Sampling aspects of approximately time-limited multiband and bandpass signals

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***Abstract*—We provide an overview of recent progress regarding the role of sampling in the study of signals that are in the image of a bandpass or multiband frequency limiting operation and have most of their energies concentrated in a given time interval. We finish considering a means to approximate essentially time-limited bandpass signals. In this case we present a new phase-locking metric that arises in the study of EEG signals.**

I. INTRODUCTION

We discuss relationships between time and band limiting and sampling, leading also to numerical computation of essentially time-limited multiband and bandpass signals. As an application we propose a method to analyze phase synchrony of bandpass projections of signals, illustrating a particular case of electroencephalographic (EEG) signals. In this introductory section we will review several elements of the theory of time and band limiting. In Section II we review connections between sampling and time and bandlimiting. In Section III we present a method to construct time- and multiband-limited signals from eigenfunctions for time and band limiting to separate bands and a numerical technique that takes advantage of sampling. In Section IV we provide a method to approximate essentially time-limited bandpass signals. In Section V we use a corresponding time localized projection to provide a new method to study phase differences of bandpass projections of signals. We illustrate the method in the context of study of EEG signals. In this setting, it is believed that relatively constant phase lag among two EEG channels can indicate involvement of the corresponding cortical regions in a distributed cognitive process.

A. *Time and band limiting*

Set $(Q_T)(f)(t) = \mathbb{1}_{[-T,T]}(t)\, f(t)$ where $\mathbb{1}_S$ denotes the function equal to one on $S \subset \mathbb{R}$ and zero outside S. Let $Q = Q_1$. Also let $(P_\Sigma)(f)(t) = (\mathbb{1}_\Sigma \widehat{f})^\vee(t)$ where $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-2\pi \mathrm{i} t\xi}\, \mathrm{d}t$. We write $P_\Omega = P_{[-\Omega/2,\Omega/2]}$ and $P = P_1$. The *Paley–Wiener* space PW_Σ is of the image of $L^2(\mathbb{R})$ under the orthogonal projection P_Σ. We write PW_Ω instead when $\Sigma = [-\Omega/2, \Omega/2]$ and simply PW when $\Omega = 1$. For compact Σ, the operator $P_\Sigma Q_T$ is compact and its trace is equal to $2T|\Sigma|$ where $|\Sigma|$ denotes the Lebesgue measure of $\Sigma \subset \mathbb{R}$. It is also self adjoint on PW_Σ while $P_\Sigma Q_S P_\Sigma$ is self adjoint on $L^2(\mathbb{R})$. Since functions in PW_Σ are real analytic, $P_\Sigma Q_S$ has no unit eigenfunctions and the discrete spectrum of $P_\Sigma Q_T$ is contained in $[0, 1)$.

B. *Prolate functions and their properties*

Prolate spheroidal wave functions, or prolates for short, on the interval $[-1, 1]$ are eigenfunctions of

$$\mathcal{P}_c\, \varphi_n(t) = \chi_n\, \varphi_n(t); \quad \mathcal{P}_c = \frac{\mathrm{d}}{\mathrm{d}t}\,(t^2-1)\,\frac{\mathrm{d}}{\mathrm{d}t} + c^2 t^2\,. \quad (1)$$

Such functions extend to elements of $L^2(\mathbb{R})$. The operator $\mathcal{P}_c$ commutes with $P_{c/\pi}Q$ and thus the prolates also form a complete orthogonal basis for $\mathrm{PW}_{c/\pi}$ and they are also eigenfunctions of the integral operator

$$(F_c f)(t) = \int_{-1}^{1} \mathrm{e}^{\mathrm{i}cst}\, f(s)\, \mathrm{d}s \, = \widehat{Qf}(-ct/2\pi)\,. \quad (2)$$

