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Prolate Spheroidal Wavelets: Translation, Convolution, and Differentiation Made Easy

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ABSTRACT. *Prolate spheroidal wavelets were previously introduced and shown to have some interesting convergence properties. In this work, several shortcomings of standard wavelets are discussed, and are shown not to be present in these new wavelets. These include invariance under arbitrary translations and differentiation of the associated multiresolution subspaces as well as similar properties of dilations.*

1. Introduction

Wavelet theory has undergone a phenomenal growth in the last decade. It has expanded in a number of different theoretical directions and has also found many applications in such diverse areas as image compression, oceanography, and statistics. The standard discrete wavelet theory [1] has, in large part, enjoyed this success because of its usefulness in data compression and noise reduction.

This wavelet theory usually begins with a scaling function ϕ whose integer translates are a Riesz basis of a subspace V_0 of $L^2(\mathbb{R})$. It, in turn, becomes part of a family of nested subspaces usually referred as a multiresolution analysis (MRA). The other spaces are obtained by dilations by factors of two: $f(t) \in V_m$, if and only, if $f(2^{-m}t) \in V_0$. An MRA will have the following properties:

1. $\cdots \subseteq V_{m-1} \subseteq V_m \subseteq \cdots \subseteq L^2(\mathbb{R})$, (1.1)
2. $\overline{\cup V_m} = L^2(\mathbb{R})$,
3. $\cap V_m = \{0\}$.

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Implicit in this definition is the existence of a *dilation equation* which relates the bases of two successive subspaces i.e., it expands ϕ in terms of $\phi(2 \cdot -n)$. This dilation equation is modified to construct a *mother wavelet* ψ which is, then used to form a basis (usually orthogonal) of $L^2(\mathbb{R})$ composed of dilations and translations $\{\psi(2^m t - n)\}$. Wavelet approximations have a strong localization property which enables them to mimic irregularities in the function they are approximating. This is in contrast to other orthogonal bases which give smooth approximations no matter what one starts with. But they do have a number of shortcomings:

- The subspaces V_m are not translation invariant for arbitrary real translations. For a function $f \in V_m$ and a a real number, the translation $f(t-a)$ may fail to be in the same space and even in none of these subspaces whatever the value of m. This means that the expansion of f depends strongly on where we choose the origin.
- The multiresolution sequence $\{V_m\}$ may fail to be dilation invariant for arbitrary positive dilations. For a function $f \in V_m$, and a positive number b, the dilation $f(bt)$ may fail to belong to any of the subspaces whatever the value of m.
- The subspaces V_m are not invariant under differentiation. For a function $f \in V_m$, the derivative may fail to be in any one of these subspaces whatever the value of m. This makes it impossible to solve differential equations within such subspaces.
- Approximations to discontinuous functions by functions in V_m lead to the excessive oscillations arising from Gibbs phenomenon.

These shortcomings do not just arise occasionally, but are endemic in wavelet theory. Most widely used wavelet systems have all four shortcomings mentioned. But the prolate spheroidal wavelets (PS wavelets) do not. These, together with the related semi-wavelets, were introduced in [8], where their convergence properties were explored. The last shortcoming was also discussed there, where it is shown that Gibbs phenomenon does not arise for approximations based on hybrid series for PS semi-wavelets.

There PS wavelets and semi-wavelets were in turn based on prolate spheroidal wave functions. These were introduced in a classic article [4] by David Slepian and his collaborators in Bell Labs as solutions of an energy concentration problem. They had previously been known as solutions of a Sturm-Liouville problem, from which many of their properties could be derived.

2. Prolate Spheroidal Wave Functions

The prolate spheroidal wave functions, (abbreviated PSWFs) $\{\varphi_{n,\sigma,\tau}(t)\}$, constitute an orthonormal basis of the space of $\sigma-$ bandlimited functions on the real line (functions whose Fourier transforms have support on the interval $[-\sigma, \sigma]$). The PSWFs are maximally concentrated on an interval $[-\tau, \tau]$ and depend on parameters σ and τ. There are several ways of characterizing them:

