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Numerical Functional Analysis and Optimization

ISSN: 0163-0563 (Print) 1532-2467 (Online) Journal homepage: http://www.tandfonline.com/loi/lnfa20

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Numerical Functional Analysis and Optimization, 28(3–4):445–466, 2007

ISSN: 0163-0563 print/1532-2467 online
DOI: 10.1080/01630560701283342

CONSTRUCTION OF PERIODIC PROLATE SPHEROIDAL WAVELETS USING INTERPOLATION

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□ *Periodic prolate spheroidal wavelets (periodic PS wavelets), based on the periodizaton of the first prolate spheroidal wave function (PSWF), were recently introduced by the authors. Because of localization and other properties, these periodic PS wavelets could serve as an alternative to Fourier series for applications in modeling periodic signals. In this paper, we continue our work with periodic PS wavelets and direct our attention to their construction via interpolation. We show that they have a representation in terms of interpolation with the modified Dirichlet kernel. We then derive a group of formulas of interpolation type based on this representation. These formulas enable one to obtain a simple procedure for the calculation of the periodic PS wavelets and finding expansion coefficients. In particular, they are used to compute filter coefficients for the periodic PS wavelets. This is done for a number of concrete cases.*

Keywords Bandlimited signal; Paley–Wiener space; Periodic wavelets; PS wavelets; Spheroidal wave functions; Wavelets.

AMS Subject Classification 42C40; 65T60; 33E10; 42C05; 94A11; 94A12.

1. INTRODUCTION

The prolate spheroidal wave functions (PSWFs), $\{\varphi_{n,\sigma,\tau}(t)\}_{n\in Z}$, achieved prominence in the 1960s as solutions of an important energy concentration problem. These solutions were exploited by a group at Bell Labs ([9] and [10]) who used them to solve problems in communications theory ([5, 8]). The problem was one of finding the σ-bandlimited function on the real line (functions whose Fourier transforms have support on the interval $[-\sigma,\sigma]$) whose energy on $[-\tau,\tau]$ was maximized. This led

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to an integral equation satisfied by the PSWFs with the sinc function, $S(t) = \sin \pi t/\pi t$, as its kernel:

$$\int_{-\tau}^{\tau} \varphi_{n,\sigma,\tau}(x)\frac{1}{T}S\left(\frac{t-x}{T}\right)dx = \lambda_{n,\sigma,\tau}\varphi_{n,\sigma,\tau}(t), \tag{1.1}$$

where $T = \pi/\sigma$. The prolate spheroidal wavelets (PS wavelets) constructed and studied in [15] were generated by the first prolate spheroidal wave function $\varphi_{0,\sigma,\tau}(t)$. More recently, periodic PS wavelets were further introduced via periodization [14]. These periodic PS wavelets have many desirable properties lacking in other periodic wavelets [13], such as the closure under differentiation and translation of the associated subspaces.

However, both the PSWFs and the periodic PS wavelets are difficult to construct. In the former, a procedure based on a differential equation that the PSWFs also satisfied (Slepian's Lucky Accident [9]) is frequently used for the construction. However, in the case of our periodic PS wavelets, we have no such differential equation available. Rather, we use an alternate procedure based on interpolation. This procedure is similar to that in [18]. But rather than the sinc function used there, here we use a type of trigonometric interpolation that requires only the values of the PS wavelet scaling function at the integers.

This article is organized as follows: We begin with a brief review of definitions for PS wavelets, periodic PS wavelets, and some related properties. In the second section, we show periodic PS wavelets have a representation in terms of interpolation with the modified Dirichlet kernel. A number of formulas of interpolation type for the scaling function and wavelets is then derived and used to find their dilation equation coefficients. These formulas enable us to obtain a simple procedure for the calculation of periodic PS wavelets themselves and for the calculation of coefficients in the series approximation of a function. We then illustrate the use of the interpolation formulas to derive the filter coefficients of the PS wavelets numerically. The article is concluded by a brief summary.

1.1. PS Wavelets

We will assume that the readers are familiar with basic wavelet theory (for general references, see [2] or [17], for example). The scaling function $\phi(t)$ of the PS wavelets on R was defined as the first PSWF $\varphi_{0,\pi,\tau}(t)$, the one with maximum concentration on $[-\tau, \tau]$ among normalized π-bandlimited functions. Many of the calculations are based on the Fourier transform of the PSWF given by:

$$\widehat{\varphi}_{0,\sigma,\tau}(\omega) = A_{\sigma,\tau}\varphi_{0,\sigma,\tau}\left(\frac{\tau\omega}{\sigma}\right)\chi_{\sigma}(\omega), \tag{1.2}$$

where $A_{\sigma,\tau} = \sqrt{\frac{2\pi\tau}{\sigma\lambda_{0,\sigma,\tau}}}$, $\lambda_{0,\sigma,\tau}$ is the first (largest) eigenvalue of the associated integral operator (1.1), and $\chi_\sigma(\omega)$ is the characteristic function of the interval $[-\sigma,\sigma]$. The Fourier integral theorem inverts the Fourier transform and enables us to express $\phi = \varphi_{0,\pi,\tau}$ as

$$\phi(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{\phi}(\omega)e^{i\omega t}d\omega$$

Because ϕ is π-bandlimited, it can be represented by the sampling theorem as

$$\phi(t) = \sum_{n=-\infty}^{\infty}\phi(n)S(t-n). \tag{1.3}$$

This can be used to derive several properties such as the discrete orthogonality (More discussions on sampling theory of the PSWF $\varphi_{0,\pi,\tau}(t)$ can be found in [16]). If $\tilde{\phi}$ denotes the biorthogonal dual function, that is, the π-bandlimited function such that

$$\int_{-\infty}^{\infty}\phi(t)\tilde{\phi}(t-j)dt = \delta_{0j},$$

then by substitution of (1.3) we find that

$$\sum_{n=-\infty}^{\infty}\phi(n)\tilde{\phi}(n-j) = \delta_{0j}.$$

The PS mother wavelet was then defined as (see [15]):

$$\psi(t) = \varphi_{0,\pi/2,\tau/2}(t)\cos\left(\frac{3\pi}{2}t\right),$$

with

$$\hat{\psi}(\omega) = \frac{1}{2}\left\{\widehat{\varphi}_{0,\pi/2,\tau/2}\left(\omega-\frac{3\pi}{2}\right)+\widehat{\varphi}_{0,\pi/2,\tau/2}\left(\omega+\frac{3\pi}{2}\right)\right\}.$$

Neither the PS scaling function nor the PS mother wavelet are orthogonal to their translates, but these integer translates constitute a Riesz basis of an appropriate subspace. They were both used to construct periodic scaling functions and wavelets.