The eigenvalues $P_{c/\pi}Q$ are nondegenerate. Denote by $\lambda_0(c) > \lambda_1(c) > \dots$ the nth eigenvalue of $P_{c/\pi}Q$ and φ_n^c the corresponding eigenfunction. That φ_n^c is an eigenfunction of (2) and some other basic properties imply that

$$D_{c/\pi}\widehat{\varphi_n^c} = \frac{\mathrm{i}^n}{\sqrt{\lambda_n}} Q\varphi_n^c \quad (3)$$

where D_a is the unitary dilation $(D_a f)(t) = \sqrt{a} f(at)$, $a > 0$. When $L^2(\mathbb{R})$-normalized, the prolates $\{\varphi_n^c\}$ form an orthonormal basis for $\mathrm{PW}_{c/\pi}$, as well as a complete, orthogonal set in $L^2[-1, 1]$ with $\lambda_n(c) = \int_{-1}^{1} |\varphi_n^c|^2$. As such, any $f \in \mathrm{PW}_{c/\pi}$ can be expanded in the form $f = \sum_{n=0}^{\infty} \alpha_n\, \varphi_n^c$ with $\|f\|^2_{L^2(\mathbb{R})} = \sum \alpha_n^2$ and $\int_{-1}^{1} |f|^2 = \sum \lambda_n \alpha_n^2$. The prolates are real valued and φ_n^c is even (odd) if n is even (odd). Further properties of prolates and justification of the facts just mentioned, which were established in the Bell Labs papers [1]–[3], can be found in [4].

C. *The $2\Omega T$ theorem*

Suppose that Σ is a union of M pairwise disjoint frequency intervals of unit length so that the total time–bandwidth product corresponding to $P_\Sigma Q_T$ is $2MT$. Denote by $\mathcal{N}(2MT, \alpha)$ the number of eigenvalues of $P_\Sigma Q_T$ larger than α. The following is a special case of a version of the "$2\Omega T$" theorem proved by Landau and Widom in [5].

Theorem 1 (Landau–Widom, 1980): As $T \to \infty$ the number of eigenvalues of $P_\Sigma Q_T$ exceeding $\alpha \in (0,1)$ satisfies

$$\mathcal{N}(2MT,\alpha) = 2MT + \frac{M}{\pi^2}\log 2T \log\Big(\frac{\alpha}{1-\alpha}\Big) + o(\log 2MT)\,.$$

II. Sampling and time and band limiting

A. Local approximation by shifted sinc functions

Walter and Shen [6] and Khare and George [7] observed

$$(PQ_T f)(t) = \sum_{n=0}^{\infty} \lambda_n \sum_{k=-\infty}^{\infty} f(k)\varphi_n(k)\,\varphi_n(t)$$

where φ_n are eigenfunctions of PQ_T. Oscillatory behavior of the prolates near the endpoints of $[-T,T]$ prohibits an estimate $\sum_{|k|>T}\varphi_n^2(k) \le C(T)(1-\lambda_n)$. However, in [8] the estimate

$$\sum_{|k|>M(T)} \varphi_n^2(k) \le C(1-\lambda_n), \quad M(T) = \pi^2 T(1+\log^\gamma(T)) \tag{4}$$

was proved for any $\gamma > 1$. The $\log^\gamma(T)$ factor arises from use of a Fourier bump function ψ, i.e., $\widehat{\psi} = 1$ on $[-1/2, 1/2]$ and $\widehat{\psi}$ has compact support. The best known temporal decay of such a function is $|\psi(t)| = o(\exp(-c|t|/\log^\gamma(t))$. It was conjectured in [8] that the log factor is not necessary. The following consequence of (4) was also established in [8].

Theorem 2: Let $f \in \mathrm{span}\,\{\varphi_n\}_{n=0}^N$, with φ_n the nth eigenfunction of PQ_T. Define $\varphi_n^T = \sum_{|k|<M(T)} \varphi_n(k)\mathrm{sinc}\,(t-k)$ with $M(T)$ as in (4). Then

$$\|Q_T(f - \sum_{n=0}^{N}\langle f, \varphi_n^T\rangle \varphi_n^T)\| \le C\|f\| \sum_{n=0}^{N}\lambda_n(1-\lambda_n)\,.$$

The last sum can be shown to be essentially a multiple of λ_N. A method to obtain accurate numerical estimates of integer samples of prolates is outlined in Hogan et al., [8].

III. Time- and multiband-limited signals

This section reviews techniques underlying numerical computation of certain time- and multiband-limited signals. We start with a method for building eigenfunctions for the case in which Σ is a finite union of intervals from appropriately modulated prolates.

