Calculus on Graphs

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1 Introduction

The purpose of this paper is to develop a "calculus" on graphs that allows graph theory to have new connections to analysis. For example, our framework gives rise to many new partial differential equations on graphs, most notably a new (Laplacian based) wave equation (see [FTa, FTc]); this wave equation gives rise to a partial improvement on the Chung-Faber-Manteuffel diameter/eigenvalue bound in graph theory (see [CFM94]), and the Chung-Grigoryan-Yau and (in a certain case) Bobkov-Ledoux distance/eigenvalue bounds in analysis (see [CGY96, CGY97, BL97]). Our framework also allows most techniques for the non-linear p-Laplacian in analysis to be easily carried over to graph theory (see [FTb]).

After developing the core "calculus on graphs" common to this and future works, we give some new variants and simpler proofs of inequalities, such as those of Federer-Fleming and Sobolev, known in graph theory (as in [DSC96, SC97, Cou92, Cou96a, BCLSC95a, CG97, Cou96b, CY95]). We also develop a notion of "split" functions that gives some

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improvements in these inequalities in certain cases in both analysis and graph theory. Yet, it might be said that the applications of "calculus on graphs" given here are secondary to the future applications to appear in [FTa, FTc, FTb].

One key point in this "calculus on graphs" is that, for what appears to be the first time, "non-linear" functions (functions that are not edgewise linear) become important; in previous approaches that unify graph theory and analysis (e.g. [Fri93] and the references there) only linear functions are ultimately used. The use of non-linear functions allows many proofs and ideas to carry over more simply from analysis to graphs and vice versa.

We caution the reader that using "non-linear" techniques can lead to a small loss in the resulting constants in the inequalities. However, it is almost always the case that the most interesting aspects of the inequalities (their dependencies on geometric constants, critical exponents, the statements of interesting theorems based on the inequalities, etc.) carry over identically. Also, when there is a loss in the constant (and this isn't always the case), one can usually easily recover this loss by slightly refining the analysis done with the simpler, non-linear technique.

Another benefit of the calculus on graphs is that it enables more analysis techniques to carry over to graphs and vice versa in a very direct and simple fashion; less intuition is obscured in technicalities that are particular to analysis or graphs.

We mention that most of the inequalities we prove in this paper can be called *gradient inequalities*, in which we lower bound the L^p of the gradient of f in terms of norms on f and constants depending on the graph ("isoperimetric constants"). A large number of well known results in graph theory such as results on the eigenvalues of the Laplacian can be viewed as gradient inequalities.

In section 2 we develop some notions of "calculus on graphs." In section 3 we discuss some preliminary remarks on gradient inequalities for "closed" graphs, i.e. finite graphs with no boundary; gradient inequalities for such graphs are only interesting when one works with functions "modulo constant functions." In section 4 we discuss Federer-Fleming theorems; these give lower bounds of the L^1 norm of the gradient. In section 5 we discuss analogues of Cheeger's inequality for graphs, namely the inequalities of Dodziuk, Alon, and Mohar, from our point of view. In section 6 we discuss the heat kernel, in preparation for section 7. In section 7 we discuss lower bounds for the L^p norm of the gradient, including Sobolev inequalities and Nash inequalities and the resulting eigenvalue inequalities. In section 7 our techniques, especially using that of "split" functions in the closed case, improves the constants in results published previously; some other material in sections 4-7 represents generalizations, alternate forms, and/or simplifications of theorems that appears in the literature (see, for example, [DSC96, SC97, Cou92, Cou96a, BCLSC95a, CG97, Cou96b, CY95]).

Notation
Throughout this paper, if $1 \leq p \leq \infty$, then p' is the dual exponent of p, meaning that $1 \leq p' \leq \infty$ with $(1/p) + (1/p') = 1$; then $(p-1)(p'-1) = 1$. We let $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$.

If Ω is a subset of a topological space, then $\overline{\Omega}$ denotes Ω's closure, and Ω^{c} denotes Ω's complement.

If $G = (V, E)$ is a graph then $e \sim \{u, v\}$ means that e is an edge whose endpoints are u and v (since we will allow multiple edges, we cannot replace the $\sim$ with an $=$). We write $v \in e$ or $e \ni v$ to mean that e has v as an endpoint. We write $u \sim v$ to indicate that u and v have an edge joining them.