1.2. Periodic PS Wavelets

The periodic PS scaling function was defined by periodizing the PS scaling function [14]. At level m, the periodic PS scaling function is defined as

$$\phi^p_{m,0}(t) = \sum_{n=-\infty}^{\infty} \phi(2^m(t-n)). \tag{1.4}$$

This was shown to have the expression for $m > 0$,

$$\phi^p_{m,0}(t) = \frac{A_{\sigma,\tau}}{2^{m+1}}\left\{\phi(-\tau)e^{2\pi i 2^{m-1}t} + 2\sum_{k=-2^{m-1}+1}^{2^{m-1}-1} \phi\left(\frac{k\tau}{2^{m-1}}\right)e^{2\pi ikt} + \phi(\tau)e^{-2\pi i 2^{m-1}t}\right\}, \quad m = 0,1,\ldots, \tag{1.5}$$

where $A_{\sigma,\tau} = \sqrt{\frac{2\pi\tau}{\sigma\lambda_{0,\sigma,\tau}}}$, $\lambda_{0,\sigma,\tau}$ is the first (largest) eigenvalue of the associated integral equation (1.1). Notice that $\phi^p_{m,0}$ are a set of trigonometric polynomials of degree 2^{m-1}. The translates by $j2^{-m}$ give us the other functions at the same scale

$$\phi^p_{m,j}(t) = \phi^p_{m,0}(t - j2^{-m}), \quad m = 1,2,\ldots;\ j = 0,1,\ldots,2^m - 1. \tag{1.6}$$

This gives us the usual multiresolution decomposition of $L^2(0,1)$, that is, a nested sequence of subspaces V^p_m spanned by $\phi^p_{m,0}(t)$, $\phi^p_{m,1}(t),\ldots,\phi^p_{m,2^m-1}(t)$ with the property that

1. $V^p_0 \subset V^p_1 \subset \cdots \subset V^p_m \subset \cdots \subset L^2(0,1)$,
2. $\bigcup_{m=0}^{\infty} V^p_m$ is dense in $L^2(0,1)$.

Clearly, V^p_m is composed of trigonometric polynomials of degree $\leq 2^{m-1}$ and can be shown to contain all trigonometric polynomials of degree $< 2^{m-1}$. Hence $\bigcup_{m=0}^{\infty} V^p_m$ contains all trigonometric polynomials, which therefore is dense in $L^2(0,1)$.

We then defined the periodic PS wavelets as [14]:

$$\psi^p_{m,j}(t) = A_{\frac{\pi}{2},\frac{\tau}{2}} 2^{-m}\left[\sum_{k=1}^{2^m-1} \varphi_{0,\pi/2,\tau/2}\left(\frac{k\tau}{2^m}\right)\cos\frac{3\pi k\tau}{2^m}\cos 2\pi k\left(t - \frac{j}{2^m}\right) + \varphi_{0,\pi/2,\tau/2}(\tau)\cos(2^m 2\pi t)\right],$$
$$m = 1,2,\ldots;\quad j = 0,1,\ldots,2^m - 1.$$

a trigonometric polynomial of degree 2^m.

Figure 1 shows the periodic PS scaling function $\phi^p_{2,0}(t)$, $\tau = 1$ and the corresponding mother wavelet. Even at this relatively coarse scale, the

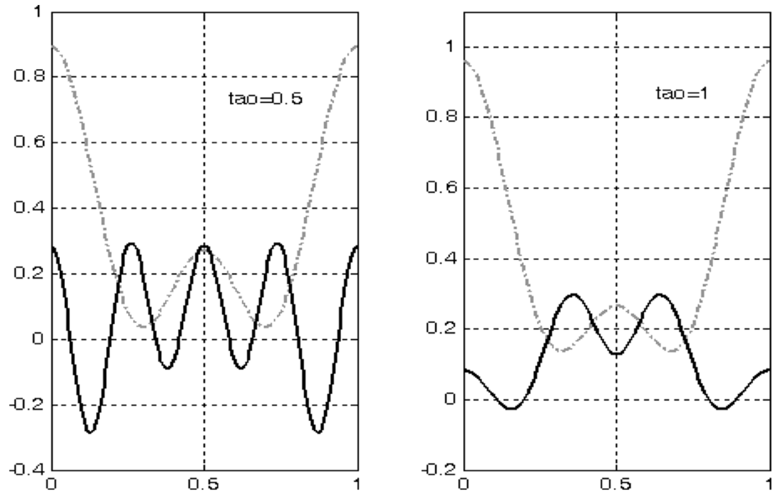

FIGURE 1 The periodic PS scaling functions $\phi^{p}_{2,0}(t)$ (solid line) and mother wavelet $\psi^{p}_{2,0}(t)$ (dotted line). Left: $\tau = 0.5$. Right: $\tau = 1$.

localization of the scaling function is evident. This is even more evident at finer scales as shown in [14].

2. INTERPOLATION OF PERIODIC PS WAVELETS

Many of the formulas involving these prolate spheroidal functions are related to the sinc function as we have seen. This function is itself a scaling function, and we can find the associated periodic scaling function and mother wavelet. The periodic extension at scale m is

$$s^{p}_{m,0}(t) = \sum_{n=-\infty}^{\infty} S(2^m(t-n)). \tag{2.1}$$

Because the Fourier transform of S is the characteristic function of the interval $[-\pi, \pi]$, (2.1) may be represented by the trigonometric polynomial

$$\begin{aligned} s^{p}_{m,0}(t) &= 2^{-1-m}\left(e^{2\pi i 2^{m-1}t} + 2\sum_{k=-2^{m-1}+1}^{2^{m-1}-1} e^{2\pi i k t} + e^{-2\pi i 2^{m-1}t}\right) \\ &= 2^{-m}\left(1 + 2\sum_{k=1}^{2^{m-1}-1}\cos(2\pi k t) + 1\cos(2^m\pi t)\right) = \frac{\sin(2^m\pi t)}{2^m\tan(\pi t)}. \end{aligned}$$

This is the modified Dirichlet kernel $D^{*}_{2^{m-1}}(2\pi t)$ of Fourier series theory ([20], p. 50). This kernel is used for trigonometric interpolation. It is very close to the scaling function of periodic PS wavelet $\phi^p_{m,0}(t)$ for small values of τ but has poor time localization compared with $\phi^p_{m,0}(t)$ for larger values of τ.

