Numerical Functional Analysis and Optimization

ISSN: 0163-0563 (Print) 1532-2467 (Online) Journal homepage: http://www.tandfonline.com/loi/lnfa20

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Numerical Functional Analysis and Optimization, 26(7–8):953–976, 2005

ISSN: 0163-0563 print/1532-2467 online
DOI: 10.1080/01630560500434260

PERIODIC PROLATE SPHEROIDAL WAVELETS

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□ *Prolate spheroidal wavelets (PS wavelets) were recently introduced by the authors. They were based on the first prolate spheroidal wave function (PSWF) and had many desirable properties lacking in other wavelets. In particular, the subspaces belonging to the associated multiresolution analysis (MRA) were shown to be closed under differentiation and translation. In this paper, we introduce periodic prolate spheroidal wavelets. These periodic wavelets are shown to possess properties inherited from PS wavelets such as differentiation and translation. They have the potential for applications in modeling periodic phenomena as an alternative to the usual periodic wavelets as well as the Fourier basis.*

Keywords Bandlimited signal; Paley-Wiener space; Periodic wavelets; Prolate spheroidal wave functions; PS wavelets; Wavelets.

Mathematics Subject Classification Primary 42C40; Secondary 33E10, 42C05, 94A11, 94A12.

1. INTRODUCTION

The prolate spheroidal wave functions (PSWFs), $\{\varphi_{n,\sigma,\tau}(t)\}_{n\in Z}$, constitute an orthonormal basis of the Paley-Wiener space of σ-bandlimited functions on the real line (functions whose Fourier transforms have support on the interval $[-\sigma,\sigma]$). They are the eigenfunctions of an integral operator with the sinc function, $S(t)=\sin \pi t/\pi t$, as its kernel:

$$\int_{-\tau}^{\tau} \varphi_{n,\sigma,\tau}(x)\frac{1}{T}S\left(\frac{t-x}{T}\right)dx = \lambda_{n,\sigma,\tau}\varphi_{n,\sigma,\tau}(t), \tag{1.1}$$

where $T=\pi/\sigma$.

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Although they had long been known as solutions to a Sturm-Liouville problem, they were shown by David Slepian and his collaborators at Bell Lab to be the solutions of an energy concentration problem, which in turn led to this integral equation. The problem, posed by Claude E. Shannon, was to find the normalized (by L^2 norm) σ-bandlimited function that possesses the maximum energy concentration on the interval $[-\tau, \tau]$. For this reason, the PSWFs are called Slepian functions. The system carries two parameters, the bandwidth σ and the width of the time concentration interval τ. The product of these two parameters, denoted by $c = \sigma\tau$, is sometimes called Slepian bandwidth in the engineering community. The properties of these PSWFs were extensively studied and reported in classic papers ([4–9, 11]) during the 1960s.

Recently, there has been renewed interest in PSWFs in part because of their sampling properties [14] as well as their multiscale properties [13]. New applications such as numerical solutions to partial differential equations and density estimations [1, 12] have arisen. Motivated by their multiscale nature, a system of wavelets based on these PSWFs was introduced in a earlier paper [13] (called PS wavelets for short). The PS wavelets were constructed in a certain way to retain the energy concentration property and were shown to have many desirable properties lacking in other wavelet systems. In this article, we use these PS wavelets to construct a new multiscale system—periodic PS wavelets. This system has similar properties to those of the PS wavelets, which distinguish them from other periodic systems, such as the Fourier system and periodic wavelets.

We organize this paper as follows. This section is followed by Section 2, in which we recall some related properties of PS wavelets. Section 3 is dedicated to the construction of the periodic PS wavelet basis. Some interesting properties of these periodic PS wavelets, such as differentiation, translation, and duality, are discussed in Section 4. Some numerical examples are given in the last section to demonstrate properties of the periodic PS wavelet system.

2. PS WAVELETS

Before constructing the periodic PS wavelets, we review the PS wavelets [13]. A scaling function $\phi = \varphi_{0,\pi,\tau}$ was first introduced, which is the π-bandlimited function of norm 1 whose energy on $[-\tau, \tau]$ is maximized. The integer translates formed a Riesz basis of the space $V_0 \subset L^2(R)$. This space V_0 turned out to be the Paley-Wiener space B_π of π-bandlimited functions no matter what the choice of τ.

This space then becomes part of the family of nested subspaces of a multiresolution analysis (MRA). The other spaces are obtained, as usual, by dilations by factors of two and consist of the Paley-Wiener spaces $V_m = B_{2^m\pi}$. Because they are entire functions, the bandlimited functions cannot have compact support in the time domain. If we denote the time concentration index as

$$\alpha(h,\tau) \equiv \left(\int_{-\tau}^{\tau} h^2(t)dt\right)^{1/2}, \quad h \in L^2(R), \ \|h\|_{L^2(R)} = 1$$

the PSWF for $n = 0$, $\varphi_{0,\pi,\tau}$ is the one that possesses the maximum concentration index α on $[-\tau,\tau]$ possible. In fact, for τ sufficiently large, its energy can be made arbitrarily small outside of the interval of concentration $[-\tau,\tau]$. For example, for $\tau = 2$, $\sigma = \pi$, the total energy outside of the interval is of the order 10^{-6}. Hence, the PS wavelets are not only superior as far as analytic properties are concerned but also similar to compactly supported wavelets for most practical computations.

We recall the following properties proved in [10] and [13].

Proposition 2.1. *Let $\phi(t) = \varphi_{0,\pi,\tau}(t)$ be a π-bandlimited PSWF with concentration interval $[-\tau,\tau]$; then $\{\phi(t-k)\}$ is a Riesz basis of B_π. The PS mother wavelet, in turn, is given by*

$$\psi(t) := \cos\left(\frac{3\pi}{2}t\right)\varphi_{0,\pi/2,\tau/2}(t), \tag{2.1}$$

which is orthogonal to all integer translates of $\phi(t)$. The translates of the mother wavelet form a Riesz basis of the orthogonal complement of V_0 in V_1.

As usual, we denote the wavelet subspace by $W_0 = \overline{span\{\psi(t-n)\}}$ with its dilations denoted by $W_m = \overline{span\{2^{\frac{m}{2}}\psi(2^m t - k)\}}$, $m \in Z$.

In most of the standard wavelets, the derivatives of the scaling function do not belong to the space V_0, nor for that matter, to any of the subspaces V_m. The exceptions are the Meyer wavelets, which do not have compact support, but even in this case the derivatives do not belong to V_0. Furthermore, none of the scaling functions with compact support belong to C^∞, so that they cannot be differentiated arbitrarily often [2].

For any wavelet system, the translation of an $f \in V_0$ by an integer is again in V_0, but if it is translated by some other real number, it no longer belongs to V_0. In fact, in most cases, it no longer belongs to any of the subspaces V_m. This, of course, becomes a problem when there are measurements based on an independent variable without a natural zero, for example if the independent variable is time. In fact, we have the following [10]:

Proposition 2.2. *Let $\phi(t) = \varphi_{0,\pi,\tau}(t)$ be the PS scaling function. Then*

(i) $\phi^{(k)} \in V_0$, *for* $k = 0, 1, \ldots$;
(ii) $\phi(t-\beta) \in V_0$ *for any* $\beta \in R$; *and*

(iii) *if* $f \in V_0$ *has the expansion*

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n \phi(t-n)$$

then $f' \in V_0$ *and*

$$f'(t) = \sum_{n=-\infty}^{\infty} a_n \phi(t-n); \tag{2.2}$$

where $a_n = \sum_{j \neq n} \frac{(-1)^{j-n}}{j-n} \alpha_j$;

(iv) *moreover, if* β *is not an integer, then*

$$f(t-\beta) = \sum_{n=-\infty}^{\infty} b_n \phi(t-n), \tag{2.3}$$

where $b_n = \sum_{j=-\infty}^{\infty} \alpha_j S(n-j-\beta)$.