2 Calculus on Graphs

2.1 The Setup

We use a similar setting as in [Fri93], and we recall this setting here. Let $G = (V, E)$ be a graph (undirected), such that with each edge, $e \in E$, we have associated a length, $\ell_e > 0$. We form the *geometric realization*, $\mathcal{G}$, of G, which is the metric space consisting of V and a closed interval of length ℓ_e from u to v for each edge $e = \{u, v\}$. When there is no confusion, we identify a $v \in V$ with its corresponding point in $\mathcal{G}$ and identify an $e \in E$ with its corresponding closed interval in $\mathcal{G}$. By an *edge interior* we mean the interior of an edge in $\mathcal{G}$.

In analysis, a Riemannian manifold may or may not have a *boundary*. However, certain concepts, such as *nodal regions* and certain eigenvalue inequalities, require the notion of a boundary *even if the original manifold has no boundary*. The graph theoretic analogues of these concepts will also require the notion of a boundary in the graph setting.

Definition 2.1 *The* boundary*, $\partial\mathcal{G}$, of a graph, $\mathcal{G}$, is simply a specified subset of its vertices. By the* interior of $\mathcal{G}$*, denoted $\mathcal{G}^{\mathrm{o}}$, we mean $\mathcal{G} \setminus \partial\mathcal{G}$; similarly the* interior vertices*, denoted V^{o}, we mean $V \setminus \partial\mathcal{G}$.*

Convention 2.2 By a *traditional graph* we mean an undirected graph $G = (V, E)$. Throughout this article we assume our graphs are always given with (1) lengths associated to each edge, (2) a specified boundary (i.e. a specified subset of vertices). Whenever an edge length is not specified, it is taken to be one. Whenever a boundary is not specified, it is taken to be empty. We refer to the geometric realization, $\mathcal{G}$, of the graph as the graph, when no confusion may arise.

Definition 2.3 *By $C^k(\mathcal{G})$ (respectively $C^k(\mathcal{G} \setminus V)$), the set of k-*times continuously differentiable functions on $\mathcal{G}$ (respectively $\mathcal{G} \setminus V$) *we mean the set of continuous functions on $\mathcal{G}$ (respectively $\mathcal{G} \setminus V$) whose restriction to each edge interior is k-times uniformly continuously differentiable (as a function on that real interval).*

We cannot differentiate functions on $\mathcal{G}$ without orienting the edges; however, we can always take the gradient of a differentiable function as long as we know what is meant by a vector field. Recall that a vector field on an interval is a section of its tangent bundle or, what is the same, a function on the interval with an orientation of the interval, where we identify f plus an orientation with $-f$ with the opposite orientation.

Definition 2.4 *By $C^k(\mathrm{T}\mathcal{G})$, the set of k-times continuously differentiable vector fields on $\mathcal{G}$, we mean those data consisting of a k-times uniformly continuously differentiable vector field on each open interval corresponding to each edge interior.*

Notice that a vector field is not defined on at a vertex, rather only on edge interiors.

Definition 2.5 *For $f \in C^k(\mathcal{G})$ we may form, by differentiation, its* gradient*, $\nabla f \in C^{k-1}(\mathrm{T}\mathcal{G})$. For $X \in C^k(\mathrm{T}\mathcal{G})$ we can form, by differentiation, its* calculus divergence*, $\nabla_{\mathrm{calc}} \cdot X \in C^{k-1}(\mathrm{T}\mathcal{G} \setminus V)$.*

Many theorems in analysis apply only to smooth functions or vector fields of *compact support*. Similarly, here, in working with infinite graphs (i.e. when either V or E or both are infinite), our theorems will only apply to a smaller class than C^k.