A. Eigenfunctions for unions

If Σ is a finite union of pairwise disjoint intervals $I_1, \dots, I_M$ then we can denote $P_\Sigma = \sum_{k=1}^M P_{I_k}$. Unlike PQ_T, the operator $P_\Sigma Q_T$ does not commute with a finite order differential operator with polynomial coefficients when Σ is a union of two or more intervals. This important fact, established by Morrison in [9], bars us from using power series methods to compute eigenfunctions.

The following results were established in [10] in a more general setting. If J is a frequency interval of unit length then the orthogonal projection onto PW_J, the Paley–Wiener subspace of $L^2(\mathbb{R})$ of functions frequency supported in J, has the form $M_{m_J} P M_{-m_J}$ where, as before, $P = P_{[-1/2,1/2]}$ and $(M_u f)(t) = \mathrm{e}^{2\pi \mathrm{i} tu} f(t)$ with m_{J_k} the midpoint of J_k. Suppose that one has M pairwise disjoint frequency intervals $J_1, \dots, J_M$ each of unit length and set $\Sigma = \cup_k J_k$. Set $m_k = m_{J_k}$. Since the J-prolates $\varphi_n^J = M_{m_J}\varphi_n$, with φ_n the corresponding eigenfunction of PQ_T, form a complete family for PW_J, any function in PW_Σ has an orthogonal decomposition $f = \sum_{k=1}^M \sum_{n=0}^\infty \langle f,\, M_{m_k}\varphi_n\rangle M_{m_k}\varphi_n$. Consider now the problem of finding an eigenvalue–eigenfunction pair $(\lambda,\, \psi)$ for $P_\Sigma Q_T$. Expanding ψ in terms of the modulated prolates $M_{m_k}\varphi_n$ and applying $P_\Sigma Q_T$ to these, one sees that one must identify the coefficients $\Gamma_{nm}^{k,\ell} = \langle Q_T M_{m_k}\varphi_n,\, M_{m_\ell}\varphi_m\rangle$. Note that $\Gamma_{mn}^{\ell,k} = \overline{\Gamma_{mn}^{k,\ell}}$, that is, if $\Gamma^{k,\ell}$ is the matrix with entries $\Gamma_{nm}^{k,\ell}$ then $\Gamma^{\ell,k} = \overline{\Gamma^{k,\ell}}$. The following lemma describes how to produce eigenvalue–eigenfunction pairs $(\lambda,\, \psi)$ for $P_\Sigma Q_T$ from the standard prolates.

Proposition 3: Suppose that $J_1, \dots, J_M$ are pairwise disjoint unit intervals with union $\Sigma = \cup_{k=1}^M J_k$. Let Λ denote the diagonal matrix with nth diagonal entry $\lambda_n(PQ_T)$ and let $\Gamma^{k,\ell}$ be the matrix with entries $\gamma_{nm}^{k,\ell} = \langle Q_T M_{m_k - m_\ell}\varphi_n,\, \varphi_m\rangle$, $k < \ell$. Then any eigenvector–eigenvalue pair ψ and λ for $P_\Sigma Q_T$ can be expressed as $\psi = \sum_{k=1}^M \sum_{n=0}^\infty \alpha_n^k M_{m_k}\varphi_n$ where the vectors $\boldsymbol{\alpha}_k = \{\alpha_n^k\}$ together form a discrete eigenvector for the block matrix eigenvalue problem

$$\lambda \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_M \end{pmatrix} = \begin{pmatrix} \Lambda & \bar{\Gamma}^{12} & \cdots & \bar{\Gamma}^{1M} \\ (\Gamma^{12})^T & \Lambda & \bar{\Gamma}^{23} & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ (\Gamma^{1M})^T & \cdots & \cdots & \Lambda \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_M \end{pmatrix}.$$

This method applies, in principle, to any family of orthogonal projections $P_1, \dots, P_M$ (i.e., $P_\nu P_\mu = \delta_{\nu\mu}\, P_\mu$) onto closed subspaces of a Hilbert space and Hermitian operator Q. However, such an approach does not necessarily lead to computable eigenproblems. In order to turn the method into a means to compute eigenvalues and eigenfunctions of $P_\Sigma Q_T$ numerically, one needs to estimate the coefficients

$$\Gamma_{nm}^{k,\ell} = \int_{-T}^{T} \mathrm{e}^{2\pi \mathrm{i}(m_k - m_\ell)t}\varphi_n(t)\,\varphi_m(t)\,\mathrm{d}t$$

and to justify truncating the matrices Λ and $\Gamma^{k,\ell}$. The matrix truncations are justified by Theorem 1.