Formula (2.2) indicates that differentiation can be reduced to a simple algebraic operation (discrete convolution) just as with Fourier series. The same is true for translations. Other operations such as dilation and

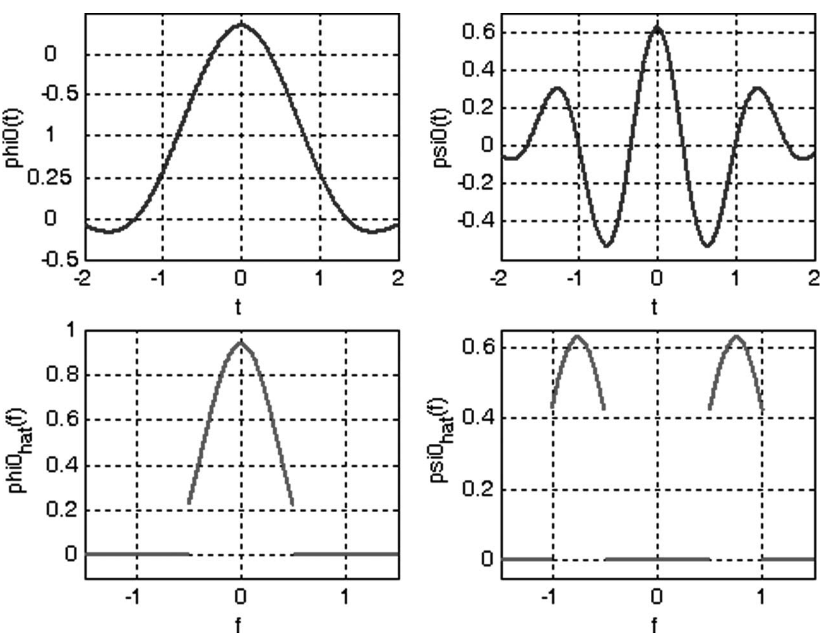

FIGURE 1 A pair of PS scaling function and its associated mother wavelet ($\tau = 1$). Top: time domain. Bottom: frequency domain.

convolution were also shown to have simple expression in terms of the coefficients. Figure 1 shows the PS scaling function and its associated mother wavelet in both time and frequency domains.

3. PERIODIC PS WAVELETS

The scaling function of the PS wavelets was defined as the first PSWF $\varphi_{0,\pi,\tau}(t)$, the one with maximum concentration on $[-\tau,\tau]$ among normalized π-bandlimited functions. We use it also to define the periodic scaling functions; many of the calculations are based on the Fourier transform of the PSWF given by

$$\hat{\varphi}_{0,\sigma,\tau}(\omega) = A_{\sigma,\tau}\varphi_{0,\sigma,\tau}\left(\frac{\tau\omega}{\sigma}\right)\chi_\sigma(\omega), \tag{3.1}$$

where $A_{\sigma,\tau} = \sqrt{\frac{2\pi\tau}{\sigma\lambda_{0,\sigma,\tau}}}$, $\lambda_{0,\sigma,\tau}$ is the first (largest) eigenvalue of the associated integral operator (1.1), and $\chi_\sigma(\omega)$ is the characteristic function of the interval $[-\sigma,\sigma]$. Just as with the PS wavelets, we restrict ourselves to $\sigma = \pi$ in the periodic case. The Fourier integral theorem inverts the Fourier transform and enables us to express $\phi = \varphi_{0,\pi,\tau}$ as

$$\phi(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{\phi}(\omega)e^{i\omega t}d\omega$$

and hence

$$\phi(t-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{\phi}(\omega)e^{i\omega t}e^{-i\omega n}d\omega.$$

This then is the nth Fourier series coefficient of $\hat{\phi}(\omega)e^{i\omega t}$ from which we deduce that

$$\sum_{n=-\infty}^{\infty}\phi(t-n)e^{i\omega n} = \hat{\phi}(\omega)e^{i\omega t}$$

where the convergence of the series is uniform for $|\omega| \le \pi - \varepsilon$. This follows from the fact that the right-hand side is continuous and differentiable on this interval.

3.1. The Periodic PS Scaling Function

We follow the standard procedure for the periodization of wavelets [15] and define the first periodic PS scaling function by

$$\phi^p_{0,0}(t) := \sum_{n=-\infty}^{\infty}\phi(2^0 t - n - 0),$$

which from the discussion above converges to the constant $\hat{\phi}(0)e^{i0t} = \hat{\phi}(0)$. Thus the first scaling function $\phi^p_{0,0}(t) = A_{\pi,\tau}\varphi_{0,\pi,\tau}(0)$ is a constant (or trigonometric polynomial of degree 0).

The second one (at level $m = 1$) is defined by

$$\phi^p_{1,0}(t) := \sum_n \phi(2^1(t-n) - 0), \tag{3.2}$$

which uses the fact that

$$\begin{aligned}
\phi(2t-2n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\phi}(\omega)e^{i\omega 2t}e^{-i\omega 2n}\,d\omega \\
&= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} \hat{\phi}(\xi/2)e^{i\xi t}e^{-i\xi n}\,d\xi/2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\phi}(\xi/2)e^{i\xi t}e^{-i\xi n}\,d\xi/2 + \frac{1}{2\pi}\int_{\pi}^{3\pi} \hat{\phi}(\xi/2)e^{i\xi t}e^{-i\xi n}\,d\xi/2 \\
&\quad + \frac{1}{2\pi}\int_{-3\pi}^{-\pi} \hat{\phi}(\xi/2)e^{i\xi t}e^{-i\xi n}\,d\xi/2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\{ \frac{1}{2}[\hat{\phi}(\xi/2)e^{i\xi t} + \hat{\phi}(\xi/2+\pi)e^{i\xi t}e^{i2\pi t} \right. \\
&\qquad \left. + \hat{\phi}(\xi/2-\pi)e^{i\xi t}e^{-i2\pi t}] \right\} e^{-i\xi n}\,d\xi,
\end{aligned}$$

that is, that $\phi(2t-2n)$ is the Fourier coefficient of the function in brackets in the integral. (Note that the integrals in the second and third lines are equivalent, as $\hat{\phi}$ has support on $[-\pi, \pi]$). Thus the defining series (2.2) converges to the average of the left and right-hand values at $\xi = 0$. Hence it is a trigonometric polynomial of degree 1 given by the expression

$$\begin{aligned}
\phi^p_{1,0}(t) &= \{\hat{\phi}(0) + \hat{\phi}(\pi^-)e^{i2\pi t}/2 + \hat{\phi}(\pi^+)e^{-i2\pi t}/2\}/2 \\
&= 2^{-2}A_{\pi,\tau}\{\phi(-\tau)e^{-2\pi it} + 2\phi(0) + \phi(\tau)e^{2\pi it}\}.
\end{aligned} \tag{3.3}$$

The third periodic scaling function is at the same level and is obtained by a translation;

$$\begin{aligned}
\phi^p_{1,1}(t) &:= \phi^p_{1,0}(t-1/2) \\
&= 2^{-2}A_{\pi,\tau}\{-\phi(-\tau)e^{-2\pi it} + 2\phi(0) - \phi(\tau)e^{2\pi it}\}.
\end{aligned} \tag{3.4}$$

Because ϕ is an even function, these functions can be expressed as

$$\phi^p_{1,0}(t) = 2^{-1}A_{\pi,\tau}\{\phi(0) + \phi(\tau)\cos 2\pi t\},$$
$$\phi^p_{1,1}(t) = 2^{-1}A_{\pi,\tau}\{\phi(0) - \phi(\tau)\cos 2\pi t\}.$$

In general, we define the 2^mth periodic PS scaling function to be

$$\phi^p_{m,0}(t) = \sum_n \phi[2^m(t-n)], \tag{3.5}$$

which again is a Fourier series of a function evaluated at $\omega = 0$.