- As the eigenfunctions of a differential operator arising from a Helmholz equation on a prolate spheroid:

$$\left(\tau^2 - t^2\right) \frac{d^2\varphi_{n,\sigma,\tau}}{dt^2} - 2t \frac{d\varphi_{n,\sigma,\tau}}{dt} - \sigma^2 t^2 \varphi_{n,\sigma,\tau} = \mu_{n,\sigma,\tau}\varphi_{n,\sigma,\tau} \ .$$

- As the maximum energy concentration of a $\sigma-$ bandlimited function on the interval

$[-\tau, \tau]$; that is $\varphi_{0,\sigma,\tau}$ is the function of total energy 1 ($= ||\varphi_{0,\sigma,\tau}||^2$) such that

$$\int_{-\tau}^{\tau} |f(t)|^2 \, dt$$

is maximized, $\varphi_{1,\sigma,\tau}$ is the function with the maximum energy concentration among those functions orthogonal to $\varphi_{0,\sigma,\tau}$, etc.

or

- as the eigenfunctions of an integral operator with kernel arising from the sinc function $S(t) = \sin(\pi t)/\pi t$:

$$\frac{\sigma}{\pi} \int_{-\tau}^{\tau} \varphi_{n,\sigma,\tau}(x) S\left(\frac{\sigma}{\pi}(t-x)\right) dx = \lambda_{n,\sigma,\tau} \varphi_{n,\sigma,\tau}(t) \,. \tag{2.1}$$

These rather surprising connections were labelled a "lucky accident" by Slepian [4], and enable one to use properties of both the differential operator and integral operator in studying properties of the PSWFs. In addition to the Equation (2.1), the $\{\varphi_{n,\sigma,\tau}\}$ satisfy an integral equation over $(-\infty, \infty)$ as well:

$$\frac{\sigma}{\pi} \int_{-\infty}^{\infty} \varphi_{n,\sigma,\tau}(x) S\left(\frac{\sigma}{\pi}(t-x)\right) dx = (\varphi_{n,\sigma,\tau} * S_\sigma)(t) = \varphi_{n,\sigma,\tau}(t) \tag{2.2}$$

with the same kernel.

As one might expect, PSWFs are closely related to the Fourier transforms. Indeed, the Fourier transform of $\varphi_{n,\sigma,\tau}$ is given by

$$\widehat{\varphi}_{n,\sigma,\tau}(\omega) = (-1)^n \sqrt{\frac{2\pi\tau}{\sigma \lambda_{n,\sigma,\tau}}} \varphi_{n,\sigma,\tau}\left(\frac{\tau\omega}{\sigma}\right) \chi_\sigma(\omega) \tag{2.3}$$

where $\chi_\sigma(\omega)$ is the characteristic function of $[-\sigma, \sigma]$.

It is possible to find the relation between these functions at different scales by using the above definitions and formulas. By a straightforward change of scale in the integral Equation (2.1), and using the fact that the eigenvalues have multiplicty one, we find that

$$\varphi_{n,\sigma\tau,1}(x) = \sqrt{\tau} \varphi_{n,\sigma,\tau}(\tau x) \,. \tag{2.4}$$

Then (2.4) leads to the following relation between scales for $n = 0$:

$$\varphi_{0,\sigma,\tau}(2x) = \left(1/\sqrt{2}\right) \varphi_{0,2\sigma,\tau/2}(x) \,. \tag{2.5}$$

Some of these properties were used to define the PS wavelets [8] with the restriction to $n = 0$, i.e., the $\sigma-$bandlimbandlimited function $\varphi_{0,\sigma,\tau}(x)$ whose concentration on the interval $[-\tau, \tau]$ is maximum. While these functions are entire functions and therefore cannot vanish on any interval, they can be made uniformly small outside of $[-\tau, \tau]$, so that computationally they behave like functions with compact support.

3. Prolate Spheroidal Wavelets

In order to construct these PSWF wavelets [8], a scaling function $\phi = \varphi_{0,\pi,\tau}$, where τ is any positive number was first introduced. The integer translates formed a Riesz basis of a space $V_0 \subset L^2(\mathbb{R})$. This space V_0 turned out to be the Paley-Wiener space B_π of $\pi-$ bandlimited functions no matter what the choice of τ.