Definition 2.6 *A subset $\Omega \subset \mathcal{G}$ is of* finite type *if it lies in the union of finitely many vertices and edges. A function on $\mathcal{G}$ is of* finite type *if its support (i.e. the closure of the set where it does not vanish) is of finite type. We set $C^k_{\mathrm{fn}}(\mathcal{G})$ to be those elements of $C^k(\mathcal{G})$ of finite type; we similarly define $C^k_{\mathrm{fn}}(\mathcal{G} \setminus V)$ and $C^k_{\mathrm{fn}}(\mathrm{T}\mathcal{G})$.*

Notice that for a finite graph, i.e. when E and V are finite, every set is of finite type and C^k_{fn} coincides with C^k. Notice also that in general a set of finite type is relatively compact in the metric space topology on $\mathcal{G}$ (i.e. its closure is compact), but not conversely—

indeed, if $\mathcal{G}$ consists of a vertex and a self-loop (i.e. edge from v to v) of length $1/n$ for each integer n, then $\mathcal{G}$ itself is compact but not of finite type.

Another further subclass of $C^k(\mathcal{G})$ will be very important.

Definition 2.7 *An $f \in C^k_{\mathrm{fn}}(\mathcal{G})$ is said to satisfy the* Dirichlet condition *if f vanishes on $\partial\mathcal{G}$. We let $C^k_{\mathrm{Dir}}(\mathcal{G})$ denote the set of such functions.*

If $\partial\mathcal{G}$ is empty then clearly $C^k_{\mathrm{Dir}}(\mathcal{G}) = C^k_{\mathrm{fn}}(\mathcal{G})$.

Finally, the positive or negative part of a smooth function will usually only be Lipschitz continuous. We therefore need the following definition.

Definition 2.8 $\mathrm{Lip}(\mathcal{G})$ *denotes the class of* Lipschitz continuous *functions on $\mathcal{G}$, i.e. those $f \in C^0(\mathcal{G})$ whose restriction to each edge interior is uniformly Lipschitz continuous. We similarly define* $\mathrm{Lip}_{\mathrm{fn}}(\mathcal{G})$ *and* $\mathrm{Lip}_{\mathrm{Dir}}(\mathcal{G})$.

2.2 Two Volume Measures

In analysis concepts such as Laplacians, Rayleigh quotients, and isoperimetric constants are defined using one volume measure; in calculus on graphs we use two "volume" measures.

Definition 2.9 *A* vertex measure, *$\mathcal{V}$, is a measure supported on V with $\mathcal{V}(v) > 0$ for all $v \in V$. An* edge measure, *$\mathcal{E}$, is a measure with $\mathcal{E}(v) = 0$ for all $v \in V$ and whose restriction to any edge interior, $e \in E$, is Lebesgue measure (viewing the interior as an open interval) times a constant $a_e > 0$.*

Traditional graph theory usually works with the traditional vertex and edge measures, $\mathcal{V}_{\mathrm{T}}$ and $\mathcal{E}_{\mathrm{T}}$, given by $\mathcal{V}_{\mathrm{T}}(v) = 1$ for all $v \in V$ and $a_e = 1$ for all $e \in E$, i.e. $\mathcal{E}_{\mathrm{T}}$ is just Lebesgue measure at each edge.

Convention 2.10 Henceforth we assume that any graph has associated with it a vertex measure, $\mathcal{V}$, and an edge measure, $\mathcal{E}$. When $\mathcal{V}$ is not specified we take it to be $\mathcal{V}_{\mathrm{T}}$; similarly, when unspecified we take $\mathcal{E}$ to be $\mathcal{E}_{\mathrm{T}}$.

In this article we write

$$\int f\, d\mathcal{E} \qquad \text{and} \qquad \int f\, d\mathcal{V}$$

for

$$\int_{\mathcal{G}} f\, d\mathcal{E} \qquad \text{and} \qquad \int_{\mathcal{G}} f\, d\mathcal{V}.$$

In this article, if μ is a measure on $\mathcal{G}$, $1 \le p \le \infty$, and f a is real or vector-valued on $\mathcal{G}$ with $|f|$ μ-measurable, then we define the usual $L^p(\mathcal{G}, \mu)$ norm

$$\|f\|_{p,\mu} = \left(\int_{\mathcal{G}} |f|^p \, d\mu \right)^{1/p}$$

(with $p = \infty$ we take the norm to be the essential supremum of $|f|$ with respect to μ).