2.1. The Transform Matrix

As was shown in [14], the PS scaling function can be recovered from its values on the integers by using (1.3). In this section, we show that a similar formula holds in the periodic case as well. We first return to the sampling theorem (1.3) and use it to find a new formula for $\phi^p_{m,0}(t)$. It is given formally by

$$\begin{aligned}
\phi^p_{m,0}(t) &= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \phi(k)S(2^m(t-n)-k) \\
&= \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \phi(k)S(2^m(t-n-k2^{-m}) \\
&= \sum_{k=-\infty}^{\infty} \phi(k)D^*_{2^m-1}(2\pi(t-k2^{-m})) \\
&= \sum_{k=0}^{2^m-1}\left\{\sum_{l=-\infty}^{\infty}\phi(k+2^m l)\right\}D^*_{2^m-1}(2\pi(t-k2^{-m})) \\
&= \sum_{k=0}^{2^m-1}\phi^p_{m,0}(k2^{-m})D^*_{2^m-1}(2\pi(t-k2^{-m})). \qquad (2.2)
\end{aligned}$$

However, the last line is an immediate consequence of the interpolation theorem from Fourier series as well. Notice that it requires only the values of the PS scaling function (the first PSWF) at the integers. This expression can also be used to find the expansion of functions in V^p_m in terms of $\phi^p_{m,j}$. Because $\phi^p_{m,j}(t)=\phi^p_{m,0}(t-j2^{-m})$, we immediately have the formula

$$\phi^p_{m,j}(t) = \sum_{k=0}^{2^m-1}\left\{\sum_{l=-\infty}^{\infty}\phi(k+2^m l)\right\}D^*_{2^m-1}(2\pi(t-(k+j)2^{-m})). \qquad (2.3)$$

Now we suppose that $f\in V^p_m$ is given by

$$f(t) = \sum_{j=0}^{2^m-1} a_j\phi^p_{m,j}(t);$$

then f may be represented by its interpolation series similar to (2.2):

$$f(t) = \sum_{k=0}^{2^m-1} f(k2^{-m}) D^*_{2^{m-1}}(2\pi(t - k2^{-m}))$$
$$= \sum_{k=0}^{2^m-1} \sum_{j=0}^{2^m-1} a_j \phi^p_{m,j}(k2^{-m}) D^*_{2^{m-1}}(2\pi(t - k2^{-m})). \quad (2.4)$$

If we restrict ourselves to the values of t given by $\{0, 2^{-m}, 2 \cdot 2^{-m}, \ldots, (1 - 2^{-m})\}$ and denote by

$$\mathbf{f} = [f(0), f(2^{-m}), \ldots, f(1 - 2^{-m})]^T,$$
$$\mathbf{a} = [a_0, a_1, \ldots, a_{2^m-1}]^T$$

and

$$\Phi_m = [\phi^p_{m,0}((k - j)2^{-m})]_{k,j=0}^{2^m-1}, \quad (2.5)$$

then formula (2.4) can be written in matrix format as

$$\mathbf{f} = \Phi_m \mathbf{a}. \quad (2.6)$$

If the matrix Φ_m (the transform matrix) is nonsingular and well conditioned, then it can be inverted to find the coefficients. That is,

$$\mathbf{a} = \Phi_m^{-1} \mathbf{f}. \quad (2.7)$$

Fortunately, Φ_m has very good computational properties. Some of them are summarized in the following proposition.

Proposition 2.1 (*Properties of the transform matrix Φ_m*). *The 2^m by 2^m matrix Φ_m defined by (2.5), $m \in Z$, satisfy the following properties:*

(i) *Φ_m is a symmetric Toeplitz matrix with generator $r_m = \{r_m(n)\}^T$, where*

$$r_m(n) = \phi^p_{m,0}\left(\frac{n}{2^m}\right), n = 0, \ldots, 2^m - 1; \quad (2.8)$$

(ii) *Φ_m is a positive definite matrix;*

(iii) *The row sums of Φ_m are all equal to $\hat{\phi}(0)$, an eigenvalue of Φ_m with eigenvector $v_1 = [1, 1, 1, \ldots, 1]^T$;*

(iv) *The eigenvalues of* Φ_m *are bounded below by* $\hat{\phi}(\pi)$ *and above by* $\hat{\phi}(0)$, *independently of the scale parameter m; they are given by*

$$\hat{\phi}(0), \hat{\phi}(2^{-m}\pi), \ldots, \hat{\phi}((2^m - 1)2^{-m}\pi);$$

(v) *The condition number is bounded uniformly with respect to the scale parameter m and*

$$\lim_{m\to\infty} cond(\Phi_m) = \frac{2\tau}{\lambda_0}\phi(0)\phi(\tau).$$

Proof. (i) The statement is clear from the construction of the transform matrix.

(ii) By the definition of the matrix Φ_m and formula (1.5), we have,

$$\begin{aligned}\Phi_m &= [\phi^p_{m,0}((n-l)2^{-m})]\\ &= \left[\frac{A_{\pi,\tau}}{2^m}\sum_{k=-2^m-1}^{2^m-1} b_k e^{2\pi i k(n-l)2^{-m}}\right]\\ &= \left[\frac{A_{\pi,\tau}}{2^m}\sum_{k=-2^m-1}^{2^m-1} u_m(n,k)\overline{u_m(k,p)}\right]\\ &= \left[\frac{A_{\pi,\tau}}{2^m}\mathbf{U}_m\mathbf{U}^*_m\right]\end{aligned}$$

where $A_{\pi,\tau} = \sqrt{\frac{2\tau}{\lambda_{0,\pi,\tau}}} > 0$, $b_k = \phi\left(\frac{k\tau}{2^{m-1}}\right)$ for $|k| < 2^m$, $b_{\pm 2^m} = \phi(\tau)/2$, $u_m(n,k) = \sqrt{b_k}e^{2\pi ikn2^{-m}}$, and $\mathbf{U}_m = [u_m(n,k)]$. If $\mathbf{x} \neq \mathbf{0}$ is a column vector of 2^m – dim, we have

$$\begin{aligned}\mathbf{x}^T\boldsymbol{\Phi}_m\mathbf{x} &= \mathbf{x}^\mathbf{T}\left(\frac{A_{\pi,\tau}}{2^m}\mathbf{U_m}\mathbf{U^*_m}\right)\mathbf{x}\\ &= \frac{A_{\pi,\tau}}{2^m}(\mathbf{x}^\mathbf{T}\mathbf{U_m})(\mathbf{U^*_m}\mathbf{x})\\ &= \frac{A_{\pi,\tau}}{2^m}\|\mathbf{U}^*_m\mathbf{x}\|^2 > 0.\end{aligned}$$

(iii) Each of the rows is given by a permutation of the generator $\{\phi^p_{m,0}(\frac{n}{2^m})\}$ and hence the row sum is

$$\sum_{n=0}^{2^m-1}\phi^p_{m,0}(n2^{-m}) = \sum_{n=0}^{2^m-1}\sum_{l=-\infty}^{\infty}\phi(k+2^m l) = \sum_{k=-\infty}^{\infty}\phi(k) = \hat{\phi}(0)$$

for each row. This last equality follow from the fact that the $\phi(k)$ are the Fourier coefficients of $\hat{\phi}(\omega)$, and the fact that the Fourier series converges at each point of continuity of $\hat{\phi}(\omega)$, in particular at $\omega = 0$. Because the row sums are just $\Phi_m[1, 1, \ldots, 1]^T$, which must equal $\hat{\phi}(0)[1, 1, \ldots, 1]^T$, we have our conclusion.

