamma)/\alpha, 1/\alpha)$ the last term is outside the support of $\widehat{f}$ so there are potentially $D-1$ nonzero terms. With this restriction in mind, the sum

$$\sum_{k\in\mathbb{Z}} \left| \sum_{\ell=0}^{\lceil 2\Omega\alpha \rceil - 1} \int_0^{1/\alpha} \widehat{f}(\xi + \ell/\alpha - \Omega)\, \mathrm{e}^{-2\pi \mathrm{i}\alpha k\xi} \varphi_n\left(\frac{\xi + \ell/\alpha}{\Omega} - 1\right) \mathrm{d}\xi \right|^2$$

is the sum of squares of Fourier coefficients (up to a factor α) of the function

$$\sum_{\ell=0}^{\lceil 2\Omega\alpha \rceil - 1} \widehat{f}(\xi + \ell/\alpha - \Omega)\, \varphi_n\left(\frac{\xi + \ell/\alpha}{\Omega} - 1\right)$$

and hence is equal to

$$\frac{1}{\alpha} \int_0^{1/\alpha} \left| \sum_{\ell=0}^{\lceil 2\Omega\alpha \rceil - 1} \widehat{f}(\xi + \ell/\alpha - \Omega)\, \varphi_n\left(\frac{\xi + \ell/\alpha}{\Omega} - 1\right) \right|^2 \mathrm{d}\xi.$$

The identity (4) follows. This completes the proof of Lemma 9 and hence of Theorem 10. □

5. Critical sampling and prolate shift Riesz bases

The arguments above show that if $2\Omega\alpha \notin \mathbb{N}$ then there will be some excess in the density of frame coefficients. When $2\Omega\alpha = N \in \mathbb{N}$ it is possible that the family $\{\varphi_n(\cdot - \alpha k) : k \in \mathbb{Z}, n = 0, \ldots, N-1\}$ may form a Riesz basis for $\mathrm{PW}_{2\Omega}$ where, as before, the duration is $2T$ with $T = 1$ so that the duration–bandwidth product is 4Ω. In this case the condition that there are $N = 2\Omega\alpha$ frame coefficients per α units of time means that there is, on average, one frame coefficient per unit time per unit bandwidth.

To consider conditions for a Riesz basis we proceed in a manner analogous to that in the previous section with the goal of showing that the quantity $f = \sum_{n=0}^{N-1} \sum_{k\in\mathbb{Z}} c_{nk}\, \varphi_n(\cdot - \alpha k)$ satisfies $\|f\|^2 \sim \sum_{n=0}^{N-1} \sum_k |c_{nk}|^2$ when $N = 2\Omega\alpha \in \mathbb{N}$ as we assume henceforth. Computing as before we see that

$$\|f\|^2 = \|\widehat{f}\|^2 = \int_{-\Omega}^{\Omega} \left| \sum_n \sum_k c_{nk} \mathrm{e}^{-2\pi \mathrm{i}\alpha k\xi} \widehat{\varphi}_n(\xi) \right|^2 \mathrm{d}\xi$$

$$= \int_{-\Omega}^{\Omega} \left| \sum_{n=0}^{N-1} \frac{(-\mathrm{i})^n}{\sqrt{\lambda_n \Omega}} \sum_k c_{nk} \mathrm{e}^{-2\pi \mathrm{i}\alpha k\xi} \varphi_n\left(\frac{\xi}{\Omega}\right) \right|^2 \mathrm{d}\xi$$

$$= \sum_{\ell=0}^{\lceil 2\Omega\alpha\rceil -1} \int_0^{1/\alpha} \left| \sum_{n=0}^{N-1} \frac{(-\mathrm{i})^n}{\sqrt{\lambda_n \Omega}} \sum_k c_{nk} \mathrm{e}^{-2\pi \mathrm{i}\alpha k\xi} \varphi_n\left(\frac{\xi+\ell/\alpha}{\Omega} - 1\right) \right|^2 \mathrm{d}\xi$$

$$= \sum_{\ell=0}^{N-1} \int_0^{1/\alpha} \left| \sum_{n=0}^{N-1} \frac{1}{\sqrt{\lambda_n \Omega}} \sum_k \widetilde{c}_{nk} \mathrm{e}^{-2\pi \mathrm{i}\alpha k\xi} \varphi_n\left(\frac{\xi+\ell/\alpha}{\Omega} - 1\right) \right|^2 \mathrm{d}\xi \qquad (7)$$

where $\widetilde{c}_{nk} = (-\mathrm{i})^n c_{nk}$ (note that $|\widetilde{c}_{nk}| = |c_{nk}|$).

Consider the matrix $Q(\xi)$:

$$Q = \sum_{\ell=0}^{N-1} Q^\ell; \qquad Q^\ell_{n,m}(\xi) = \frac{1}{\Omega\sqrt{\lambda_n\lambda_m}} \varphi_n\left(\frac{\xi+\ell/\alpha}{\Omega} - 1\right) \varphi_m\left(\frac{\xi+\ell/\alpha}{\Omega} - 1\right), \quad 0 \le n, m < N \qquad (8)$$