Convention 2.11 If $f : \mathcal{G} \to \mathbf{R}$ is measurable, then $\|f\|_p$ means $\|f\|_{p,\mathcal{V}}$ unless otherwise mentioned. For $X \in C^0(\mathrm{T}\mathcal{G})$, $\|X\|_p$ means $\|X\|_{p,\mathcal{E}}$.

2.3 Three Rayleigh Quotients

In this subsection we pause to give an example of translating results from analysis to results in graph theory. By a *Rayleigh quotient* we mean a functional

$$\mathcal{R}(f) = \frac{\int |\nabla f|^2 \, d\mu_1}{\int |f|^2 \, d\mu_2}$$

defined on some class of functions, f, in a setting where the above quotient has some reasonable interpretation. We shall now give three precise settings where this Rayleigh quotient makes sense.

1. **Analysis or Riemannian geometry:** the above Rayleigh quotient appears with $d\mu_1 = d\mu_2 = dV_g$, the Riemannian volume (or the usual volume in $\mathbf{R}^n$):

$$\mathcal{R}(f) = \frac{\int |\nabla f(x)|^2 \, dV_g(x)}{\int f^2(x) \, dV_g(x)}.$$

2. **Traditional graph theory:** here a function is a function on the vertices of the graph. To define ∇f we (arbitrarily and unnaturally) fix an orientation for each edge; we declare ∇f on an edge oriented (u, v) to be $f(v) - f(u)$; the gradient

is therefore a real number. The measures $d\mu_i$, $i = 1, 2$ are taken to be the edge counting and vertex counting measures. The Rayleigh quotient becomes:

$$\mathcal{R}(f) = \frac{\sum_{\{u,v\}\in E} \bigl(f(u) - f(v)\bigr)^2}{\sum_{v\in V} f^2(v)}$$

In this setting notions from analysis such as those of nodal regions, level sets of a function, etc. do not have an exact translation; proofs of theorems in traditional graph theory that implicitly use such notions may be unnecessarily awkward.

3. **Calculus on graphs:** here a function is a function on the geometric realization. ∇f is defined as above, and $d\mu_1 = d\mathcal{E}$, $d\mu_2 = d\mathcal{V}$, and the Rayleigh quotient is

$$\mathcal{R}(f) = \frac{\int |\nabla f|^2 \, d\mathcal{E}}{\int |f|^2 \, d\mathcal{V}}.$$

   Many more concepts and theorems in analysis translate almost immediately to this setting. It is usually easy to see that theorems in graph theory are the same whether one states them in this setting or in traditional graph theory.

   For example, consider a minimizer, f, of $\mathcal{R}(f)$ subject to certain conditions on f's values at the vertices. It is easy to see that such a minimizer must be *edgewise linear*, i.e. linear when restricted to any edge interior (see proposition 3.1); hence by restricting f to its values at the vertices and taking the Rayleigh quotient of traditional graph theory we get the same Rayleigh quotient as here.

Consider a Rayleigh quotient in traditional graph theory that we wish obtain in calculus on graphs when restricted to edgewise linear functions. Then the a_e/ℓ_e's and $\mathcal{V}(v)$'s are determined up to a multiplicative constant, since

$$\int |f|^2 \, d\mathcal{V} = \sum_v |f^2(v)| \mathcal{V}(v) \quad \text{and} \quad \int |\nabla f|^2 \, d\mathcal{E} = \sum_{e=\{u,v\}\in E} |f(u) - f(v)|^2 (a_e/\ell_e)$$

for edgewise linear f. However, we have the freedom to set either a_e or ℓ_e as we please; at times there are reasons to set one or the other to 1.

2.4 Half-degree and simple inequalities

If f is *edgewise linear*, i.e. f is continuous and its restriction to each edge interior is a linear function, then clearly

$$\int_e f \, d\mathcal{E} = \Bigl(f(u) + f(v)\Bigr) \mathcal{E}(e)/2$$

for each edge $e = \{u, v\}$. Hence

$$\int f \, d\mathcal{E} = \int f \rho \, d\mathcal{V}, \tag{1}$$

where

$$\rho(v) = \mathcal{V}(v)^{-1} \sum_{e \ni v} \mathcal{E}(e)/2. \tag{2}$$

Definition 2.12 *The* half-degree *of a vertex, v, is $\rho(v)$ as defined above. We denote the infimum and supremum of ρ's values by $\rho_{\inf}$ and $\rho_{\sup}$.*

In traditional graph theory, where $\mathcal{E} = \mathcal{E}_{\mathrm{T}}$ and $\mathcal{V} = \mathcal{V}_{\mathrm{T}}$ we have that $\rho(v)$ is just one-half the degree of v.