This gives us, after calculations similar to those above, the general expression for $m > 0$

$$\phi^p_{m,0}(t) = \frac{A_{\pi,\tau}}{2^{m+1}}\left\{\phi(-\tau)e^{2\pi i 2^{m-1}t} + 2\sum_{k=-2^{m-1}+1}^{2^{m-1}-1}\phi\left(\frac{k\tau}{2^{m-1}}\right)e^{2\pi ikt} + \phi(\tau)e^{-2\pi i 2^{m-1}t}\right\},$$

a trigonometric polynomial of degree 2^{m-1}. The translates by $j2^{-m}$ give us the other functions at the same scale

$$\phi^p_{m,j}(t) = \frac{A_{\pi,\tau}}{2^{m+1}}\left\{\phi(-\tau)e^{2\pi i 2^{m-1}(t-\frac{j}{2^m})} + 2\sum_{k=-2^{m-1}+1}^{2^{m-1}-1}\phi\left(\frac{k\tau}{2^{m-1}}\right)e^{2\pi ik(t-\frac{j}{2^m})} + \phi(\tau)e^{-2\pi i 2^{m-1}(t-\frac{j}{2^m})}\right\},$$
$$m = 1, \ldots;\ \ j = 0, \ldots, 2^m - 1. \tag{3.6}$$

The terms in the expression (3.6) can be written more concisely as

$$\phi^p_{m,j}(t) = \sum_{k=-2^{m-1}}^{2^{m-1}} c_{m,j,k}e^{2\pi ikt}, \quad m = 0, 1, \ldots;\ \ j = 0, \ldots, 2^m - 1.$$

where

$$c_{m,j,k} = \frac{A_{\pi,\tau}}{2^m}\begin{cases}\frac{1}{2}\phi(\tau)e^{\pm\pi ji}, & k = \mp 2^{m-1},\\ \phi\left(\frac{k\tau}{2^{m-1}}\right)e^{2\pi\frac{kj}{2^m}i}, & k = -2^{m-1}+1, \ldots, 2^{m-1}-1\end{cases}$$

are the Fourier coefficients obtained from (3.6) for $m > 0$, or from the first expression for $m = 0$. Notice that because ϕ is real and even, we have

$$c_{m,j,k} = \overline{c_{m,j,-k}}, \quad \text{and} \quad c_{m,-j,k} = \overline{c_{m,j,k}}.$$

The series (3.6) can also be rewritten in real form as

$$\phi^p_{m,j}(t) = \frac{a_{m,j,0}}{2} + \sum_{k=1}^{2^{m-1}} [a_{m,j,k}\cos(2\pi kt) + b_{m,j,k}\sin(2\pi kt)]$$

where $a_{m,j,0} = 2c_{m,j,0}$, $a_{m,j,k} = 2\,\mathrm{Re}(c_{m,j,k})$, and $b_{m,j,k} = 2\mathrm{Im}(c_{m,j,k})$. More precisely, we have,

$$a_{m,j,k} = \begin{cases} \dfrac{A_{\pi,\tau}}{2^{m-1}}\phi\left(\dfrac{k\tau}{2^m}\right)\cos\left(\dfrac{kj}{2^{m-1}}\pi\right), & k = 0, \dots, 2^{m-1} - 1, \\ \dfrac{A_{\pi,\tau}}{2^m}\phi(\tau)\cos(\pi j), & k = 2^{m-1}, \end{cases}$$

and

$$b_{m,j,k} = \begin{cases} \dfrac{A_{\pi,\tau}}{2^{m-1}}\phi\left(\dfrac{k\tau}{2^m}\right)\sin\left(\dfrac{kj}{2^{m-1}}\pi\right), & k = 1, \dots, 2^{m-1} - 1, \\ 0, & k = 2^{m-1}. \end{cases}$$

Thus we have finally

$$\begin{aligned} \phi^p_{m,j}(t) = \frac{A_{\pi,\tau}}{2^m}\Bigg\{ & \phi(0) + \sum_{k=1}^{2^{m-1}} \phi(2^{-m}k\tau)\cos(2^{1-m}\pi jk)\cos(2\pi kt) \\ & + \sum_{k=1}^{2^{m-1}-1} \phi(2^{-m}k\tau)\sin(2^{1-m}\pi jk)\sin(2\pi kt)\Bigg\}. \end{aligned} \tag{3.7}$$

This gives us the usual multiresolution decomposition of $L^2(0,1)$, that is, a nested sequence of subspaces V^p_m, with the property that

1. $V^p_0 \subset V^p_1 \subset \cdots \subset V^p_m \subset \cdots \subset L^2(0,1)$,
2. $\bigcup_{m=0}^{\infty} V^p_m$ is dense in $L^2(0,1)$.

This follows from the fact that V^p_m is composed of trigonometric polynomials of degree $\leq 2^{m-1}$. Moreover, because the 2^m functions $\{\phi^p_{m,j}\}_{j=0}^{2m-1}$ are linearly independent and have the form (3.7), V^p_m contains the trigonometric functions $\cos(2\pi kt)$ for $0 \leq k \leq 2^{m-1}$ and $\sin(2\pi kt)$ for $0 < k < 2^{m-1}$ as well. Because there are exactly 2^m of these linearly independent functions, they form an alternate basis of V_m^p, which therefore contains, in particular, all trigonometric polynomials of degree $<2^{m-1}$. Hence $\bigcup_{m=0}^{\infty} V_m^p$ contains all trigonometric polynomials, which therefore is dense in $L^2(0,1)$.

Figure 2 shows the periodic PS scaling function $\phi_{m,0}^p(t)$ $\tau = 1$, $m = 2$ and its translates $\phi_{m,j}^p(t)$, $j = 0, 1, 2, 3,$. Even at this relatively coarse scale, the localization is evident. This is even more evident in Figure 3 which shows the periodic PS scaling function $\phi_{m,0}^p(t)$, at level $m = 1, 2, 3$ and 4.

The parameter τ serves as a "tuning parameter;" it controls how compact the scaling function is in the time domain while keeping the "bandwidth" in frequency domain fixed. This is an unique property that provides flexibility for different applications in electrical engineering, such as filtration, classification, denoising, and so forth. Figure 4 demonstrates this property. Both scaling functions have the same "bandwidth" π but different time concentration indices. Notice the improved time localization as τ increases.

Remark 3.1. Many of the formulas involving these prolate spheroidal functions are related to the sinc function as we have seen. This function is itself a scaling function, and we can find the associated periodic scaling

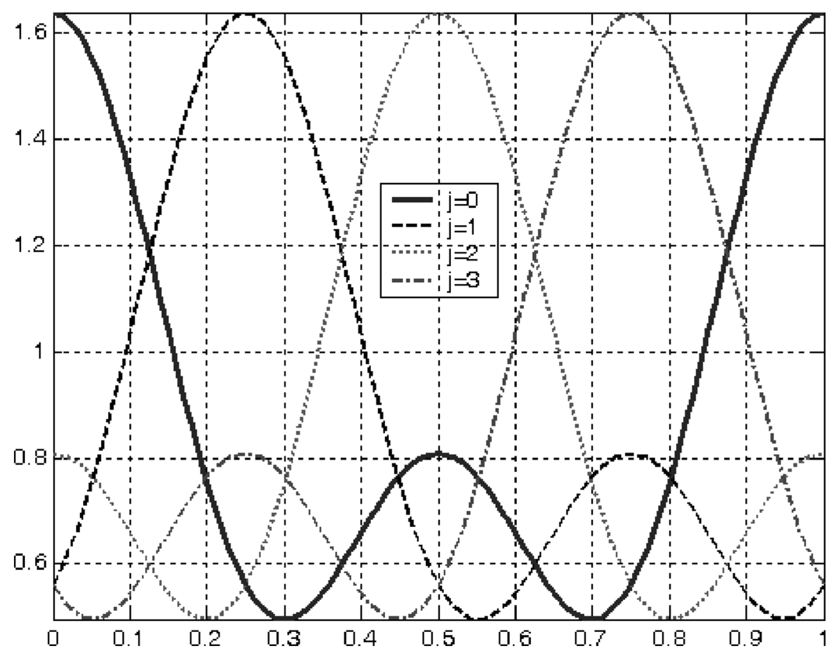

FIGURE 2 The periodic PS scaling function $\phi_{m,0}^p(t)$ for $\tau = 1$, $m = 2$ and its translates $\phi_{m,j}^p(t)$, $j = 0, 1, 2, 3$.