The corresponding Γ-matrix entries can be expressed via the inner products

$$\langle Q_T M_{m_I}\varphi_n,\, M_{m_J}\varphi_m\rangle = \sum_k \sum_\ell \varphi_n(k)\varphi_m(\ell) A(T; I, J)_{k\ell};$$

$$A(T; I, J)_{k\ell} = \int_{-T}^{T} \mathrm{e}^{2\pi \mathrm{i}(m_I - m_J)t}\,\mathrm{sinc}\,(t-k)\mathrm{sinc}\,(t-\ell)\,\mathrm{d}t\,.$$

The inner products can be computed using the following proposition derived in [10].

Lemma 4: As a bilinear form acting on the pair of sequences $\{\varphi_n(k)\}$, $\{\varphi_m(\ell)\}$, the matrix $A(T; I, J)_{k\ell}$ coincides with $\mathrm{i}^{n+m}\sqrt{\lambda_m\lambda_n}\,\mathrm{sinc}\,(2T(m_J - m_I) + k - \ell)$.

An eigenfunction ψ of $P_\Sigma Q_T$ will be called a *time- and multiband-limiting eigenfunction* (TMBLE). If ψ is a TMBLE with eigenvalue $\lambda > 1/2$ then ψ should be, at least nearly, in the span of those eigenfunctions φ_n^I, where $\Sigma = \cup I$,

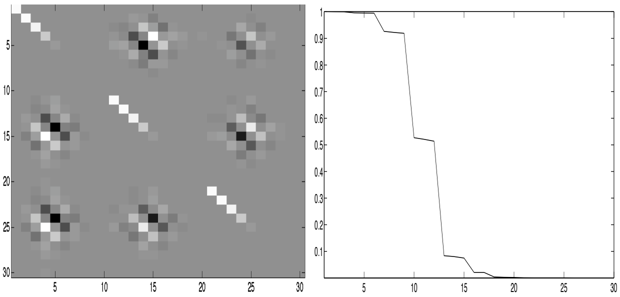

Fig. 1. *Matrix in Proposition 3 for $T = 2$, $I = [-1/2, 1/2]$, $J = [2, 3]$, and $K = [5, 6]$.* Intensity plot of the real part of the matrix in Proposition 3. Each $\Gamma^{\mu\nu}$ term is truncated to size 10×10. On the right is a plot of the moduli of the eigenvalues of the same matrix.

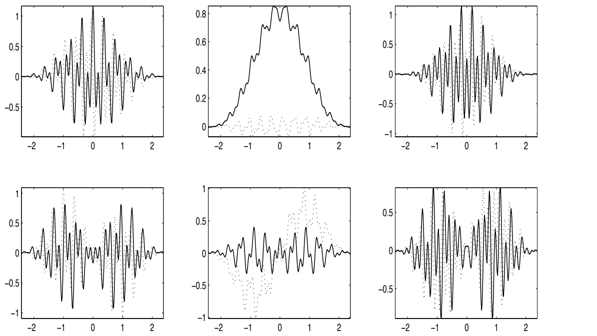

Fig. 2. *TMBLEs for $T = 2$, $I = [-1/2, 1/2]$, $J = [2, 3]$ and $K = [5, 6]$.* Plotted are the TMBLMS corresponding to $n = 0, 1, 2, 3, 4, 5$ respectively for three frequency bands. Real parts solid, imaginary parts dashed.

corresponding to the eigenvalues of $P_I Q_T$ larger than 1/2, hence, of eigenfunctions φ_n^I corresponding to $n \leq 2T$. In this case, φ_n^I can be approximated accurately on $[-T, T]$ by sinc interpolating its samples $\varphi_n^I(k)$ where $|k| \leq M(T)$ above.