By convention a graph, $\mathcal{G}$, encompasses the specification of $\mathcal{V}$ and $\mathcal{E}$; hence we may write $\rho_{\sup}(\mathcal{G})$ and $\rho_{\inf}(\mathcal{G})$ without ambiguity.

Definition 2.13 *We say that a graph $\mathcal{G}$ is r-*regular *if $\rho_{\inf}(\mathcal{G}) = \rho_{\sup}(\mathcal{G}) = r/2$.*

Clearly we have:

Proposition 2.14 *Let $f \in C^0_{\mathrm{fn}}(\mathcal{G})$ be an* edgewise convex *function, i.e. its restriction to each edge is convex*[1] *(not necessary strictly convex); further assume that f is non-negative on all vertices. Then*

$$\int f \, d\mathcal{E} \le \rho_{\sup} \int f \, d\mathcal{V}. \tag{3}$$

Similarly for a non-negative edgewise concave function we have

$$\int f \, d\mathcal{E} \ge \rho_{\inf} \int f \, d\mathcal{V}. \tag{4}$$

If $\mathcal{G}$ is regular then we can drop the requirement that f be non-negative at the vertices.

Proof We will show equation 3; equation 4 follows similarly.

[1] We mean $f\big(\alpha x + (1-\alpha)y\big) \le \alpha f(x) + (1-\alpha) f(y)$ for all $\alpha \in [0,1]$ and x, y on the open edge interval. So f does not have to be C^2 or C^1; e.g. the function $|x - 1/2|$ is convex on $[0,1]$.

Let $\widetilde{f}$ be the edgewise linear function whose values at the vertices agree with those of f. Then $\widetilde{f} \geq f$ and so $\int f \, d\mathcal{E} \leq \int \widetilde{f} \, d\mathcal{E}$. The non-negativity of $\widetilde{f}$ at the vertices implies that

$$\int \rho \widetilde{f} \, d\mathcal{V} \leq \rho_{\mathrm{sup}} \int \widetilde{f} \, d\mathcal{V}.$$

These inequalities and equation 1 imply

$$\int f \, d\mathcal{E} \leq \int \widetilde{f} \, d\mathcal{E} = \int \rho \widetilde{f} \, d\mathcal{V} \leq \rho_{\mathrm{sup}} \int \widetilde{f} \, d\mathcal{V} = \rho_{\mathrm{sup}} \int f \, d\mathcal{V},$$

using the fact that f and $\widetilde{f}$ agree on the vertices. We conclude equation 3. Furthermore, if $\mathcal{G}$ is regular, then $\int \rho \widetilde{f} \, d\mathcal{V} = \rho_{\mathrm{sup}} \int \widetilde{f} \, d\mathcal{V}$ for any $\widetilde{f}$, and we can drop the requirement that f's values be non-negative at the vertices.

□

The above proposition has one special case that we will often need in establishing inequalities.

Corollary 2.15 *For any edgewise linear f we have*

$$\|f\|_{p,\mathcal{E}} \leq \rho_{\mathrm{sup}}^{1/p} \|f\|_{p,\mathcal{V}} \tag{5}$$

for any $p \geq 1$.

Proof It suffices to show that

$$\int |f|^p \, d\mathcal{E} \leq \rho_{\mathrm{sup}} \int |f|^p \, d\mathcal{V}.$$

But $|f|^p$ is edgewise convex for $p \geq 1$.

□

Note that in analysis one has $\mathcal{V}$ and $\mathcal{E}$ replaced by the same measure, and so the two integrals above are the same. The closest we can seem to come to this in graph theory is

$$\int f \, d\mathcal{E} = \int f \, d\mathcal{V}$$

for 2-regular graphs and edgewise linear functions, f. Of course, a d-regular graph becomes a 2-regular graph if either $\mathcal{E}$ or $\mathcal{V}$ is scaled appropriately.