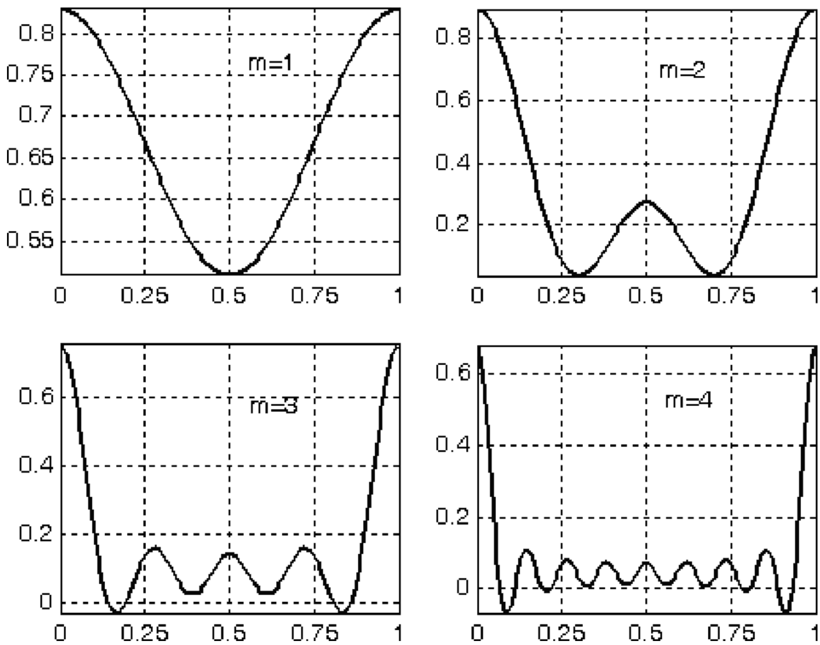

FIGURE 3 The periodic PS scaling functions $\phi^p_{m,0}(t),\ m = 1, 2, 3, 4,\ \tau = 1$.

function and mother wavelet. The periodic extension at scale m is

$$s^p_{m,0}(t) = \sum_{n=-\infty}^{\infty} S[2^m(t-n)],$$

which, because the Fourier transform of S is the characteristic function of the interval $[-\pi, \pi]$, may be represented by the trigonometric polynomial

$$\begin{aligned} s^p_{m,0}(t) &= 2^{-1-m}\left\{ e^{2\pi i 2^{m-1}t} + 2\sum_{k=-2^{m-1}+1}^{2^{m-1}-1} e^{2\pi ikt} + e^{-2\pi i 2^{m-1}t} \right\} \\ &= 2^{-m}\left\{ 2^{-1} + \sum_{k=1}^{2^{m-1}-1} \cos(2\pi kt) + 2^{-1}\cos(2^m\pi t) \right\} \\ &= \frac{\sin(2^m\pi t)}{2^{m+1}\tan(\pi t)}. \end{aligned}$$

This is the modified Dirichlet kernel $D^*_{2^{m-1}}(2\pi t)$ of Fourier series theory ([16], p. 50]). It is very close to our scaling function $\phi^p_{m,0}(t)$ for small values

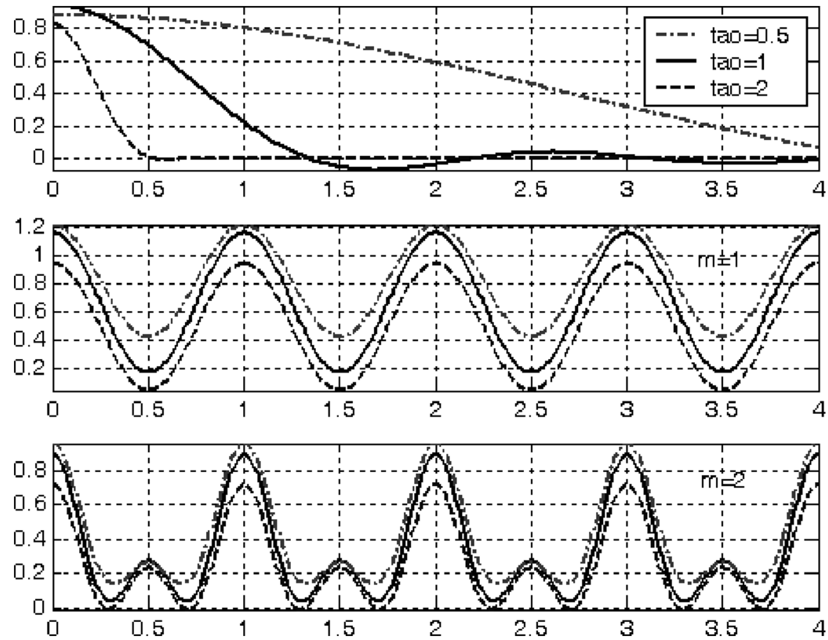

FIGURE 4 The (nonperiodic) PS scaling functions $\phi(t)$ (top) and the periodic PS scaling functions $\phi^{p}_{1,0}(t)$ (middle, $m=1$, bottom $m=2$), $\tau=\frac{1}{2},1,2$.

of τ but has poor time localization compared with our scaling function for larger values of τ.

3.2. Periodic PS Wavelets

We now turn our attention to the PS mother wavelet, which we shall use to define the periodic PS wavelets. It was defined in [13] as:

$$\psi(t) := \varphi_{0,\pi/2,\tau/2}(t)\cos\left(\frac{3\pi}{2}t\right)$$
$$\hat{\psi}(\omega) = \frac{1}{2}\left\{\hat{\varphi}_{0,\pi/2,\tau/2}(\omega-\frac{3\pi}{2})+\hat{\varphi}_{0,\pi/2,\tau/2}(\omega+\frac{3\pi}{2})\right\}.$$

By using formula (2.3), we have

$$\hat{\psi}(\omega) = \frac{1}{2}A_{\pi/2,\tau/2}\left\{\varphi_{0,\pi/2,\tau/2}\left[\frac{\tau}{\pi}\left(\omega-\frac{3\pi}{2}\right)\right]\chi_{[\pi,2\pi]}(\omega)\right.$$
$$\left.+\varphi_{0,\pi/2,\tau/2}\left[\frac{\tau}{\pi}\left(\omega+\frac{3\pi}{2}\right)\right]\chi_{[-2\pi,-\pi]}(\omega)\right\}, \qquad (3.8)$$

where $\hat{\psi}$ now has its support on $[-2\pi,-\pi]\cup[\pi,2\pi]$.

The first periodic PS wavelet is given formally by

$$\psi^p_{0,0}(t) = \sum_{n=-\infty}^{\infty} \psi(t-n),$$

where we may again proceed as in the last subsection and use the fact that $\psi(t-n)$ is the nth Fourier coefficient of a particular function. Indeed, by the Fourier integral theorem, we have

$$\begin{aligned}
\psi(t-n) &= \frac{1}{2\pi}\int_{\pi}^{2\pi} \hat{\psi}(\omega)e^{i\omega t}e^{-i\omega n}\,d\omega + \frac{1}{2\pi}\int_{-2\pi}^{-\pi} \hat{\psi}(\omega)e^{i\omega t}e^{-i\omega n}\,d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi} \hat{\psi}(\omega+\pi)e^{i(\omega+\pi)t}e^{-i(\omega+\pi)n}\,d\omega + \\
&\quad + \frac{1}{2\pi}\int_{-\pi}^{0} \hat{\psi}(\omega-\pi)e^{i(\omega-\pi)t}e^{-i(\omega-\pi)n}\,d\omega.
\end{aligned}$$

By observing that $\hat{\psi}(\omega+\pi) = 0$ on the interval $[-\pi, 0]$ and $\hat{\psi}(\omega-\pi) = 0$ on the interval $[0, \pi]$, we have

$$\begin{aligned}
\psi(t-n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \{\hat{\psi}(\omega+\pi)e^{i(\omega+\pi)t} + \hat{\psi}(\omega-\pi)e^{i(\omega-\pi)t}\}e^{-i(\omega+\pi)n}\,d\omega \\
&= \frac{1}{2\pi}\int_{0}^{2\pi} \{\hat{\psi}(\omega)e^{i\omega t} + \hat{\psi}(\omega-2\pi)e^{i(\omega-2\pi)t}\}e^{-i\omega n}\,d\omega.
\end{aligned}$$

The periodic extension of the function in brackets in the last integral has jump discontinuities at 0, π, and 2π. Its Fourier series (on $[0, 2\pi]$) converges to the average of the left and right values of this periodic extension at these discontinuities. In particular at $\omega = 0$, it converges to average of the right-hand value at 0 and the left-hand value at 2π. That is,

$$\begin{aligned}
\psi^p_{0,0}(t) &= \sum_{n=-\infty}^{\infty} \psi(t-n) \\
&= \{\hat{\psi}[(2\pi)^-]e^{i2\pi t} + \hat{\psi}(0^+)e^{i(-2\pi)t}\}/2
\end{aligned}$$

Notice that the expression for $\psi^p_{0,0}(t)$ is a linear combination of $e^{i2\pi t}$ and $e^{-i2\pi t}$, each of which belongs to V^p_1 and is orthogonal to V^p_0.