B. Numerical estimation of TMBLEs

Accurate numerical estimation of the TMBLEs is obtained via estimation of the entries of suitable truncations of the Γ matrices and eigenvectors of the corresponding truncation of the eigenproblem in Proposition 3. Details are given in [10]. Figure 1 illustrates the case with three frequency intervals.The corresponding eigenfunctions are plotted in Fig. 2.

IV. Time- and bandpass-limited signals

Given $0 < c' < c$ denote by $\mathrm{PW}^{\pi}_{c',c}$ the orthogonal complement of $\mathrm{PW}_{c'/\pi}$ inside $\mathrm{PW}_{c/\pi}$, that is, the closed subspace of $L^2(\mathbb{R})$ of functions whose Fourier transforms $\widehat{f}(\xi)$ are supported in $c'/\pi \leq |\xi| \leq c/\pi$, and by $P^{\pi}_{c'c}$ the orthogonal projection onto $\mathrm{PW}^{\pi}_{c'c}$. The eigenfunctions of the operator $P^{\pi}_{c'c}Q$ corresponding to time truncation of a function in $L^2(\mathbb{R})$ to a finite interval—$[-1, 1]$ in this work—followed by frequency limiting to frequencies $|\omega| \in [c', c]/\pi$ will be called *bandpass prolates* here. Numerical approximation of the most time concentrated bandpass limited signals (called *bandpass prolates* here) was studied recently by SenGupta et al., [11] by expressing the kernel of the bandpass limiting operator in terms of Legendre polynomials, then identifying the bandpass prolates through their Legendre coefficients. Alternatively, Proposition 5 proved in [12], produces the coefficients of the bandpass prolates, expressed as superpositions of full-band prolates, from *partial inner products* of full-band prolates. As explained below, these partial inner products can be computed directly from pointwise values of φ_n^c and $\varphi_n^{c'}$ where, as before, φ_n^c is the nth eigenfunction of $P_{c/\pi}Q$.

Denote by $R = R(c', c)$ the matrix with entries $R_{jk} = \frac{\mathrm{i}^{k-j}}{\sqrt{\lambda_j \lambda_k}} \int_{-c'/c}^{c'/c} \varphi_k^c(\xi)\, \varphi_j^c(\xi)\, \mathrm{d}\xi$. The matrix R is real symmetric, a consequence of the parity properties of the φ_n^c. Let $\Lambda = \Lambda(c)$ be the diagonal matrix with nth diagonal entry $\lambda_n(c)$.

Proposition 5: If $\psi = \sum \alpha_n \varphi_n^c \in \mathrm{PW}_{c/\pi}$ then

$$P^{\pi}_{c'c}\, Q\, \psi = \sum_k \alpha_k \lambda_k \left(\varphi_k^c - \sum_j R_{jk}\, \varphi_j^c \right).$$

In particular, if $\psi = \sum \alpha_n \varphi_n^c$ is an eigenfunction of $P^{\pi}_{c'c}Q$ with eigenvalue λ then, with $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{\infty}$,

$$\lambda \alpha_n = \lambda_n \alpha_n - \sum_k \lambda_k\, \alpha_k\, R_{nk} \quad \textit{i.e.} \quad \lambda \boldsymbol{\alpha} = (I - R)\Lambda \boldsymbol{\alpha}\,.$$

The discrete eigenvectors $\boldsymbol{\alpha}$ of the matrix $(I - R)\Lambda$ thus give rise to eigenfunctions of $P^{\pi}_{c'c}Q$ and the eigenvalue λ measures the concentration of ψ in $[-1, 1]$ just as in the case of standard prolates. The proof uses the identities (2) and (3).

The partial inner products can be calculated by virtue of the prolate differential equation and integration by parts. If $n \neq m$ then, with χ_n as in (1) and $-1 \leq a \leq b \leq 1$,

$$\big(\chi_n - \chi_m\big) \int_a^b \varphi_n(t)\varphi_m(t)\mathrm{d}t = \Big[(t^2-1)(\varphi_n'\varphi_m - \varphi_m'\varphi_n)(t)\Big|_a^b.$$

Approximate bandpass prolates are obtained from finite size truncations of the eigenproblem in Proposition 5, see [12].