(iv) For any scale parameter m, we go through similar calculations. Let

$$\mathbf{v}_m = [v_0, v_1, \ldots, v_{2^m-1}]^T$$

be an eigenvector with eigenvalue λ_m. Then we have

$$\boldsymbol{\Phi}_m \mathbf{v}_m = \left[\sum_{k=0}^{2^m-1} \phi^p_{m,0}((n-k)2^{-m}) v_k \right]^T = \lambda_m [v_0, v_1, \ldots, v_{2^m-1}]^T.$$

Now by taking the discrete Fourier transform at the 2^m discrete values $t = 0, 2^{-1}\pi, \ldots, (2^m - 1)2^{-m}\pi$, we find

$$\sum_{n=0}^{2^m-1} \sum_{k=0}^{2^m-1} \phi^p_{m,0}((n-k)2^{-m}) v_k e^{int} = \lambda_m \sum_{n=0}^{2^m-1} v_n e^{int},$$

or, because of the 2^m periodicity of $\phi^p_{m,0}(2^{-m}t)$ and e^{int},

$$\sum_{n=0}^{2^m-1} \phi^p_{m,0}(n2^{-m}) e^{int} \sum_{k=0}^{2^m-1} v_k e^{ikt} = \lambda_m \sum_{n=0}^{2^m-1} v_n e^{int} = \lambda_m v_m(t). \tag{2.9}$$

Now $v_m(t)$ cannot be equal to 0 for all 2^m values of t because it is a trigonometric polynomial of degree $2^m - 1$ with at most $2^m - 1$ zeros in $[0, 2\pi)$. Hence, because it is not identically equal to 0, we can divide both sides of (2.9) by $v(t)$ and as $e^{ik2^m t} = 1$ for each of these values of t, we have

$$\begin{aligned}
\lambda_m &= \sum_{n=0}^{2^m-1} \phi^p_{m,0}(n2^{-m}) e^{int} \\
&= \sum_{n=0}^{2^m-1} \sum_{k=-\infty}^{\infty} \phi(n - 2^m k) e^{i(n-k2^m t)} \\
&= \sum_{n=-\infty}^{\infty} \phi(n) e^{inj2^{-m}\pi}, \quad j = 0, 1, \ldots, 2^m - 1.
\end{aligned}$$

The sampling expansion of the function $e^{ij2^{-m}x}$ is $\sum_{n=-\infty}^{\infty} S(x-n)$ $e^{inj2^{-m}\pi}$, and hence we have

$$\lambda_m = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} S(t-n)e^{inj2^{-m}\pi}\phi(t)\,dt$$
$$= \int_{-\infty}^{\infty} e^{ij2^{-m}\pi t}\phi(t)\,dt = \hat{\phi}(j2^{-m}\pi). \tag{2.10}$$

Using the duality of PSWFs and their Fourier transform,

$$\widehat{\varphi}_{n,\sigma,\tau}(\omega) = (-1)^n\sqrt{\frac{2\pi\tau}{\sigma\lambda_{n,\sigma,\tau}}}\varphi_{n,\sigma,\tau}\left(\frac{\tau\omega}{\sigma}\right)\chi_\sigma(\omega), \tag{2.11}$$

(2.10) can be rewritten as

$$\lambda_m = \sqrt{\frac{2\tau}{\lambda_{0,\pi,\tau}}}\varphi_{0,\pi,\tau}\left(\frac{j\tau}{2^m}\right), \quad j = 0, \ldots, 2^m - 1. \tag{2.12}$$

(v) Follows from (iv). □

The generators r_m, $m = 3, 4, 5, 6$ of the matrix Φ_5 are shown in Figure 2 and the matrix Φ_5 is shown in Figure 3. Eigenvalues of matrices Φ_m,

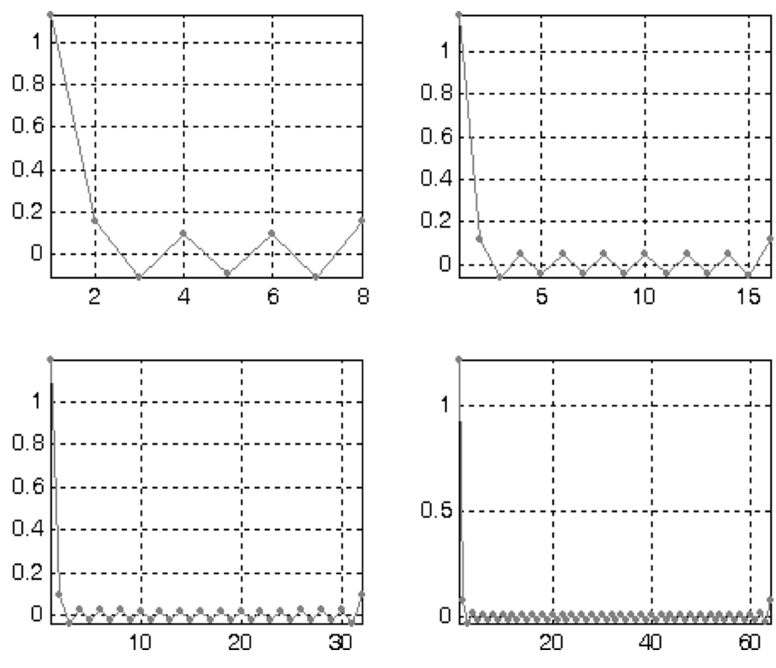

FIGURE 2 The generators $\mathbf{r}_m$, $m = 3, 4, 5, 6$ of the matrix $\mathbf{\Phi}_5$.

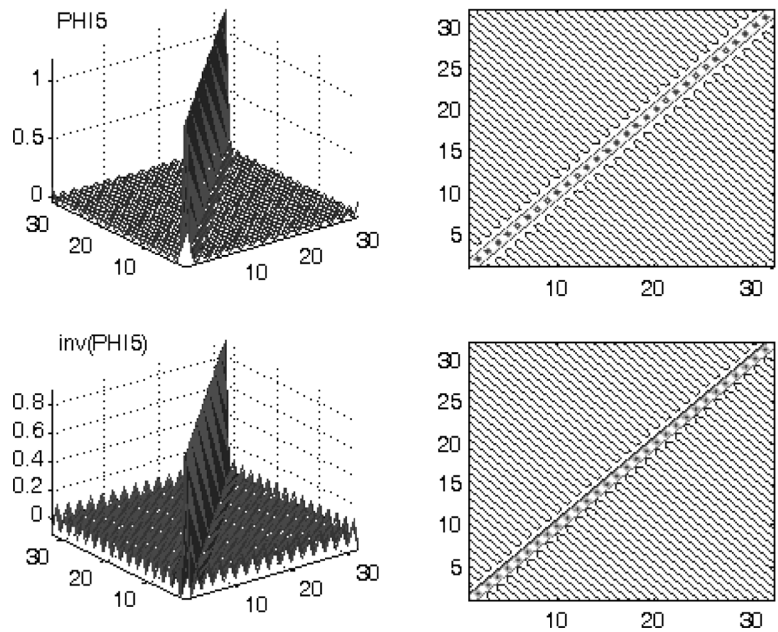

FIGURE 3 The matrix Φ_5 (top) and its inverse Φ_5^{-1} (bottom).