In general, then, in analogy to the periodic scaling function definition, we define the periodic PS wavelet to be

$$\psi^p_{m,j}(t) = \sum_{n=-\infty}^{\infty} \psi(2^m(t-n)-j), \quad m = 0, 1, \ldots;\; j = 0, \ldots, 2^m - 1.$$

This, again, is a translate of $\psi^p_{m,0}$, given by $\psi^p_{m,j}(t) = \psi^p_{m,0}(t - j2^{-m})$ and

$$\begin{aligned}
\psi(2^m(t-n)) &= \frac{1}{2\pi}\int_0^{2\pi} \{\hat{\psi}(\omega)e^{i\omega 2^m t} + \hat{\psi}(\omega - 2\pi)e^{i(\omega-2\pi)2^m t}\}e^{-i\omega 2^m n}\, d\omega \\
&= \frac{1}{2\pi}\int_0^{2^{m+1}\pi} \{\hat{\psi}(2^{-m}\omega)e^{i\omega t} \\
&\qquad + \hat{\psi}(2^{-m}\omega - 2\pi)e^{i(\omega t - 2\pi 2^m t)}\}2^{-m}e^{-i\omega n}\, d\omega \\
&= \frac{1}{2\pi}\sum_{k=0}^{2^m-1}\int_{k2\pi}^{(k+1)2\pi} \{\hat{\psi}(2^{-m}\omega)e^{i\omega t} \\
&\qquad + \hat{\psi}(2^{-m}\omega - 2\pi)e^{i(\omega t - 2\pi 2^m t)}\}2^{-m}e^{-i\omega n}\, d\omega \\
&= \frac{1}{2\pi}\sum_{k=0}^{2^m-1}\int_{k2\pi}^{(k+1)2\pi} H(m,t,\omega)e^{-i\omega n}\, d\omega \\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{2^m-1} H(m,t,\omega+2\pi k)e^{-i\omega n}\, d\omega.
\end{aligned}$$

Hence $\psi^p_{m,0}(t) = \sum_n \psi(2^m(t-n))$ is the Fourier series of $\sum_{k=0}^{2^m-1} H(m,t,\omega + 2\pi k)$ at $\omega = 0$. Because the latter is a piecewise continuous and differentiable function, the series is convergent to the average of the left- and right-hand values of the periodic extension at $\omega = 0$,

$$\psi^p_{m,0}(t) = \sum_{k=1}^{2^m-2} H(m,t,2\pi k) + \{\hat{\psi}(2\pi^-)e^{i\pi 2^{m+1}t} + \hat{\psi}(-2\pi^+)e^{-i\pi 2^{m+1}t}\}2^{-(m+1)}$$

and similarly for $\psi^p_{m,j}(t)$.

Thus we have, in general, by using the expression for $H(m,t,\omega)$ and the formula (2.1) again, an expression involving the PS wavelet functions

$$\begin{aligned}
\psi^p_{m,j}(t) &= A_{\frac{\pi}{2},\frac{\tau}{2}}2^{-m-1}\sum_{1\le|k|<2^m}\varphi_{0,\pi/2,\tau/2}\left(\frac{k\tau}{2^m}\right)\cos\frac{3\pi k\tau}{2^m}e^{2\pi ik(t-2^{-m}j)} \\
&\quad + A_{\frac{\pi}{2},\frac{\tau}{2}}2^{-m}\varphi_{0,\pi/2,\tau/2}(\tau)\cos(2^m 2\pi t) \\
&= A_{\frac{\pi}{2},\frac{\tau}{2}}2^{-m}\left[\sum_{k=1}^{2^m-1}\varphi_{0,\pi/2,\tau/2}\left(\frac{k\tau}{2^m}\right)\cos\frac{3\pi k\tau}{2^m}\cos 2\pi k\left(t - \frac{j}{2^m}\right)\right. \\
&\qquad \left. + \varphi_{0,\pi/2,\tau/2}(\tau)\cos(2^m 2\pi t)\right]
\end{aligned} \tag{3.9}$$

a trigonometric polynomial of degree 2^m. For computational purpose, we can use the change of parameter formula (see [13]):

$$\varphi_{0,\sigma,\tau}(x) = \sqrt{1/a}\varphi_{0,a\sigma,\tau/a}(x/a), \tag{3.10}$$

to get

$$\varphi_{0,\pi/2,\tau/2}(t) = \sqrt{2}\varphi_{0,\pi/4,\tau}(2t) = \sqrt{\frac{2}{\tau}}\varphi_{0,\frac{\pi\tau}{4},1}\left(\frac{2t}{\tau}\right).$$

Formula (3.8) can be rewritten as:

$$\psi^p_{m,j}(t) = \sqrt{\frac{2}{\tau}}A_{\frac{\pi}{2},\frac{\tau}{2}}2^{-m}\left[\sum_{k=1}^{2^m-1}\varphi_{0,\frac{\pi\tau}{4},1}\left(\frac{k}{2^{m-1}}\right)\cos\frac{3\pi k\tau}{2^m}\cos 2\pi k\left(t-\frac{j}{2^m}\right)\right.$$
$$\left.+\varphi_{0,\frac{\pi\tau}{4},1}(2)\cos(2^m 2\pi t)\right]. \tag{3.11}$$

Figure 5 shows the periodic PS mother wavelet together with its translates at level $m = 2$ as well as the periodic PS scaling function. Figure 6 shows the PS mother wavelets generated from Slepian functions with fixed bandwidth $\sigma = \pi$ but different time parameter τ. It can also be observed

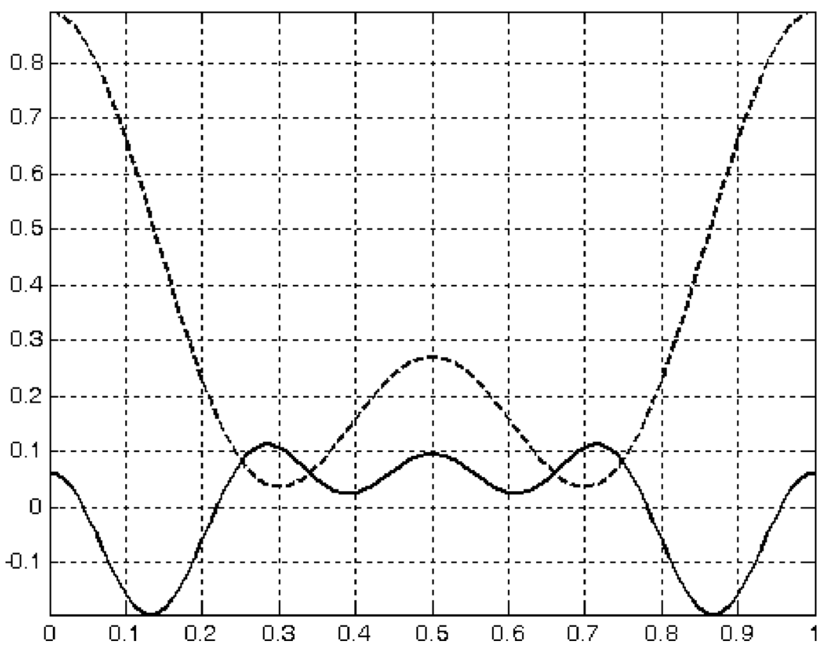

FIGURE 5 Periodic PS mother wavelets $\psi^p_{2,j}(t)$ and the PS scaling function (dotted line) $\phi^p_{2,0}$, $\tau = 1$.