Khare [13] also considered the problem of numerical evaluation of bandpass prolates, focusing instead on the role of the interpolating function (sinc multiplied by a suitably dilated cosine) and establishing that the bandpass prolate samples form a discrete eigenvector of the matrix of partial integrals on $[-1, 1]$ of shifts of the interpolating kernel, cf. also Hogan et al., [8]. Khare did not investigate dependence on c'/c.

V. Phase synchrony and an application to EEG

In this final section we want to discuss briefly an application of bandpass prolates in the study of phase synchrony, particularly in the context of EEG signals. It has long since been argued that, in mental tasks that recruit different regions of neural cortex, communication between the regions is manifest in phase synchrony of neural firing patterns, e.g., [14], [15]; in particular, attention focusing tasks are hypothesized to manifest such synchrony in the *gamma band*, e.g., [16]. Different methods have been proposed to measure band specific synchrony in EEG channel signals, including filtered analytic signals, convolutions with modulated Gaussians [17], and others. Each of these methods that have been proposed to

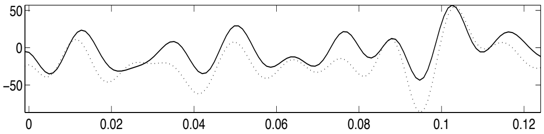

Fig. 3. *EEG channel data* $1/8$ second record of two concurrent EEG channel measurements, digitally sampled at 1024 samples per second.

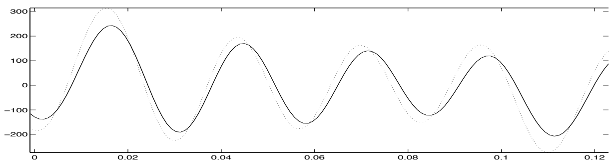

Fig. 4. *Approximate γ projections.* Projections of channel measurements onto the span of the six top eigenvectors of time-limiting to $1/8$ second and bandpass limiting to 24–40 Hz.

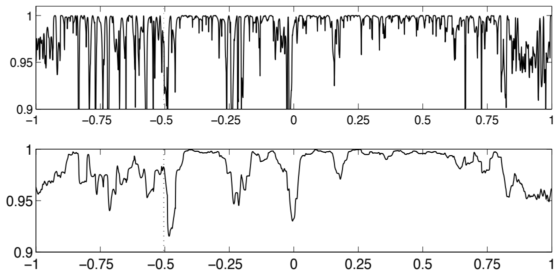

Fig. 5. *Phase locking.* Top shows PLVs of projections in Fig. 4 computed over the full two second record. Bottom shows PLVs time averaged over 20 consecutive time shifts.

quantify phase locking in EEG is subject to criticism for different reasons.

To assign a temporal phase-locking value to a pair of signals in a given band, one averages instantaneous phase difference over a duration that accounts for several oscillations, say three to five, in order that an average phase difference makes sense, but not so many oscillations that one is unable to distinguish episodes of phase synchrony from asynchronous epochs.

We consider here a new phase-locking metric computed through the following steps. *Step 1*: define the duration and frequency band for which synchrony is to be measured. *Step 2*: define the projection onto the span of the bandpass prolates whose eigenvalues are close to one or, at least, not much smaller than one half. *Step 3*: compute the analytic signal for this projection, and divide by its amplitude to get its unimodular factor. *Step 4*: For a pair of such signals, multiply the unimodular part of one by the conjugate of that of the other, integrate over the given duration, and compute the modulus. This is the *phase locking value* (PLV).

We implemented this algorithm as follows to produce Fig. 5. To analyze the gamma band of EEG signals, we chose the frequency range from 24 to 40 Hz. In order to compute the PLV over 3 to 5 oscillations of signals in this range, we took the duration of interest to be $1/8$ second. The time bandwidth product in this case is $2(40-24)/8 = 4$. The corresponding time- and bandpass-limiting operator has six eigenvalues "not much smaller than $1/2$." We successively chose $1/8$ second blocks of the EEG channels and computed the projections onto the span of the first six eigenfunctions. We then computed the analytic signal using the `matlab` builtin `hilbert`. A PLV was computed for each successive $1/8$-second segment of the two EEG channels.