where, again, $N = 2\Omega\alpha \in \mathbb{N}$. This matrix is different from the matrix $M(\xi)$ considered previously in that the roles of the shift parameter ℓ and the prolate parameter n are reversed. As before, though, each $Q^\ell(\xi)$ is nonnegative definite since it has the form $U_\ell(\xi)\, U_\ell^T(\xi)$ for a real vector $U_\ell(\xi)$ in $\mathbb{R}^N$ thought of as a column vector whose nth coordinate is $\varphi_n(\frac{\xi+\ell/\alpha}{\Omega} - 1)/\sqrt{\lambda_n \Omega}$. In order that $Q(\xi)$ is strictly positive definite for a given value of ξ, it must be the case that there is no unit vector $\vec{c}$ in $\mathbb{R}^N$ such that $\vec{c}$ is orthogonal to $U_\ell(\xi)$ for each $\ell = 0, \ldots, N-1$. But such orthogonality would imply that $\vec{c}$, thought of as a column vector, is in the kernel of the matrix whose ℓth row is $U_\ell(\xi)$. That is, the $N \times N$ matrix whose (ℓ, n)th entry is $\varphi_n(\frac{\xi+\ell/\alpha}{\Omega} - 1)/\sqrt{\lambda_n \Omega}$ must be a singular matrix. This contradicts that the functions $\{\varphi_n\}_{n=0}^{N-1}$ (and any nonzero constant multiples of them) form a Chebyshev system on $[-1, 1]$ just as before. As in Proposition 1, by compactness, we conclude that the spectrum of $Q(\xi)$, as a function of $\xi \in [0, 1/\alpha]$, is uniformly bounded below in $(0, \infty)$.

The following paraphrases [17, Corollary 3.4].

Theorem 11. *Let $Q : [0, 1/\alpha) \to \mathbb{C}^{N\times N}$ be a positive definite matrix-valued function. The trigonometric system $\{\mathrm{e}^{2\pi \mathrm{i}k\alpha\xi}\, \mathbf{e}_n\}_{n=0,\, k\in\mathbb{Z}}^{N-1}$ forms a Riesz basis for $L^2([0, 1/\alpha), \mathbb{C}^N; Q)$ with squared-norm $\int_0^{1/\alpha} (F^*QF)(\xi)\, \mathrm{d}\xi$, if and only if there exist constants $0 < c \le C < \infty$ such that the spectrum $\sigma(Q(\xi))$ of $Q(\xi)$ satisfies*

$$c \le \min \sigma\bigl(Q(\xi)\bigr) \quad \textit{and} \quad C \ge \max \sigma\bigl(Q(\xi)\bigr), \quad \textit{a.e. } \xi \in [0, 1/\alpha).$$

In (7) one associates to $f = \sum_{n=0}^{N-1} \sum_{k\in\mathbb{Z}} c_{nk} \varphi_n(\cdot - \alpha k)$ the vector function $F(\xi) = \sum_n \sum_k \widetilde{c}_{nk}\, \mathrm{e}^{2\pi \mathrm{i}k\alpha\xi}\, \mathbf{e}_n$ with $\mathbf{e}_n$ the nth standard basis vector. In this case, $\|f\|^2 = \|F\|_{L^2([0,1/\alpha),\, \mathbb{C}^N; Q)}$ with Q in (8). By Theorem 11 and the observations made on Q in (8), we conclude that the functions $\{\mathrm{e}^{2\pi \mathrm{i}k\alpha\xi} \mathbf{e}_n\}_{n=0,\, k\in\mathbb{Z}}^{N-1}$ form a Riesz basis for $L^2([0, 1/\alpha), \mathbb{C}^N; Q)$ and, therefore, that for this F one has $\|f\|^2 = \|F\|^2_{L^2([0,1/\alpha),\, \mathbb{C}^N; Q)} \sim \sum_{nk} |\widetilde{c}_{nk}|^2 = \sum_{nk} |c_{nk}|^2$. That is, the functions $\{\varphi_n(\cdot - \alpha k)\}_{n=0,\, k\in\mathbb{Z}}^{N-1}$ form a Riesz basis for $\mathrm{PW}_{2\Omega}$. We have proved the following theorem.

Theorem 12. *Let φ_n denote the nth prolate bandlimited to $[-\Omega, \Omega]$ and time-concentrated in $[-1, 1]$. If $N = 2\Omega\alpha \in \mathbb{N}$ then the functions $\{\varphi_n(\cdot - \alpha k)\}_{n=0, k\in\mathbb{Z}}^{N-1}$ form a Riesz basis for $\mathrm{PW}_{2\Omega}$. That is, there exist constants $0 < A \le B < \infty$ such that for any sequence $\{c_{nk}\}_{n=0,\, k\in\mathbb{Z}}^{N-1} \in \ell^2(\mathbb{Z}^N)$ one has*

$$A \sum_{nk} |c_{nk}|^2 \le \Bigl\| \sum_{nk} c_{nk}\, \varphi_n(\cdot - \alpha k) \Bigr\|^2 \le B \sum_{nk} |c_{nk}|^2.$$

Technically, the arguments above show that $\{\varphi_n(\cdot - \alpha k)\}_{n=0, k\in\mathbb{Z}}^{N-1}$ forms a Riesz basis for its span. However, by Theorem 10 they form a frame for $\mathrm{PW}_{2\Omega}$ and hence span $\mathrm{PW}_{2\Omega}$. As in the case of Theorem 10, the

Riesz basis bounds depend on the spectrum of the matrix Q—the constant A in Theorem 12 can be taken as $A = \inf_\xi \min(\sigma(Q(\xi)))/\alpha$. However, so far there is no simple analytical estimate for this lower bound.

In the case $\alpha = 1/2\Omega$ one has $N = 1$. In this case the theorem states that the shifts $\varphi_0(\cdot - k/2\Omega)$ form a Riesz basis for $\mathrm{PW}_{2\Omega}$ and the lower Riesz basis bound can be computed explicitly as the infimum of $|\varphi_0(t)|^2/\lambda_0$ on the interval $[-1, 1]$ as in Corollary 8, cf., [27]. This lower bound is very small for large values of Ω, cf. the bottom curve in Fig. 1 for $N = 3$. It appears that one obtains better frame bounds using $N \approx 4\Omega$ and $\alpha \approx 2$, see Fig. 2.

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