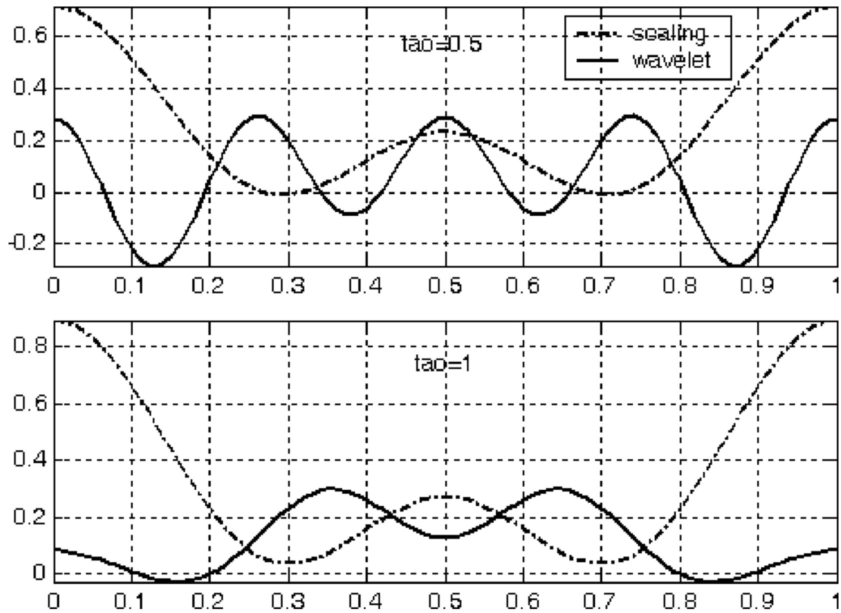

FIGURE 6 Periodic PS mother wavelets $\psi^p_{2,0}(t)$ (solid line), with associated scaling functions (dashed line) $\phi^p_{2,0}$, $\tau = 1/2$ (top), and $\tau = 1$ (bottom).

that $\psi^p_{m,j}(t)$ has average zero and is orthogonal to its associated scaling function $\phi^p_{m,0}(t)$.

These wavelets, for fixed m, span a subspace W^p_m, which is orthogonal to V^p_m and in fact $V^p_{m+1} = V^p_m \oplus W^p_m$, just as for wavelets on the entire real line. This leads to the fact that $L^2(0,1)$ can be expressed as an orthogonal direct sum:

$$L^2(0,1) = V^p_0 \bigoplus_{m=0}^{\infty} W^p_m.$$

4. PROPERTIES OF THE PERIODIC PS WAVELET SYSTEM

4.1. Dual Basis

Because the periodic PS scaling functions $\{\phi^p_{m,j}\}$ are not orthogonal, we need to introduce a dual basis in order to find the expansion coefficients. To get this dual basis, we look for a sequence of functions in V^p_m, $\{\widetilde{\phi}^p{}_{m,k}(t)\}$ such that

$$\left\langle \phi^p_{m,j}, \widetilde{\phi}^p{}_{m,k} \right\rangle = \delta_{jk}. \tag{4.1}$$

The case where $m = 0$ is obvious because we have only one term and hence $\widetilde{\phi}^p{}_{0,0} = 1/c_{0,0,0}$. For $m = 1$, we get something more typical of the general case. We suppose

$$\widetilde{\phi}^p{}_{1,j}(t) = a_{1,0,j} + a_{1,1,j}\cos 2\pi t$$

then the inner product becomes

$$\begin{aligned}\left\langle \phi^p_{1,j}, \tilde{\phi}^p{}_{1,n} \right\rangle &= \int_0^1 2^{-1} A_{\pi,\tau}\{\phi(0) + (-1)^j \phi(\tau)\cos 2\pi t\}\{a_{1,0,n} + a_{1,1,n}\cos 2\pi t\}\,dt \\ &= 2^{-1} A_{\pi,\tau}\{\phi(0) a_{1,0,n} + (-1)^j \phi(\tau) a_{1,1,n}/2\} = \delta_{j,n}.\end{aligned}$$

We now substitute the values

$$\begin{aligned}2^{-1} A_{\pi,\tau}\phi(0) &= \frac{1}{2}\sqrt{\frac{2\tau}{\lambda_{0,\pi,\tau}}}\phi(0) = \gamma_0, \\ 2^{-1} A_{\pi,\tau}\phi(\tau)/2 &= \frac{1}{4}\sqrt{\frac{2\tau}{\lambda_{0,\pi,\tau}}}\phi(\tau) = \gamma_1.\end{aligned}$$

This gives us the matrix equation

$$\begin{bmatrix} \gamma_0 & \gamma_1 \\ \gamma_0 & -\gamma_1 \end{bmatrix} D = I,$$

where $D = [a_{1,k,n}]$. We may further write the first matrix as

$$\begin{bmatrix} \gamma_0 & \gamma_{1.} \\ \gamma_0 & -\gamma_{1.} \end{bmatrix} = \begin{bmatrix} \gamma_0 & 0 \\ 0 & \gamma_{1.} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

each of which is easy to invert. Hence we have

$$D^T = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1/\gamma_0 & 0 \\ 0 & 1/\gamma_{1.} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1/\gamma_0 & 1/\gamma_0 \\ 1/\gamma_1 & -1/\gamma_1 \end{bmatrix},$$

and the dual mother wavelets are given by

$$\begin{bmatrix} \widetilde{\phi}^p{}_{1,0}(t) \\ \widetilde{\phi}^p{}_{1,1}(t) \end{bmatrix} = D^T \begin{bmatrix} 1 \\ \cos 2\pi t \end{bmatrix}$$

or

$$\begin{aligned}\widetilde{\phi}^p{}_{1,0}(t) &= 1/2\gamma_0 + 1/2\gamma_1 \cos 2\pi t \\ \widetilde{\phi}^p{}_{1,1}(t) &= 1/2\gamma_0 - 1/2\gamma_1 \cos 2\pi t.\end{aligned}$$

The procedure we have followed here works in general; we have

$$\begin{aligned}\widetilde{\phi}^{p}{}_{m,j}(t) &= a_{0,m,j} + \sum_{k=1}^{2^{m-1}} a_{k,m,j}\cos(2\pi kt) + \sum_{k=1}^{2^{m-1}-1} b_{k,m,j}\sin(2\pi kt) \\ &= \sum_{k=-2^{m-1}+1}^{2^{m-1}-1} d_{m,j,k} e^{2\pi ikt} + d_{2^{m-1},m,j}\cos(2^m \pi t). \qquad (4.2)\end{aligned}$$

which satisfies a similar but more complex equation. The calculations are straightforward and need only be done once at the mth scale. Rather than the trigonometric form, we use the exponential form of the Fourier series to find $\widetilde{\phi}^{p}{}_{m,j}(t)$. It is

$$\begin{aligned}\phi^{p}_{m,j}(t) &= \frac{A_{\pi,\tau}}{2^{m+1}}\Bigg\{\phi(-\tau)e^{2\pi i 2^{m-1}(t-\frac{j}{2^m})} \\ &\qquad + 2\sum_{k=-2^{m-1}+1}^{2^{m-1}-1} \phi\left(\frac{k\tau}{2^m}\right) e^{2\pi ik(t-\frac{j}{2^m})} + \phi(\tau)e^{-2\pi i 2^{m-1}(t-\frac{j}{2^m})}\Bigg\} \\ &= \sum_{k=-2^{m-1}+1}^{2^{m-1}-1} \gamma_{m,k} e^{2^{1-m}\pi ikj} e^{2\pi ikt} + \gamma_{2^{m-1}}(-1)^j \cos 2^m \pi t, \qquad (4.3)\end{aligned}$$