$m = 2, 3, 4, 5$ are shown in Figure 4. Notice that the matrix is very close to the identity and hence we could get a reasonable approximation to the expansion coefficients a_n by using the sampling coefficients $f(n2^{-m})$.

2.2. The Dilation Equations

The dilation equation is used to connect the coefficients at different scales. For periodic wavelets, we need a separate equation at each scale in contrast with the nonperiodic case in which the same equation works at each scale. Fortunately, as we have shown earlier in this section, some computational-related properties of the transform matrix and filter coefficients are independent of the scale parameter m (Proposition 2.1). We begin with the dilation equations for the scaling function.

2.2.1. *Dilation Equation for the Scaling Function*

The equation at scale m is of the form

$$\phi^p_{m,j}(t) = \sum_{k=0}^{2^{m+1}-1} c_{m,j,k}\phi^p_{m+1,k}(t) \tag{2.13}$$

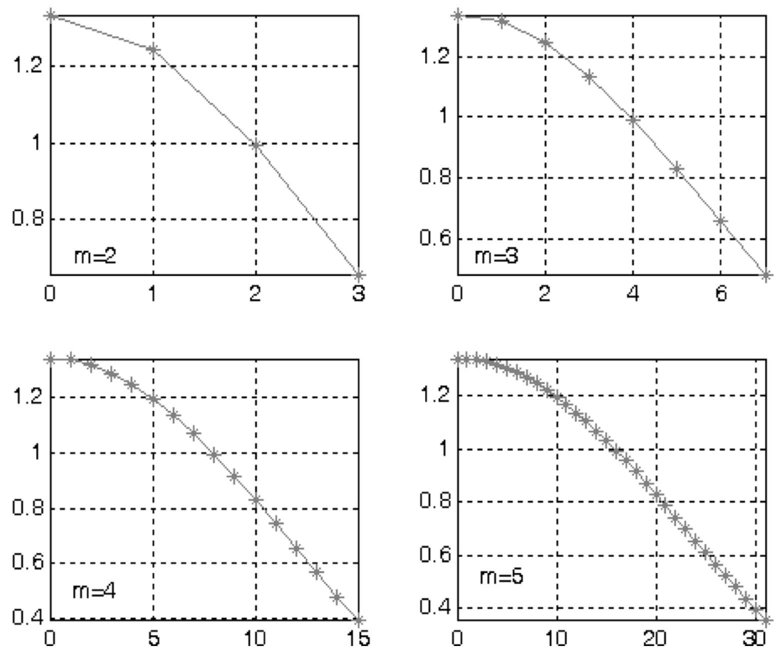

FIGURE 4 Eigenvalues of Φ_m, $m = 2, 3, 4, 5$.

However, we need only consider the equation for $j = 0$ because

$$\begin{aligned}
\phi^p_{m,j}(t) &= \phi^p_{m,0}(t - j2^{-m}) \\
&= \sum_{k=0}^{2^{m+1}-1} c_{m,0,k}\phi^p_{m+1,k}(t - 2j2^{-m-1}) \\
&= \sum_{k=0}^{2^{m+1}-1} c_{m,0,k}\phi^p_{m+1,0}(t - k2^{-m-1} - 2j2^{-m-1}). \\
&= \sum_{k=0}^{2^{m+1}-1} c_{m,0,k-2j}\phi^p_{m+1,k}(t).
\end{aligned}$$

To get the coefficients in (2.13), we need to invert the matrix Φ_{m+1} again

$$\phi^p_{m,0}\left(\frac{n}{2^{m+1}}\right) = \sum_{k=0}^{2^{m+1}-1} c_{m,0,k}\phi^p_{m+1,0}\left(\frac{n-k}{2^{m+1}}\right).$$

To do so, we denote by $\mathbf{c}_m = \{c_{m,0,k}\}_{k=0}^{2^{m+1}-1}$, then

$$c_m = \Phi^{-1}_{m+1} \cdot [r_m(n/2)]_{n=0}^{2^{m+1}-1}. \tag{2.14}$$

The filter coefficient vector $\mathbf{c}_m$ is shown in Figure 5 for $m = 2, 3, 4, 5$.

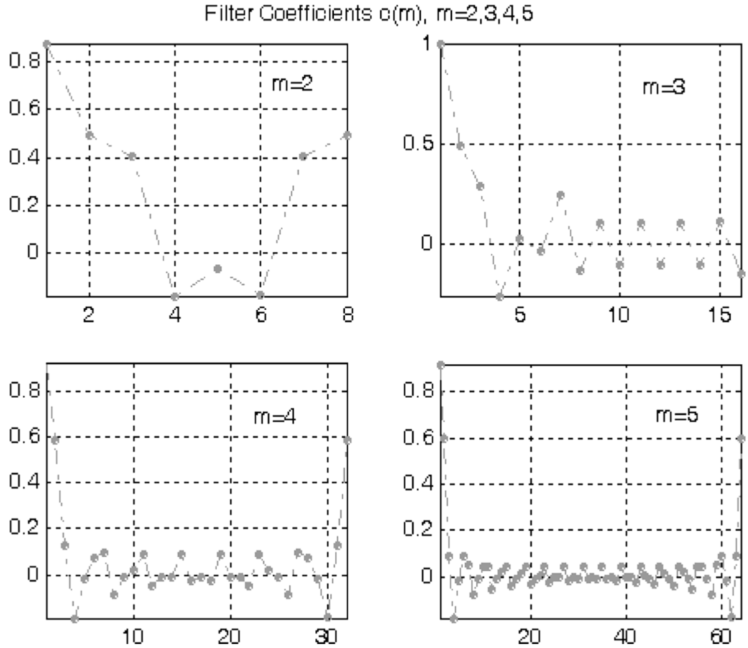

FIGURE 5 The filter coefficient vector $\mathbf{c}_m$, $m = 2, 3, 4,$ and 5.