This space, then becomes part of the family of nested subspaces of a multiresolution analysis (MRA). The other spaces are obtained, as usual, by dilations by factors of two and consist of the Paley-Wiener spaces $V_m = B_{2^m\pi}$. This MRA has been widely studied and has as its standard scaling function the sinc function $S(t) = \sin \pi t/\pi t$ mentioned above. This function has very good frequency localization, but not very good time localization. This has limited its use as a wavelet basis in comparison to the Daubechies wavelets which have compact support in the time domain.

Because they are entire functions, the bandlimited functions cannot have compact support in the time domain. However, the PSWFs for $n = 0$ are as close to it as one can get, and in fact, for τ sufficiently large, can be made arbitrarily small outside of the interval of concentration. For example, for $\tau = 2$, the total energy outside of the interval is of the order of 10^{-6}. Hence, the PS wavelets should be similar to the Daubechies wavelets for practical computations and superior to the sinc functions.

This new scaling function is not orthogonal to its translates, but is very close since its translates constitute a Riesz basis of V_0. The calculations are not too difficult (see [8]) and give us:

Proposition 1. *Let $\varphi_{0,\pi,\tau}(t)$ be a $\pi-$bandlimited PSWF with concentration interval $[-\tau, \tau]$; then*

(i) $\{\varphi_{0,\pi,\tau}(t-n)\}$ *is a Riesz basis of* B_π *and*

(ii) *its dual basis is given by* $\{\widetilde{\varphi}_{0,\pi,\tau}(t-n)\}$, *whose Fourier transform (for* $n = 0$) *is*

$$\widehat{\widetilde{\varphi}}_{0,\pi,\tau}(\omega) := \frac{\widehat{\varphi}_{0,\pi,\tau}(\omega)}{\sum_k \left|\widehat{\varphi}_{0,\pi,\tau}(\omega - 2\pi k)\right|^2} . \tag{3.1}$$

Such a dual Riesz basis has the property that $\{\widetilde{\varphi}_{0,\pi,\tau}(t-n)\}$ is biorthogonal to $\{\varphi_{0,\pi,\tau}(t-n)\}$ and in this case is again a Riesz basis of V_0.

The PS mother wavelet is given by

$$\psi(t) := \cos(3\pi/2t)\varphi_{0,\pi/2,\tau/2}(t) , \tag{3.2}$$

which is orthogonal to all integer translates of $\varphi_{0,\pi,\tau}(t)$. The PS father wavelet is denoted from now on by

$$\phi(t) := \varphi_{0,\pi,\tau}(t) .$$

Both are shown in Figure 1. In both cases we have suppressed the parameters 0, π and τ. The translates of the father wavelet form a Riesz basis of the orthogonal complement of V_0 in V_1, which, as is usual, we denote by W_0 with its dilations denoted by W_m.

For the most part we shall work only with the scaling function approximation. We consider the approximation of a function in L^2 by an element of the space V_m. This will be done in two different ways. The first involves the projection onto V_m. This is obtained by forming the series

$$f_m(t) = \sum_n < f, \widetilde{\phi}\left(2^m \cdot -n\right) > 2^m \phi\left(2^m t - n\right) ,$$

where $\widetilde{\phi}$ is the dual scaling function whose Fourier transform is obtained by substituting (2.3) in (3.1)

$$\widehat{\widetilde{\phi}}(\omega) = \frac{\varphi_{0,\pi,\tau}\left(\frac{\tau\omega}{\pi}\right)\chi_\pi(\omega)}{\sqrt{\frac{2\tau}{\lambda_{0,\pi,\tau}}} \sum_k \left[\varphi_{0,\pi,\tau}\left(\frac{\tau}{\pi}\ (\omega + 2\pi k)\chi_\pi(\omega + 2\pi k)\right]^2} .$$