We remark that given $\mathcal{E}$ such that for any vertex, v, we have

$$\sum_{e \ni v} \mathcal{E}(e) \ < \infty,$$

there is a unique measure $\mathcal{V}$ such that $\mathcal{G}$ is 2-regular. We call $\mathcal{V}$ the *natural* measure with respect to $\mathcal{E}$.

We finish by discussing ρ in other settings.

1. If G is viewed as an irreducible, reversible Markov chain, with stationary distribution π and transition probabilities K, then for the typical Rayleigh quotient used we have $\mathcal{V}_{\mathrm{P}}(u) = \pi(u)$ and $a_e/\ell_e = \pi(u)K(u,v) = \pi(v)K(v,u)$. If we insist upon $\ell_e = 1$, then we have $\mathcal{E}_{\mathrm{P}}(e) = a_e\ell_e = a_e = \pi(u)K(u,v) = \pi(v)K(v,u)$. Since $\sum_v K(u,v) = 1$ for fixed u, we have that $(\mathcal{V}_{\mathrm{P}}, \mathcal{E}_{\mathrm{P}})$ is 1-regular.

2. In [DK86, DK88, Chu93], the denominator of the traditional Rayleigh quotient is modified from the sum of $f^2(v)$ to that of $d_v f^2(v)$, where d_v is the degree of v. This corresponds to taking $\mathcal{E} = \mathcal{E}_{\mathrm{T}}$ and $\mathcal{V}(v) = d_v$. In this case $\rho(v) = 1/2$ for all v, i.e. $\mathcal{G}$ is 1-regular.

2.5 Remarks on Square Norm Inequalities

In this subsection we make two remarks on the inequality in equation 5 for $p = 2$, namely

$$\|f\|_{2,\mathcal{E}} \le \rho_{\sup}^{1/2} \|f\|_{2,\mathcal{V}} \tag{6}$$

This inequality comes up a lot in Laplacian eigenvalues and therefore Cheeger's inequality.

Our first remark is that while the analogue of this inequality cannot be improved upon in analysis (for there $\mathcal{E} = \mathcal{V}$), it can be improved, in a certain sense, in graph theory. Namely, if f is linear on an edge $e = \{u, v\}$ and $f(u) = b$ and $f(v) = c$, then it is easy to see that

$$\int_e f^2 \, d\mathcal{E} = a_e(b^2 + bc + c^2)/3, \quad \text{and} \quad \int_e |\nabla f|^2 \, d\mathcal{E} = a_e(b^2 - 2bc + c^2),$$

provided that all edge lengths are 1. We easily conclude that

$$\|f\|_{2,\mathcal{E}}^2 + (1/6)\|\nabla f\|_2^2 = \int \rho f^2 \, d\mathcal{V} \le \rho_{\sup} \|f\|_{2,\mathcal{V}}^2, \tag{7}$$

with equality if $\mathcal{G}$ is regular. This improvement to equation 6 is interesting in Laplacian eigenvalues and Cheeger's inequality, for there it is precisely $\|\nabla f\|_2$ that one is interested in and bounding from below. This type of improvement seems to have been first exploited by Mohar (see [Moh88, Moh89]).

Our second remark is that later in this article we will be interested in upper bounding $\|fX\|_{2,\mathcal{E}}/\|f\|_{2,\mathcal{V}}$ for a function f and an edgewise constant vector field, X. To do so we simply remark that

$$\|fX\|_{2,\mathcal{E}} \leq \rho_{\sup}^{1/2}(\mathcal{G}^{|X|^2})\|f\|_{2,\mathcal{V}}, \tag{8}$$

where $\mathcal{G}^{|X|^2}$ is the graph obtained from $\mathcal{G}$ by multiplying the edge weights, a_e, by $|X(e)|^2$.

2.6 The Divergence

The divergence of a vector field and the Laplacian of a function can be defined in terms of concepts that are already fixed, namely a graph (encompassing measures $\mathcal{E}$ and $\mathcal{V}$) and the gradient. Interestingly enough, the divergence turns out to be different from the "calculus divergence" described earlier.

Before defining the divergence we record a "divergence theorem" for the calculus divergence.