2.2.2. *Dilation Equation for the Mother Wavelet*

In order to get an expression for the dilation equation of the mother wavelets, we need to find coefficients $d_{m,k}$ such that

$$\psi^p_{m,0}(t) = \sum_{k=0}^{2^{m+1}-1} d_{m,k}\phi^p_{m+1,k}(t). \tag{2.15}$$

Just as before, we can take the values at discrete points to get

$$\psi^p_{m,0}(n2^{-m-1}) = \sum_{k=0}^{2^{m+1}-1} d_{m,k}\phi^p_{m+1,0}((n-k)2^{-m-1}).$$

Again in matrix notation, we find that $\mathbf{d}_m = \{d_{m,k}\}_{k=0}^{2^{m+1}-1}$ is given by

$$\mathbf{d}_m = \phi^{-1}_{m+1} \cdot \left[\psi^p_{m,0}\left(\frac{n}{2^{m+1}}\right)\right]_{n=0}^{2^{m+1}-1}. \tag{2.16}$$

The expressions for these two dilation equations (2.14 and 2.15) are similar. However, in the case of the mother wavelet, we cannot use the same simplifications. Rather, in order to simplify (2.16), we first notice the fact that the original PS mother wavelet

$$\psi(t) = \varphi_{0,\pi/2,\tau/2}(t)\cos\left(\frac{3\pi}{2}t\right),$$

is the product of a $\frac{\pi}{2}$-bandlimited function and a cosine function. Hence it can be rewritten as:

$$\psi(t) = \sum_{k=-\infty}^{\infty} \varphi_{0,\pi/2,\tau/2}(2k)S(t/2-k)\cos\left(\frac{3\pi}{2}t\right)$$

and $\psi^p_{m,0}$ becomes, for $m > 1$,

$$\begin{aligned}
\psi^p_{m,0}(t) &= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \varphi_{0,\pi/2,\tau/2}(2k)S(2^{m-1}(t-n)-k)\cos\left(\frac{3\pi}{2}2^m(t-n)\right)\\
&= \sum_{k=-\infty}^{\infty} \varphi_{0,\pi/2,\tau/2}(2k)\sum_{n=-\infty}^{\infty} S\left(2^{m-1}\left(t-n-\frac{k}{2^{m-1}}\right)\right)\cos\left(\frac{3\pi}{2}2^m t\right)\\
&= \sum_{k=-\infty}^{\infty} \varphi_{0,\pi/2,\tau/2}(2k)D^*_{2^m-2}\left(2\pi\left(t-\frac{k}{2^{m-1}}\right)\right)\cos(3\pi 2^{m-1}t)
\end{aligned}$$

Thus the values for the dilation equation in (2.16) can be found from the integer values of a PSWF, though not those of a PS scaling function. It can be converted to one by using the change of scale formula for PSWF (see [15]):

$$\varphi_{0,\pi/2,\tau/2}(2k) = \varphi_{0,\pi,\tau/4}(k)/\sqrt{2}$$

which is a PS scaling function because it is π-bandlimited (though it has different concentration interval). The filter coefficient vector $\mathbf{d}_m$, $m = 2, 3, 4$, and 5, is shown in Figure 6.

3. NUMERICAL IMPLEMENTATION

Most of the standard methods of computing PSWFs involve an expansion in Legendre polynomials for small values of t and expansion in Bessel functions for large values. In practice, we rely on published tabulated values [1, 3, 4, 11] to construct the original PSWFs. Although some computer programs for evaluating the PSWFs are available [12, 19], many are not portable or have not been tested thoroughly. In this section, we will discuss some computation issues of the periodic PS wavelets. The interpolation formulas (2.3) and (2.4) allow us to construct the periodic PS wavelets and compute the expansion coefficients by using only the integer values of the

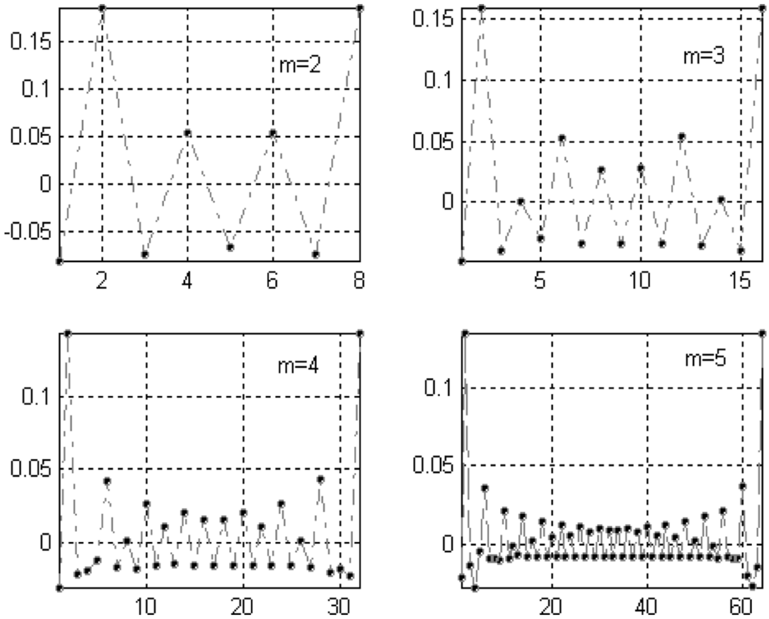

FIGURE 6 The filter coefficient vector $\mathbf{d}_m$, $m = 2, 3, 4,$ and 5.

PSWF $\phi_{0,\pi,\tau}$. To find these integer values, we employ a new algorithm that was recently developed in [18]. When we move from one scale to the next, there are two different ways to go: using the interpolating formulas directly or using the dilation equations developed in the last section. No matter which way we go, we only need the integer values of $\phi_{0,\pi,\tau}$. In what follows, we will illustrate the implementation for the interpolation approach. We begin with reviewing the method in [18] briefly.

3.1. The Eigenvector Algorithm

This method is based on identifying the relation between of energy concentration problem and a discrete optimization problem. As is well-known, the energy concentration problem is to maximize the ratio:

$$\rho = \frac{\int_{-\tau}^{\tau} |f(t)|^2 dt}{\int_{-\infty}^{\infty} |f(t)|^2 dt}, \tag{3.1}$$

for π-bandlimited functions f. Such functions may be represented by the Shannon sampling theorem as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) S(t-n),$$

where $S(t)$ is again the sinc function. We now denote by A_τ, the doubly infinite matrix,

$$A_\tau := [a_\tau(n,k)] = \left[\int_{-\tau}^{\tau} S(t-n)S(t-k)\,dt\right]. \tag{3.2}$$

Thus the ratio in (3.1) can be expressed as

$$\rho = \frac{\langle \mathbf{f}, A_\tau \mathbf{f}\rangle}{\langle \mathbf{f}, \mathbf{f}\rangle}, \tag{3.3}$$

where $\mathbf{f}$ now denotes the sequence $\{f(n)\}$ and the inner product is just the l^2 inner product.

Now if we expand the PS scaling function as

$$\phi(t) = \sum_{n=-\infty}^{\infty} \phi(n) S(t-n), \tag{3.4}$$

by Shannon sampling theory, the following proposition was shown to hold in [18].