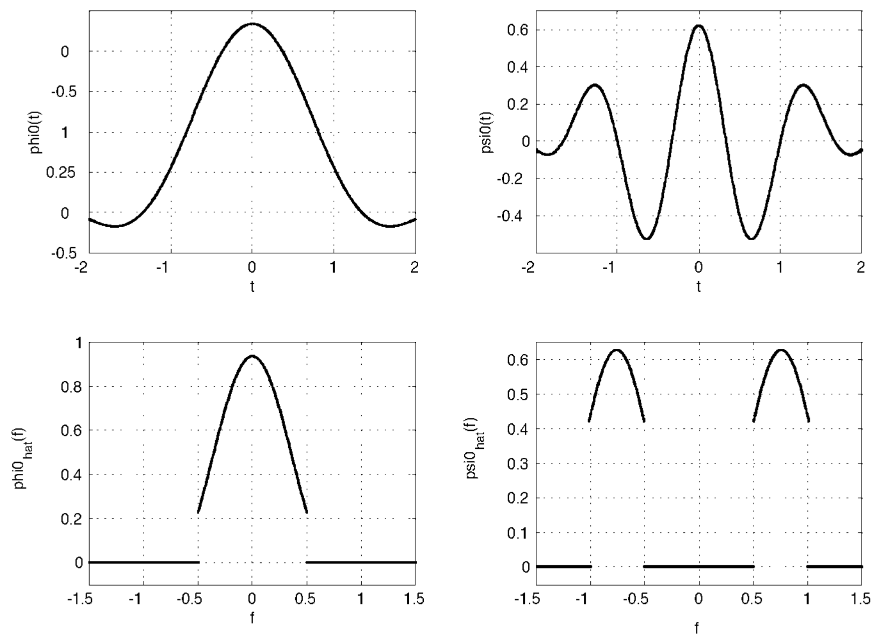

FIGURE 1 The PS wavelet scalling function $\phi(t)$ and its associated mother wavelet $\psi(t)$ for $\tau = 1$.

The other approximation is valid for continuous functions and involves the "hybrid series"

$$f_m^h(t) = \frac{2\pi}{\widehat{\phi}(0)} \sum_n f\left(2^{-m}n\right) \phi\left(2^m t - n\right) ,$$

which is frequently used for wavelet approximation in signal processing. Both series have been shown to converge rapidly for f in Sobolev spaces as $m \to \infty$ [8]. The former converges more rapidly than the latter, which, however, avoids the excessive oscillations associated with Gibbs phenomenon.

4. Differentiation

In most of the standard wavelets, the derivatives of the scaling function do not belong to the space V_0, nor for that matter, to any of the subspaces V_m. The exceptions are the Meyer wavelets which do not have compact support, but even in this case the derivatives do not belong to V_0. Furthermore, none of the scaling functions with compact support belong to C^∞, so that they cannot be differentiated arbitrarily often [1]. In our case the scaling function is analytic so that all the power of complex analysis is available. Furthermore, there are simple expressions for these derivatives.

We first show that our scaling function not only has derivatives of all orders, but that these derivatives are also in L^2. We repeat the fact that the Fourier transform is

$$\widehat{\varphi}_{0,\pi,\tau}(\omega) = \sqrt{\frac{2\tau}{\lambda_{0,\pi,\tau}}} \varphi_{0,\pi,\tau}\left(\frac{\tau\omega}{\pi}\right) \chi_\pi(\omega) .$$

Proposition 2. *Let $\phi(t) = \varphi_{0,\pi,\tau}(t)$ be the PS scaling function, then $\phi^{(k)} \in L^2(\mathbb{R})$ for $k = 0, 1, \ldots$.*

Proof. We use the Fourier integral theorem to represent, ϕ in terms of its inverse Fourier transform.

$$\phi(t) = \varphi_{0,\pi,\tau}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{\varphi}_{0,\pi,\tau}(\omega)e^{i\omega t}\, d\omega \,.$$

Since the interval of integration is bounded and all the derivatives of the integrand are uniformly bounded, we may differentiate under the integral sign to get

$$\varphi^{(k)}_{0,\pi,\tau}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{\varphi}_{0,\pi,\tau}(\omega)(i\omega)^k e^{i\omega t}\, d\omega = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\widehat{\varphi}_{0,\pi,\tau}(\omega)(i\omega)^k\right)' \frac{e^{i\omega t}}{it}\, d\omega$$
$$+ \frac{1}{2\pi}\frac{\widehat{\varphi}_{0,\pi,\tau}(\pi)}{it}\left[(i\pi)^k e^{i\pi t} + (-i\pi)^k e^{-i\pi t}\right] ,$$