Fig. 5 shows PLVs of projections of $1/8$-seconds of the two EEG channels onto the space generated by the six eigenfunctions of time limiting to $1/8$-second duration and bandpass limiting to 24–40 Hz most concentrated to the given duration. The PLVs were computed for $1/8$-second duration. In the data presented, a visual stimulus was shown to the subject after a half second. An initial interval of synchrony then presumably reflects response of the visual cortex. The subsequent interval of synchrony after "$t = 0$" presumably then corresponds to the subject maintaining a mental representation of the stimulus.

REFERENCES

[1] D. Slepian and H. Pollak, "Prolate spheroidal wave functions, Fourier analysis and uncertainty. I," *Bell System Tech. J.*, vol. 40, pp. 43–63, 1961.

[2] H. Landau and H. Pollak, "Prolate spheroidal wave functions, Fourier analysis and uncertainty. II," *Bell System Tech. J.*, vol. 40, pp. 65–84, 1961.

[3] ——, "Prolate spheroidal wave functions, Fourier analysis and uncertainty. III. The dimension of the space of essentially time- and band-limited signals." *Bell System Tech. J.*, vol. 41, pp. 1295–1336, 1962.

[4] J. Hogan and J. Lakey, *Duration and Bandwidth Limiting*. Birkhäuser/Springer, New York, 2012.

[5] H. Landau and H. Widom, "Eigenvalue distribution of time and frequency limiting," *J. Math. Anal. Appl.*, vol. 77, pp. 469–481, 1980.

[6] G. Walter and X. Shen, "Sampling with prolate spheroidal wave functions," *Sampl. Theory Signal Image Process.*, vol. 2, pp. 25–52, 2003.

[7] K. Khare and N. George, "Sampling theory approach to prolate spheroidal wavefunctions," *J. Phys. A*, vol. 36, pp. 10 011–10 021, 2003.

[8] J. Hogan, S. Izu, and J. Lakey, "Sampling approximations for time- and bandlimiting," *Sampl. Theory Signal Image Process.*, pp. 91–117, 2010.

[9] J. A. Morrison, "On the eigenfunctions corresponding to the bandpass kernel, in the case of degeneracy," *Quart. Appl. Math.*, vol. 21, pp. 13–19, 1963.

[10] J. D. Lakey and J. A. Hogan, "On the numerical computation of certain eigenfunctions of time and multiband limiting." *Numer. Funct. Anal. Optim.*, vol. 33, no. 7-9, pp. 1095–1111, 2012.

[11] I. SenGupta, B. Sun, W. Jiang, G. Chen, and M. Mariani, "Concentration problems for bandpass filters in communication theory over disjoint frequency intervals and numerical solutions," *J. Fourier Anal. Appl.*, vol. 18, no. 1, pp. 182–210, 2012.

[12] J. Hogan and J. Lakey, "Letter to the editor: On the numerical evaluation of bandpass prolates," *J. Fourier Anal. Appl.*, pp. 1–8, 2013.

[13] K. Khare, "Bandpass sampling and bandpass analogues of prolate spheroidal functions," *Signal Process.*, vol. 86, no. 7, pp. 1550–1558, 2006.

[14] D. Hansel, G. Mato, and C. Meunier, "Synchrony in excitatory neural networks." *Neural Computation*, vol. 7, no. 2, pp. 307–337, 1995.

[15] F. Varela, J. P. Lachaux, E. Rodriguez, and J. Martinerie, "The brainweb: phase synchronization and large-scale integration." *Nature reviews. Neuroscience*, vol. 2, pp. 229–239, 2001.

[16] C. Tallon-baudry, O. Bertrand, F. Peronnet, and J. Pernier, "Induced gamma-band activity during the delay of a visual short-term memory task in humans," *J. Neurosci*, vol. 18, pp. 4144–4154, 1998.

[17] M. L. V. Quyen, J. Foucher, J. Lachaux, E. Rodriguez, A. Lutz, J. Martinerie, and Varela, "Comparison of Hilbert transform and wavelet methods for the analysis of neuronal synchrony," *J Neurosci Methods*, no. 111, pp. 83–98, 2001.