Proposition 3.1. *Let A_τ be the operator on l^2 given by* (3.2). *Then*

(i) *A_τ is a self-adjoint, positive definite, and compact; its eigenvalues are simple and positive and satisfy*

$$1 > \lambda_{0,\pi,\tau} > \lambda_{1,\pi,\tau} > \cdots > \lambda_{n,\pi,\tau} > \cdots > 0.$$

In addition,

(ii) *The principal eigenvalue of A_τ is the concentration index, that is, $\lambda_{0,\pi,\tau} = \int_{-\tau}^{\tau} |\phi(t)|^2 dt$.*

(iii) *The associated eigenvector $\{\phi_n\}$ consists of the integer values of the PS scaling function, that is, $\phi_n = \phi(n)$, $n \in Z$.*

Practically, it is impossible to find exact eigenpairs of A_τ because it is a doubly infinite matrix. We consider rather a finite projection defined by the approximation matrix $A_\tau^n = [a_\tau(l,k)]$ with entries:

$$a_\tau^n(l,k) = \begin{cases} a_\tau(l,k), & \text{if } |l| \le n, \ |k| \le n. \\ 0, & \text{otherwise} \end{cases} \tag{3.5}$$

We also use the same notation A_τ^n for the truncated version A_τ, the matrix consisting of $a_\tau(l,k)$, $|l| \le n, |k| \le n$, which is a $2n+1$ square

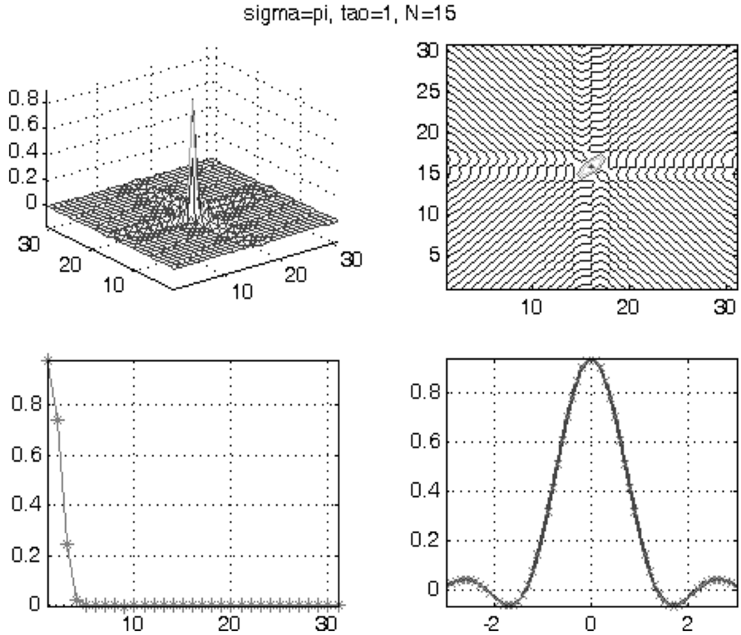

FIGURE 7 Top: the matrix A_1^{15} and its contour plot. Bottom: Eigenvalues of A_1^{15} and the eigenvector associated with $\lambda_{1,0}^{15}$.

matrix. This can also be used to get bounds on the approximation to the principal eigenvalue, which is the only one we are interested in here. Let $\tau \geq 2$; then we know that the energy of $\phi(t)$ outside of the interval $[-\tau, \tau]$ is less than $\varepsilon_\tau \leq \varepsilon_2 < 0.00006$, and is very small indeed for larger values of τ.

The eigenvalues and eigenvectors of A_1^n are then used to approximate those of A_1. Those of A_1^{15} are shown in Figure 7.

The principal eigenvalues for A_1^{10} and A_1^{15} compared with those of A_1 are shown in Table 1. The first prolate function $\phi_{\pi,1,0}(t)$, recovered by using truncated Shannon expansions (3.4) via its integer values (the eigenvector corresponding with the principal eigenvalue of the truncated matrix A_τ^n), is illustrated in Figure 8, $n = 10$ and $n = 15$.

TABLE 1 Principal eigenvalues of matrices $\mathbf{A}_1^n$, $n = 10, 15$

n	$\lambda_{\pi,1,0}^{10}$	$\lambda_{\pi,1,0}^{15}$	$\lambda_{\pi,\tau,0}$
0	.9810381982	.9810435577	.981045699

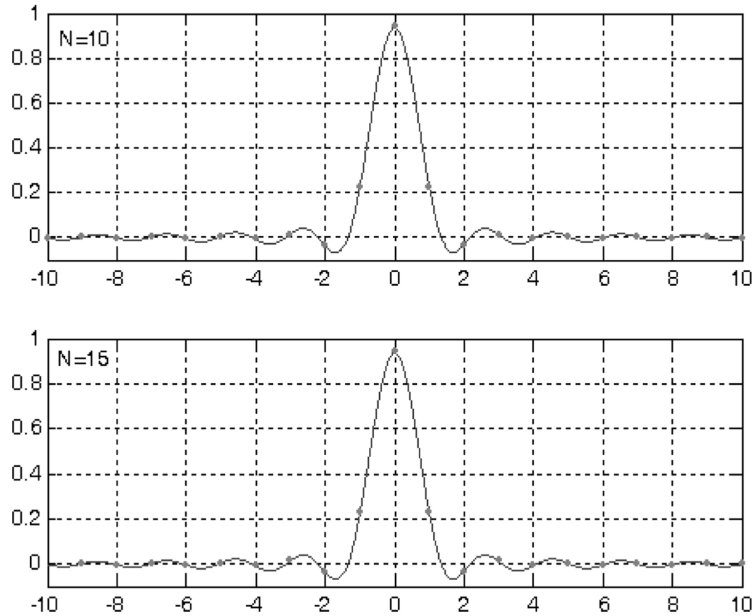

FIGURE 8 The eigenvector associated with the principal eigenvalue of matrix A_1^n (comparing with $\phi_{\pi,1,0}(t)$, the solid line), $n = 10$ (top panel), $n = 15$ (bottom panel), respectively.

3.2. Algorithm for Periodic PS Wavelets

Now we turn to the computation of the periodic PS scaling functions $\phi^p_{m,0}(t)$. By using the values of PS scaling function already computed in last subsection (by the eigenvector algorithm), the periodic PS scaling function can be computed by means of a truncated version of the Shannon theorem substituted in the interpolation formula (2.3). More precisely, the computation can be described by the following algorithm:

Algorithm 3.2 (Compute periodic PS wavelets)**.**

Step 1. Compute the sequence $\{\phi(k)\}_{k\in Z}$, the function values of PS scaling function ϕ at the integers, using the eigenvector algorithm [18] described above.