by integration by parts and hence

$$\left|\varphi^{(k)}_{0,\pi,\tau}(t)\right| \le \sqrt{\frac{2\tau}{\lambda_{0,\pi,\tau}}}\frac{1}{2\pi|t|}\left[\int_{-\pi}^{\pi}\left|\varphi'_{0,\pi,\tau}\left(\frac{\tau\omega}{\pi}\right)\frac{\tau}{\pi}\omega^k\right| d\omega\right.$$
$$\left. + \int_{-\pi}^{\pi}\left|\varphi_{0,\pi,\tau}\left(\frac{\tau\omega}{\pi}\right)k\omega^{k-1}\right| d\omega\right] + \text{ integrated term}$$
$$\le \frac{1}{|t|}\pi^{k-1}(k+1)C_\tau \;,$$

where C depends on τ but not k, or t. Since $\varphi^{(k)}_{0,\pi,\tau}(t)$ is also bounded in a neighborhood of the origin, it must be in L^2. □

By using similar formulae, we can find the exact expression of the expansion of ϕ' in terms of $\phi(t-n)$. Indeed, the expansion coefficients are given by

$$a_n = <\phi', \widetilde{\phi}(\cdot - n)> = \frac{1}{2\pi}\int_{-\pi}^{\pi} i\omega\widehat{\phi}(\omega)\widehat{\widetilde{\phi}}(\omega)^{i\omega n}\, d\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{\frac{2\tau}{\lambda_{0,\pi,\tau}}}\varphi_{0,\pi,\tau}\left(\frac{\omega\tau}{\pi}\right)(i\omega)\frac{\varphi_{0,\pi,\tau}\left(\frac{\tau\omega}{\pi}\right)\chi_\pi(\omega)}{\sqrt{\frac{2\tau}{\lambda_{0,\pi,\tau}}}\sum_k\left[\varphi_{0,\pi,\tau}\left(\left(\frac{\tau}{\pi}\right)(\omega+2\pi k)\right)\right]^2}e^{i\omega n}\, d\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_{0,\pi,\tau}\left(\frac{\omega\tau}{\pi}\right)(i\omega)\frac{\varphi_{0,\pi,\tau}\left(\frac{\tau\omega}{\pi}\right)\chi_\pi(\omega)}{\left[\varphi_{0,\pi,\tau}\left(\left(\frac{\tau}{\pi}\right)(\omega)\right)\right]^2}e^{i\omega n}\, d\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(i\omega)e^{i\omega n}\, d\omega$$
$$= \left\{\begin{array}{c}\frac{(-1)^n}{n},\ n\neq 0\\ 0, n=0\end{array}\right\} .$$

By repeating this argument we see that the following holds:

Proposition 3. *Let $\phi(t) = \varphi_{0,\pi,\tau}(t)$ be the PS scaling function, then $\phi^{(k)} \in V_0$ and has expansion coefficients given by*

$$a^{(k)}_n = \left\{\begin{array}{c}\frac{k!(-1)^{n+k+1}}{n^k}, n\neq 0\\ \frac{(i\pi)^k\left(1-(-1)^k\right)}{2(k+1)}, n=0\end{array}\right\} .$$

The fact that $\phi^{(k)} \in V_0$ is obvious since its Fourier transform is given by

$$\widehat{\phi^{(k)}}(\omega) = \sqrt{\frac{2\tau}{\lambda_{0,\pi,\tau}}}(i\omega)^k \varphi_{0,\pi,\tau}\left(\frac{\tau\omega}{\pi}\right)\chi_\pi(\omega)$$

which is bounded and has support on $[-\pi, \pi]$. Therefore it inverse Fourier transform belongs to the Paley-Wiener space $B_\pi = V_0$.

This proposition enables us to find the coefficients of the derivative of a function in V_0, e.g., if $f \in V_0$ has the expansion

$$f(t) = \sum_n \alpha_n \phi(t-n)$$

then $f' \in V_0$ and

$$\begin{aligned} f'(t) &= \sum_n \alpha_n \phi'(t-n) \\ &= \sum_n \alpha_n \sum_{k \neq 0} \frac{(-1)^k}{k} \phi(t-n-k) \\ &= \sum_j \left\{ \sum_{n \neq j} \frac{(-1)^{j-n}}{j-n} \alpha_n \right\} \phi(t-j) \, . \end{aligned}$$

That is, differentiation is reduced to a simple algebraic operation just as with Fourier series. The same is true for higher order derivatives and constant coefficient differential operators. Furthermore, both series can be truncated since ϕ, and ϕ' are both small outside of the concentration interval $[-\tau, \tau]$.