where

$$\gamma_{m,k} = \frac{A_{\pi,\tau}}{2^m}\phi\left(\frac{k\tau}{2^m}\right).$$

Again, we must find a matrix D such that

$$[\gamma_{m,k} e^{2^{1-m}\pi ikj}]D = I.$$

Now we may factor the first matrix into

$$\begin{aligned}[\gamma_{m,k} e^{2^{1-m}\pi ikj}] &= \begin{bmatrix} \gamma_{-2^{m-1}+1} & 0 & \cdots & 0 \\ 0 & \gamma_{-2^{m-1}+2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \gamma_{2^{m-1}} \end{bmatrix} \\ &\quad \times \begin{bmatrix} 1 & w^{1-2^{m-1}} & \cdots & w^{(1-2^{m-1})(2^m-1)} \\ 1 & w^{2-2^{m-1}} & \cdots & w^{(2-2^{m-1})(2^m-1)} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & w^{2^{m-1}} & \cdots & w^{2^{m-1}(2^m-1)} \end{bmatrix} = \Gamma F,\end{aligned}$$

where $w = e^{2^{1-m}\pi i}$ and use the fact that

$$FF^* = [e^{2^{1-m}\pi ikj}][e^{-2^{1-m}\pi ijk}] = 2^m I$$

to conclude that

$$D = 2^{-m} F^* \Gamma^{-1}.$$

Hence, the dual function will be

$$\widetilde{\phi^p_{m,j}}(t) = 2^{-m} \sum_{k=-2^{m-1}+1}^{2^{m-1}-1} \frac{1}{\gamma_{m,k}} e^{-2^{1-m}\pi ikj} e^{2\pi ikt} + \frac{2^{-m}}{\gamma_{m,2^{m-1}}} (-1)^j \cos 2^m \pi t. \quad (4.4)$$

Notice that because $\gamma_{m,k} = \gamma_{m,-k}$, we also have

$$\widetilde{\phi^p_{m,j}}(t) = 2^{-m} \left\{ \frac{1}{\gamma_0} + 2 \sum_{k=1}^{2^{m-1}-1} \frac{1}{\gamma_{m,k}} \cos \frac{k\pi}{2^{m-1}} (2^m t - j) + \frac{1}{\gamma_{m,2^{m-1}}} (-1)^j \cos 2^m \pi t \right\}. \quad (4.5)$$

The dual function at different resolution m is shown in Figure 6.

We now return to the problem of finding the derivatives of an element of V^p_m.

4.2. Differentiation

It was shown in [10] that the PS scaling function $\phi = \varphi_{0,\pi,\tau}$ had derivatives of all orders belonging to the space spanned by its translates, that is,

$$\phi^{(k)}(t) = \sum_n a^k_n \phi(t-n) \in V_m.$$

This was also true of translations by an arbitrary real number.

This is not an approximation as with other wavelets systems, but is exact. We shall see that similar results hold for our periodic PS scaling function expansions. In fact, it is even more obvious in this case because $\phi^p_{m,j}$ is a finite order trigonometric polynomial given by (3.6). Furthermore, because V^p_m contains all trigonometric polynomials of degree $<2^{m-1}$and is contained in the space of all trigonometric polynomials of degree $\le 2^{m-1}$, it follows that all derivatives and real transactions of functions in V^p_m belong to the space V^p_{m+1}. In fact, because of the special form of the $\phi^p_{m,j}$, they belong to V^p_m for even derivatives. The coefficients of the Fourier series can be found as well and are as follows.

Proposition 4.1. *Let $f \in V_m^p$ with scaling function expansion $\sum_{k=0}^{2^m-1} a_k \phi_{m,k}^p(t)$. Then*

(i) $f^{(k)} \in V_{m+1}^p$ *and is given by*

$$\begin{aligned}
f^{(k)}(t) &= \sum_{j=0}^{2^m-1} a_j \phi_{m,j}^{p(k)}(t) = \sum_{j=0}^{2^m-1} a_j \sum_{n=-2^{m-1}}^{2^{m-1}} (2\pi i n)^k c_{m,j,n} e^{2\pi i t n} \\
&= \sum_{n=-2^{m-1}+1}^{2^{m-1}-1} \left\{ (2\pi i n)^k \sum_{j=0}^{2^m-1} a_j c_{m,j,n} \right\} e^{2\pi i t n} \\
&\quad + \sum_{j=0}^{2^m-1} a_j \Big\{ c_{m,j,2^{m-1}} (2\pi i 2^{m-1})^k e^{2\pi i t 2^{m-1}} \\
&\qquad\qquad + c_{m,j,-2^{m-1}} (-2\pi i 2^{m-1})^k e^{-2\pi i t 2^{m-1}} \Big\};
\end{aligned}$$

also, if $k = 2q$ is even, the last term and hence all of the terms of $f^{(2q)}$ belong to V_m^p. Furthermore,

(ii) *the translation by a real number α, $f(\cdot - \alpha) \in V_{m+1}^p$ and is given by*

$$f(t-\alpha) = \sum_{j=0}^{2^m-1} a_j \phi_{m,j}^p(t-\alpha) = \sum_{j=0}^{2^m-1} a_j \sum_{n=-2^{m-1}}^{2^{m-1}} e^{-2\pi i \alpha n} c_{m,j,n} e^{2\pi i t n}.$$

However, rather than a Fourier series expansion, we would like the derivative and translation expressed in terms of the scaling function expansion. In order to do this, we have to invert the Fourier coefficient matrix of the scaling functions. This would be easy if our scaling functions were orthogonal with respect to translations, but they are not. Rather, we must use the dual scaling function basis to get the expansion coefficients.

First, we consider $f^{(2q)} = \phi_{m,k}^{p(2q)}(t)$; then

$$\begin{aligned}
\phi_{m,k}^{p(2q)}(t) &= 2^{-m} \sum_{p=0}^{2^m-1} \left\{ \sum_{n=-2^{m-1}+1}^{2^{m-1}} -1(2\pi i n)^{2q} c_{m,k,n} \overline{d_{m,p,n}} \right\} \phi_{m,p}^p(t) \\
&= 2^{-m} \sum_{p=0}^{2^m-1} \Bigg\{ \sum_{n=-2^{m-1}+1}^{2^{m-1}-1} (2\pi i n)^{2q} e^{2\pi i 2^{-m} n(p-k)} \\
&\qquad + (-1)^q (2^m \pi)^{2q} \cos(p-k)\pi \Bigg\} \phi_{m,p}^p(t) \qquad (4.6)
\end{aligned}$$

because the $\gamma_{m,k}$ and $\gamma_{m,k}$ cancel each other. This gives us the following:

Proposition 4.2. *Let $f \in V_m^p$ with scaling function expansion $\sum_{k=0}^{2^m-1} a_k \phi_{m,k}^p(t)$, then $f^{(2q)} \in V_m^p$ and is given by*

$$
\begin{aligned}
f^{(2q)}(t) &= \sum_{k=0}^{2^m-1} a_k \phi_{m,k}^{p(2q)}(t) \\
&= \sum_{p=-2^{m-1}+1}^{2^{m-1}} \left\{ \sum_{n=-2^{m-1}+1}^{2^{m-1}} (2\pi i n)^{2q} \right. \\
&\quad \left. \times \sum_{k=0}^{2^m-1} a_k 2^{-m} e^{2\pi i 2^{-m} n(p-k)} (-1)^q (2^m p)^{2q} cos(p-k) \right\} \phi_{m,p}^p(t).
\end{aligned}
$$

For odd derivatives, the cosine in the last expression must be replaced by the sine, and a similar result holds.

Everything we have done for V_m^p can be applied to the subspaces W_m^p because they are also finite dimensional subspaces that have a basis consisting of trigonometric functions. Even derivatives again belong to this space, but the expression for odd derivatives is more complicated because the W_m^p are not nested.

The calculations involving the derivative can be immediately extended to translation as well and can then be used to get a result similar to Proposition 4.1 involving the coefficients of the scaling functions expansion.