Let $X \in C^1(\mathrm{T}\mathcal{G})$. For any edge $e = \{u, v\}$ let $X|_e$ denote X restricted to the interior of e and then extended to u and v by continuity. We clearly have

$$\int_e \nabla_{\mathrm{calc}} \cdot X \, d\mathcal{E} = a_e\Big(\mathbf{n}_{e,u} \cdot X|_e(u) + \mathbf{n}_{e,v} \cdot X|_e(v)\Big),$$

where $\mathbf{n}_{e,u}, \mathbf{n}_{e,v}$ denote outward pointing unit (normal) vectors. Hence we obtain:

Proposition 2.16 *For all $X \in C^1_{\mathrm{fn}}(\mathcal{G})$ we have*

$$\int \nabla_{\mathrm{calc}} \cdot X \, d\mathcal{E} = \int \widetilde{\mathbf{n}} \cdot X \, d\mathcal{V}, \tag{9}$$

where

$$(\widetilde{\mathbf{n}} \cdot X)(v) = \mathcal{V}(v)^{-1} \sum_{e \ni v} a_e \mathbf{n}_{e,v} \cdot X|_e(v).$$

Equation 9 shows that ∇_{calc} cannot be the right notion of a divergence. Indeed, in analysis the analogue to $\widetilde{\mathbf{n}} \cdot X$ is integrated over the *boundary*, and we do not wish to consider every vertex of $\mathcal{G}$ as a boundary point. Fortunately the notion of the divergence is essentially forced upon us by previously fixed concepts.

Let $C^k_{\mathrm{Dir}}(\mathcal{G})$ denote those functions in $C^k_{\mathrm{fn}}(\mathcal{G})$ that vanish on the boundary of $\mathcal{G}$.

Definition 2.17 *For a vector field, X, its* divergence functional *is the linear functional $\mathcal{L}_X \colon C^\infty_{\mathrm{Dir}}(\mathcal{G}) \to \mathbf{R}$ given by,*

$$\mathcal{L}_X(g) = -\int X \cdot \nabla g \, d\mathcal{E}.$$

Proposition 2.18 *For any $X \in C^1(\mathrm{T}\mathcal{G})$ and $g \in C^\infty_{\mathrm{Dir}}(\mathcal{G})$ we have*

$$\mathcal{L}_X(g) = \int (\nabla_{\mathrm{calc}} \cdot X) g \, d\mathcal{E} - \int (\widetilde{\mathbf{n}} \cdot X) g \, d\mathcal{V},$$

i.e. the divergence functional of X is represented by $(\nabla_{\mathrm{calc}} \cdot X)\mathcal{E} - (\widetilde{\mathbf{n}} \cdot X)\mathcal{V}$ (viewed as a linear functional via integration).

Proof We substitute Xg for X in equation 9, and note that $\nabla_{\mathrm{calc}} \cdot (Xg) = g\nabla_{\mathrm{calc}} \cdot X + X \cdot \nabla g$.

Definition 2.19 *For $X \in C^1(\mathrm{T}\mathcal{G})$ we define its* divergence, *$\nabla \cdot X$, to be the measure*

$$(\nabla_{\mathrm{calc}} \cdot X) d\mathcal{E} - (\widetilde{\mathbf{n}} \cdot X) d\mathcal{V}.$$

If X is edgewise constant, so that $\nabla_{\mathrm{calc}} \cdot X = 0$, we will also refer to

$$-\widetilde{\mathbf{n}} \cdot X$$

(a function defined only on vertices) as its divergence, and write $\nabla \cdot X$ for it.