Step 2. Compute the sequence $\{\phi^p_{m,0}(k2^{-m})\}_{k=0}^{2^m}$, the function values of the periodic PS scaling function $\phi^p_{m,0}$ at the points $t = k2^{-m}$, using the values $\phi(k)$ in Step 1 and the truncated formulas (2.2) and (2.3):

$$\phi^p_{m,0}(k2^{-m}) = \sum_{j=-\infty}^{\infty} \phi(k + 2^m j) \approx \sum_{j=-N}^{N} \phi(k + 2^m j),$$

where N is the truncation parameter.

Step 3. Compute the other values of $\phi^p_{m,0}$ using the interpolation formula (2.3):

$$\phi^p_{m,0}(t) = \sum_{k=0}^{2^m} \phi^p_{m,0}(k2^{-m})D^*_{2^m-1}(2\pi(t - k2^{-m})).$$

where $D^*_{2^m-1}(u) = \dfrac{\sin(2^m \pi u)}{2^m \tan(\pi u)}$.

In this algorithm, we have to truncate the series in Step 2. We should check that the error is not too great. Because it is clear from the formula for the Fourier transform and the fact that $\hat{\phi}$ is even, we have,

$$\phi(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\phi}(\omega)e^{i\omega k}d\omega = O(k^{-2}),$$

it follows that the truncation error satisfies

$$\begin{aligned} e^m_{k,N} &= \left| \sum_{|l|\geq N+1}^{\infty} \phi(k + 2^m l)D^*_{2^m-1}(2\pi(t - k2^{-m}))\right| \\ &\leq \sum_{l=N+1}^{\infty} (|\phi(2^m l - k)| + |\phi(2^m l + k)|) \\ &\leq C\sum_{l=N+1}^{\infty} ((2^m l - k)^{-2} + (2^m l + k)^{-2}) \\ &\leq \frac{C}{2^{2m}(N+1)}. \end{aligned}$$

Figure 9 shows $\phi^p_{m,0}(t)$, $m = 3, 4$ computed by Algorithm 3.2, where the truncation parameter $N = 3$. In this case, the truncation error is $O(\frac{1}{2^{2m}})$.

3.3. Calculation of the Filter Coefficients

In this subsection, we will compute filter coefficient vectors $\mathbf{c}_m$ and $\mathbf{d}_m$ defined by (2.14) and (2.16) for the dilation equations (2.13) and (2.15), respectively. Before we start, it is worth to remark that the filter coefficients are dependent on the scale parameter m!

As in Algorithm 3.2, we assume that the dyadic values of the periodic PS scaling function are available (computed by using Algorithm 3.2). The algorithm to compute the filter coefficients $\mathbf{c}_m$ can be described as the following,

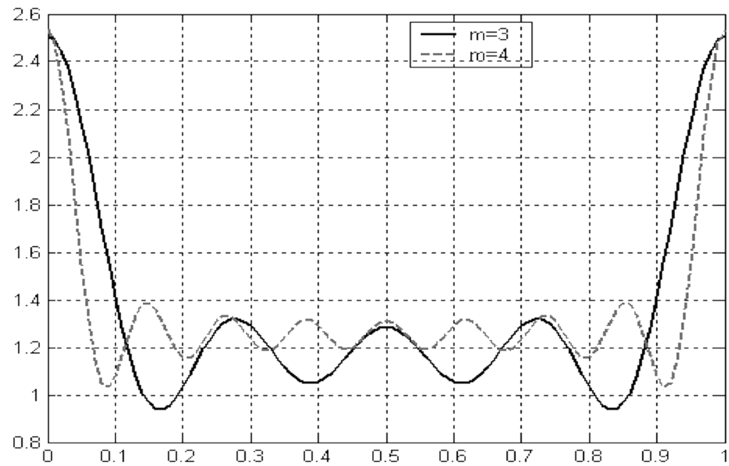

FIGURE 9 The periodic PS scaling functions $\phi^{p}_{m,0}(t), m = 3, 4,$ constructed by using the interpolation formula (2.3), truncated to $n = 10, 15$.

Algorithm 3.3 (Compute Filter Coefficients $\mathbf{c}_m$).

Step 1. Compute inverse of the transform matrix $\mathbf{\Phi}_{m+1}$ using formula (2.8);

Step 2. Compute $r_m(\frac{n}{2}), n = 0, \ldots, 2^{m+1} - 1$, using formula (2.8);

Step 3. Compute $\mathbf{c}_m$, using formula (2.14).

Remark 3.4. In general, to compute the inverse of a Toeplitz matrix requires n^3 operations. Some algorithms for inverse QR factorization for Toeplitz matrices using n^2 are also available, see [7], for example. Notice that the transform matrix is well conditioned with small condition number. In our case, the condition number is bounded by

$$\begin{aligned} cond(\Phi_m) &\leq \frac{2\tau}{\lambda_0}\phi(0)\phi(\tau) \leq \frac{2*1}{\lambda_0}\phi(0)\phi(1) \\ &\leq \frac{2}{0.98105} * 0.936576 * 0.224680 = .42899. \end{aligned}$$

Remark 3.5. The algorithm to compute the filter coefficients $\mathbf{d}_m$ (2.16) can be derived by a simple modification of Algorithm 3.3 and therefore we omit it.

Examples for filter coefficients computed by using the above algorithms are shown in Table 2.

TABLE 2 Filter coefficients

	$m = 2$		$m = 3$	
n	$c_{2,n}$	$d_{2,n}$	$c_{3,n}$	$d_{3,n}$
0	0.87860000	−.08071762	1.0025000	−.04790569
1	0.48990000	0.18695579	0.48770000	0.15999924
2	0.40500000	−.07385376	0.28490000	−.03968472
3	−.17990000	0.05429668	−.26190000	0.00114755
4	−.06860000	−.06623164	0.02720000	−.02922145
5	−.17990000	0.05429668	−.04160000	0.05284673
6	0.40500000	−.07385376	0.24580000	−.03422436
7	0.48990000	0.18695579	−.13160000	0.02715859
8			0.10680000	−.03450731
9			−.10490000	0.02719346
10			0.10460000	−.03433055
11			−.10480000	0.05325124
12			0.10580000	−.03609781
13			−.10810000	0.00155206
14			0.11520000	−.03979091
15			−.14980000	0.16003410

4. CONCLUSIONS

In this paper, we have continued our previous work with periodic prolate spheroidal wavelets and directed our attention to their construction via interpolation. We show that these periodic wavelets have a representation in terms of the interpolation with the modified Dirichlet kernel. We then derived a group of formulas of interpolation type based on this representation. These formulas enable one to obtain a simple procedure for the calculation of the periodic PS wavelets and their associated filter coefficients. All of our procedures avoid integration and thus should be easier to implement. They also avoid the use of the associated differential equations for the PSWFs. These procedures could also be extended to several dimensions where the PSWFs do not have such a differential equation.

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