5. NUMERICAL EXAMPLES

In this section, we present a few numerical examples to demonstrate the properties of periodic PS wavelets introduced in previous sections. Because the purpose is to illustrate analytical properties by graphs, all computations were performed on MATLAB (Version 6.1.0.450, Release 12.1) on an IBM ThinkPad with Intel Pentium III Mobile with 1133 MHz CPU and 32 Mb RAM memory. It should be also pointed out that the parameter τ is chosen as 1 for the purpose of simplicity only. It can be tuned to make the scaling function and the wavelet as compact as needed.

Example 5.1. In this example, we consider the sawtooth function $f(t)$, $t \in R$ as the periodic extension of the function defined on $[0, 1]$ by:

$$
f(t) = \begin{cases} 2t, & 0 < t < 1/2, \\ -2(t-1), & 1/2 < t < 1. \end{cases} \tag{5.1}
$$

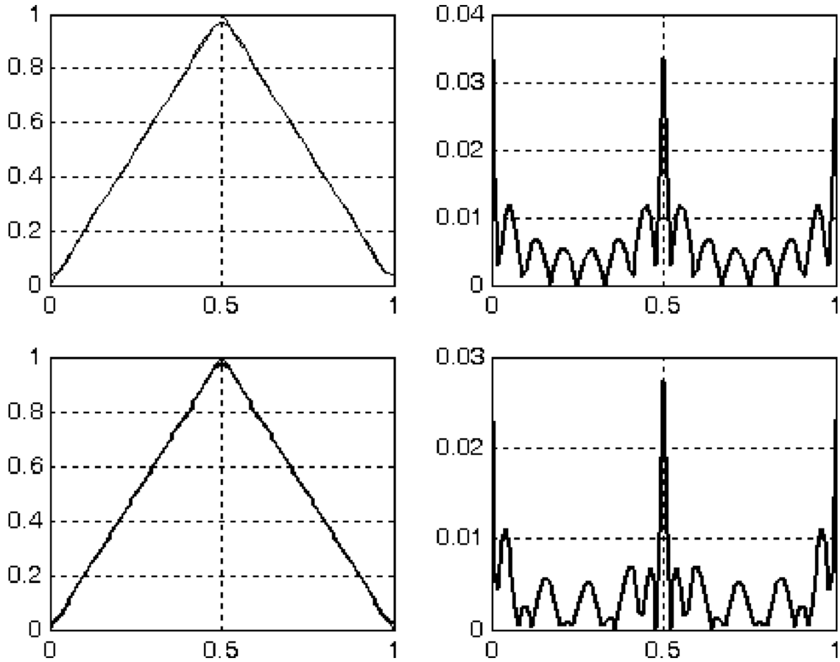

FIGURE 7 Top row: The sawtooth function, its Fourier series, and the associated error function. Bottom row: The sawtooth function, its periodic PS scaling function series, and the associated error function. Both series truncated to 2^4 terms.

The Fourier series of f is given by

$$f(t) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-4}{\pi^2(2n-1)^2} \cos 2(2n-1)\pi t$$

The function f, its Fourier series expansion (truncated to 2^4 terms), and absolute error are shown in the top row of Fig. 7.

To expand the function f at the mth scale, we write

$$f_m(t) = \sum_{k=0}^{2^m-1} a_{m,k}\phi^p_{m,k}(t),$$

where $a_{m,k} = \int_0^1 f(t)\widetilde{\phi^p_{m,k}}(t)\,dt$. The triangular function, its PS series expansion (truncated to 2^4 terms), and the absolute error are shown in the bottom row of Fig. 7. We observe that there is very little difference in the approximations except at the finest scale. This is because both approximations are projections onto a subspace of trigonometric polynomials of the same degree. The latter differs in that the highest order has a special form.

Example 5.2 (Function with Jump Discontinuity). In this example, we consider the periodic square wave $h(t) = h(t+1)$, defined on interval $[0, 1]$ as with its Fourier series is given by,

$$h(t) = \begin{cases} -1, & 0 < t < 1/2, \\ 0, & t = 0, \\ 1, & 1/2 < t < 1, \end{cases} \tag{5.2}$$

$$h(t) = \sum_{n=1}^{\infty} \frac{-4}{\pi(2n-1)} \sin 2(2n-1)\pi t \tag{5.3}$$

The square wave function, its Fourier series expansion (truncated to 2^6 terms), and absolute error are shown in the top row of Fig. 8. Similar to Example 5.1, we illustrate the PS series expansion and its error function in the bottom row of Fig. 8. We observe that both the Fourier series and the PS series show the well-known Gibbs phenomenon around the jump discontinuity at $t = \frac{1}{2}$ as expected. However, one can reduce the Gibbs phenomenon considerably by using a hybrid series as in [13].

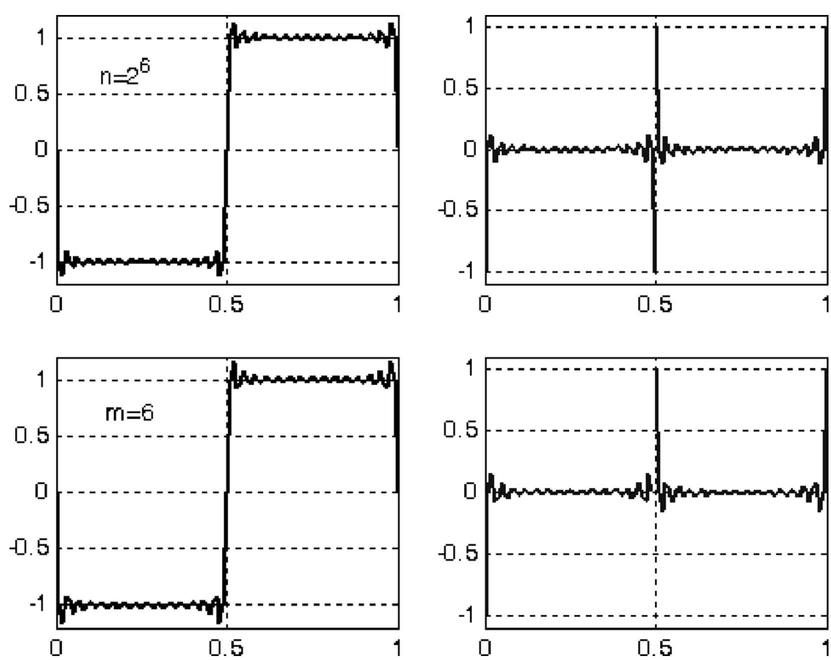

FIGURE 8 Top row: The square wave function, its Fourier series, and the associated error function. Bottom row: The square wave function, its PS wavelet series, and the associated error function. Both series truncated to 2^6 terms.

6. CONCLUSIONS

We have shown that for this new family of periodic wavelets, even order differentiation and translation can be carried in the subspaces V_m^p and furthermore have formulae given in closed form. Hence, computations involving these two operations are strictly algebraic operations on the expansion coefficients. Because the prolate spheroidal scaling functions are non-negative in the interval of concentration and are negligible outside of it, so are these periodized functions.

Because these periodic PS wavelets are trigonometric polynomials, their expansions can be obtained by rearranging the Fourier series of a given function. Thus one might ask, why not just use the Fourier series itself as an approximation? The answer is that the wavelets have a multiscale structure that is useful in constructing filter banks. They also have a much stronger localization property than the Fourier series and thus pick up irregularities in the signal much better, provided the coefficients are chosen properly. Also, the matrices arising from the wavelet approximation will be well conditioned because of this localization. Gibbs phenomenon can also be reduced to a certain extent by using the hybrid series studied in [13]. This is not seen in the examples because the biorthogonal expansion is used rather than the hybrid series. We should also like to point out the potential for flexibility of the parameterized system; we are free to choose τ to give us whatever degree of localization we want. We will discuss the convergence and other properties, as well as their applications, in a future work.

ACKNOWLEDGMENT

The second author was partially supported by Professor Naoki Saito's grant ONR YIP N00014-00-1-0469 while completing this paper.

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