5. Translation

For any wavelet system the translation of an $f \in V_0$ by an integer is again in V_0, but, if it is translated by some other real number, it no longer belongs to V_0. In fact, in most cases, it no longer belongs to any of the subspaces V_m. This, of course, becomes a problem when there are measurements based on an independent variable without a natural zero, for example, if the independent variable is time. This problem does not arise in trigonometric Fourier series; the partial sum of a trigonometric series with N terms is, after translation, still a sum with N terms. Fortunately, there is a wavelet system in which V_0 is invariant under all real translations, the PS wavelet system.

Let the translation operator by an amount β be denoted by T_β, $T_\beta f(t) = f(t-\beta)$. We shall see that it has properties similar to those of the previous operator.

Proposition 4. *Let $\phi(t) = \varphi_{0,\pi,\tau}(t)$ be the PS scaling function, then $T_\beta\phi \in V_0$ and the expansion of coefficients of $T_\beta\phi$ are given by*

$$< \phi(\cdot - \beta), \widetilde{\phi}(\cdot - n) > = \frac{\sin \pi(n-\beta)}{\pi(n-\beta)}, n \in \mathbb{Z}, \beta \notin \mathbb{Z} \, .$$

The proof again is simple, if we recognize that the Fourier transform of $T_\beta\phi$ has compact support on the interval $[-\pi, \pi]$ whatever the value of β, and hence $T_\beta\phi \in V_0$.

The coefficients are given, for $\beta \neq$ an integer, by

$$< \phi(\cdot - \beta), \widetilde{\phi}(\cdot - n) > = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{\phi}(\omega) e^{-i\omega\beta} \widehat{\widetilde{\phi}}(\omega) e^{i\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\omega\beta + i\omega n} \, d\omega = \frac{\sin \pi(n-\beta)}{\pi(n-\beta)} \; ,$$

and hence we have the expansion

$$T_\beta \phi(t) = \sum_n \frac{\sin \pi(n-\beta)}{\pi(n-\beta)} \phi(t-n) \; .$$

When β is an integer, say, k, then the expansion coefficients are just $\delta_{n,k}$ no matter what the wavelet system, in particular for ours.

This proposition gives us again a similar result to the one for differentiation of functions $f \in V_0$, that is

$$\begin{aligned} T_\beta f(t) &= \sum_n \alpha_n \phi(t - \beta - n) \\ &= \sum_n \alpha_n \sum_k \frac{\sin \pi(k-\beta)}{\pi(k-\beta)} \phi(t-n-k) \\ &= \sum_j \left\{ \sum_n \alpha_n \frac{\sin \pi(j-n-\beta)}{\pi(j-n-\beta)} \right\} \phi(t-j) \; . \end{aligned}$$

These results also hold for any of the spaces V_m and enable use to deduce that this operator commutes with projection operator P_m of $f \in L^2$ onto V_m, i.e., for $f \in L^2$, the projection is given by the series

$$P_m f(t) = f_m(t) = \sum_n < f, \widetilde{\phi}\left(2^m \cdot - n\right) > 2^m \phi\left(2^m t - n\right)$$

and

$$T_\beta f_m(t) = \sum_n < f, \widetilde{\phi}\left(2^m \cdot - n\right) > 2^m \phi\left(2^m (t-\beta) - n\right) \; .$$

Then we use the fact that the reproducing kernel of V_m is

$$\frac{\sin \pi 2^m (x-t)}{\pi(x-t)} = \sum_n \widetilde{\phi}\left(2^m x - n\right) 2^m \phi\left(2^m t - n\right)$$

to conclude that

$$T_\beta f_m(t) = < f, \frac{\sin \pi 2^m (\cdot - t + \beta)}{\pi(\cdot - t + \beta)} > = P_m T_\beta f(t) \; ,$$

just as with trigonometric Fourier series.

6. Other Operators

Many other operators beside the two mentioned above have a nicer behavior in the case of our PS wavelets than they do for other wavelets. We mention operators arising from dilation, multiplication by trigonometric functions, convolution, and multiplication by polynomials.

6.1 Dilation

One could not in general expect the dilation operator to be invariant in V_0, since it shrinks, or expands the support of a function in both the time and frequency domain. If we denote this operator by D_γ, i.e., $D_\gamma f(t) = f(\gamma t)$ for $\gamma > 0$, then $D_2 f \in V_1$, but will not be in V_0 in general. However, we can ask how $D_\gamma f$ behaves for, say, $0 < \gamma < 1$. Again, for many of the standard wavelets, this operator carries $f \in V_0$ completely out of any of the spaces V_m. In our case, it does not.

Proposition 5. *Let $\phi(t) = \varphi_{0,\pi,\tau}(t)$ be the PS scaling function, let $0 < \gamma < 1$, then $D_\gamma \phi \in V_0$.*

The proof is immediate since the Fourier transform of $D_\gamma \phi$ has support in $[-\pi, \pi]$ and is bounded. Similar results hold for other values of the dilation operator, but, then the operator takes ϕ to a different V_m.

6.2 Multiplication by Trigonometric Functions

Just as with the other operators mentioned here, multiplication by a trigonometric function takes a function in V_0 completely out of the space for general wavelets. This operation is important in signal processing in the case of modulated signals which have the form $T(t)f(t)$ where T is a trigonometric function consisting of the carrier while f is the modulating function. We consider the case where $T(t) = e^{i\alpha t}$, a complex trigonometric function.

Proposition 6. *Let $\phi(t) = \varphi_{0,\pi,\tau}(t)$ be the PS scaling function, let $-\pi < \alpha \leq \pi$, then $e^{i\alpha t}\phi(t) \in V_1$.*

The proof again is based on the fact that the support of the Fourier transform $\widehat{\phi}(\omega - \alpha)$ of $e^{i\alpha t}\phi(t)$ is on $[-2\pi, 2\pi]$. This may be extended to other frequencies beyond the range considered here, but the result will be that the Fourier transform has support in a larger interval and hence the product will belong to a space V_m farther up the ladder of subspaces. This holds as well, if $T(t)$ is any trigonometric polynomial since it can be written as a linear combination of these exponential functions.

6.3 Convolution

Convolution is a central operation in two of the principal consumers of wavelet theory, signal processing and statistics. In the former, if $h(t)$ is the impulse response of a linear system and $x(t)$ is the input, then the output $y(t)$ is the convoluton $y(t) = (x*h)(t) := \int x(t-s)y(s)\,ds$.

In statistics, if X and Y are independent random variables with densities $f(x)$ and $g(x)$, respectively, then the density of the sum $X+Y$ is $(f*g)(x)$. We consider the operator consisting of convolution with

- a fixed bounded L^1 function,
- a function in V_0,
- a singular unbounded function.

Then in the first case, the convolution with ϕ exists and is in L^2. But there's more.

Proposition 7. *Let $\phi(t) = \varphi_{0,\pi,\tau}(t)$ be the PS scaling function,*

1. *let h be a function in $L^1 \cap L^\infty$, then the convolution $h * \phi \in V_{0,}$*

2. *let $h \in V_0$, then the convolution $h * \phi \in V_0$ and is given by*

$$h * \phi(t) = \sum_n h(n)\phi(t-n) ,$$

3. *let $h(t) = -1/\pi t$, then*

$$h * \phi(t) = \sum_{n \neq 0} \frac{\left(1-(-1)^n\right)}{n} \phi(t-n) .$$

The proof of the first part is based on the fact that $\widehat{h}$ is a continuous bounded function on $\mathbb{R}$ and convolution is transformed into pointwise multiplication in the Fourier transform domain. The product $\widehat{h}(\omega)\widehat{\phi}(\omega)$ has compact support on $[-\pi, \pi]$ and is bounded there. Hence, the convolution belongs to V_0.

If the function $h \in V_0$, further results hold. They involve finding the expansion coefficients of $h * \phi$ which are.

$$\begin{aligned} < h * \phi, \widetilde{\phi}(\cdot - n) > &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{h}(\omega)\widehat{\phi}(\omega)\widehat{\widetilde{\phi}}(\omega) e^{i\omega n}\, d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{h}(\omega) e^{i\omega n}\, d\omega = h(n) . \end{aligned}$$

The last inequality follows from the fact that $\widehat{h}(\omega)$ has support on $[-\pi, \pi]$ and hence this last integral is the inverse Fourier transform of $\widehat{h}$ evaluated at the integers.

This convolution property can also be extended to singular functions. In the third result the function arises from the *Hilbert transform,*

$$Hf(t) := \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t-x}\, dx ,$$

which may be written as $Hf(t) = (h * f)(t)$, where $h(t) = -1/\pi t$. The Fourier transform of h does not exist in the usual sense, but must be taken in the sense of tempered distributions [2], for which it may be found to be $\widehat{h}(\omega) = \operatorname{sgn}(\omega)$. Again the product $\widehat{h}(\omega)\widehat{\phi}(\omega)$ has compact support which gives us the last conclusion.

This second result allows us to express a convolution of h with any function $f \in V_0$ by means of a simple series. It follows from the fact that translation behaves nicely with respect to convolution

$$T_\alpha(f * h) = (T_{-\alpha} f * h) .$$

Hence, if f has the expansion

$$f(t) = \sum_n a_n \phi(t-n) ,$$

then it follows by continuity that

$$\begin{aligned} h * f(t) &= \sum_n a_n (h * \phi(\cdot - n))(t) \\ &= \sum_n a_n (h(\cdot + n) * \phi)(t) \\ &= \sum_n a_n \left(\sum_k h(n+k)\phi(t-k) \right) \\ &= \sum_k \left(\sum_n a_n h(n+k) \right) \phi(t-k) \,. \end{aligned}$$

Thus, a convolution equation $h * f = g$ can be solved in V_0 by inverting the matrix $[h(n+k)]$.

6.4 Multiplication by a Polynomial

Multiplication by even as simple a polynomial as $p(t) = t$ causes problems since the product $t\phi(t)$ does not belong to $L^2(\mathbb{R})$. However, we can carry out such a multiplication, if we extend our definition of MRA from $L^2(\mathbb{R})$ to more general spaces, spaces of tempered distributions [7]. This more general MRA is composed of spaces $\{T_m\}$ which, in our case, consist of functions of exponential type which are of polynomial growth on $\mathbb{R}$. Their Fourier transforms, taken in the sense of tempered distributions, have compact support on $[-2^m\pi, 2^m\pi]$.

Again, by our now familiar argument, the product $p(t)\phi(t)$, which, since it is at most of polynomial growth, has a distributional Fourier transform with support on $[-\pi, \pi]$, for each polynomial $p(t)$. This, however, is no longer a function necessarily, but must be treated as a distribution with compact support. The space of all distributions with compact support was denoted by L. Schwartz [2] as $\mathcal{E}'$ and was shown (p. 90) to be the inductive limit of $\mathcal{E}'_m$, the space of distributions with support on $[-2^m\pi, 2^m\pi]$. The inverse Fourier transform of $\mathcal{E}'_m$, i.e., T_m, is composed of entire functions of exponential type just as the V_m, but having at most polynomial growth on the real axis. Thus, functions in V_m are also in T_m. This gives us the following:

Proposition 8. *Let $\phi(t) = \varphi_{0,\pi,\tau}(t)$ be the PS scaling function, let $p(t)$ be a polynomial, then the product $p(t)\phi(t) \in T_0$.*

Unfortunately, we cannot use the same tricks as before, since our space is no longer a Hilbert space. But we can find a series expansion of $p(t)\phi(t)$ in terms of $\phi(t-n)$ which is convergent in the sense of S', but not in the sense of L^2. For example, the expansion of $t\phi(t)$ is obtained by taking the Fourier transform of both sides of

$$t\phi(t) = \sum_n a_n \phi(t-n)$$

to get

$$\widehat{\phi}'(\omega) = \sum_n a_n \widehat{\phi}(\omega) e^{-i\omega n} \,,$$

and, then expanding $\widehat{\phi}'(\omega)/\widehat{\phi}(\omega)$ in a Fourier series. But the poor convergence limits the usefulness of this approach.

7. Conclusions

We have shown that the new family of wavelets, the PS wavelets, do not share some of the shortcomings of other families of wavelets. In particular, they are well behaved with respect to differentiation, translation, and convolution, all of which can be carried out in a fixed subspace V_0 of the MRA. Other operations such as dilation and multiplication by a polynomial, or by a trigonometric function, cannot be carried within the subspace but still have better behavior than that of other wavelets.

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