Definition 2.19 clearly involves some amount of foresight and/or cheating. Indeed, since $\mathcal{L}_X$ is defined only on functions that vanish on $\partial\mathcal{G}$, we have no business defining $\nabla \cdot X$ on a boundary vertex. Pedantically, for $g \in C^1_{\mathrm{fn}}(\mathcal{G})$ we should now search for the missing term in

$$-\int X \cdot \nabla g \, d\mathcal{E} = \int_{\mathcal{G} \setminus \partial\mathcal{G}} (\nabla \cdot X) g + \text{ missing term}.$$

But by equation 9 (substituting Xg for X) we have

$$\text{missing term } = -\int_{\partial\mathcal{G}} (\widetilde{\mathbf{n}} \cdot X) g \, d\mathcal{V} = \int_{\partial\mathcal{G}} (\nabla \cdot X) g.$$

We conclude:

Proposition 2.20 *For any $g \in C^1_{\mathrm{fn}}(\mathcal{G})$ and $X \in C^1_{\mathrm{fn}}(\mathrm{T}\mathcal{G})$ we have*

$$\int_{\mathcal{G}} (\nabla \cdot X) g + \int X \cdot \nabla g \;\; d\mathcal{E} = 0.$$

To make this look more like analysis we can write this as:

$$\int_{\mathcal{G} \setminus \partial\mathcal{G}} (\nabla \cdot X) g + \int X \cdot \nabla g \, d\mathcal{E} = \int_{\partial\mathcal{G}} (\widetilde{\mathbf{n}} \cdot X) g \, d\mathcal{V}.$$

2.7 The Laplacian

In graph theory we usually define positive semidefinite Laplacians. So we define

$$\Delta f = -\nabla \cdot (\nabla f).$$

By using proposition 2.20 we obtain

Proposition 2.21 *For all $f \in C^2_{\rm fn}(\mathcal{G})$ and $g \in C^1_{\rm fn}(\mathcal{G})$ we have*

$$\int (\Delta f) g = \int \nabla f \cdot \nabla g \, d\mathcal{E}. \tag{10}$$

If also $g \in C^2_{\rm fn}(\mathcal{G})$ we have

$$\int (\Delta f) g = \int (\Delta g) f. \tag{11}$$

Proposition 2.22 *For $f \in C^2_{\rm fn}(\mathcal{G})$ which is edgewise linear we have $\nabla_{\rm calc} \cdot \nabla f = 0$ and so $\Delta f = \widetilde{\mathbf{n}} \cdot \nabla f \, d\mathcal{V}$. Viewing Δf as a function on vertices we therefore have:*

$$(\Delta f)(v) = \mathcal{V}(v)^{-1} \sum_{e \sim \{u,v\}} a_e \frac{f(v) - f(u)}{\ell_e}. \tag{12}$$

When restricting to edgewise linear functions, it is common (in graph theory) to write Δ as $D - A$, where D is the diagonal matrix or operator (classically the "degree" matrix) whose v, v entry is:

$$L(v) = \mathcal{V}(v)^{-1} \sum_{e \sim \{u,v\}} a_e / \ell_e,$$

where we omit e's that are self-loops from the summation, and where A is the "adjacency" matrix or operator given by

$$(Af)(v) = \mathcal{V}(v)^{-1} \sum_{e \sim \{u,v\}} (a_e / \ell_e) f(u),$$

again omitting self-loops, e.

A standard and easy application of Cauchy-Schwartz shows that

$$|(Af, f)| \leq (Df, f)$$

when f is edgewise linear (allowing for the possibility that one or both sides is $+\infty$).

We will now view Δ as an operator, and bound its norm. To simplify matters, we shall assume for the rest of this subsection that our graph is locally finite, i.e. each vertex is incident upon only finitely many edges.

Let $L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$ be the subspace of edgewise linear functions which vanish on $\partial\mathcal{G}$ and which lie in $L^2(\mathcal{G}, \mathcal{V})$. We make Δ operate on $f \in L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$ by taking Δf to be 0 on $\mathcal{G}$, to be defined by equation 12 on other vertices, and edgewise linear; as such, Δf may not lie in $L^2(\mathcal{G}, \mathcal{V})$, and in this case we view Δ as undefined on f.

Consider the norm of Δ as an operator on $L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$,

$$\|\Delta\| = \sup_{f \in L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})} \|\Delta f\| / \|f\|.$$

Just as in traditional graph theory (see [Moh82]) we have:

Proposition 2.23 *$L_{\mathrm{sup}} \le \|\Delta\| \le 2L_{\mathrm{sup}}$, where L_{sup} is the supremum of the $L(v)$ (over $v \notin \mathcal{G}$). In particular, Δ is bounded iff L_{sup} is.*

Proof For $v \in V$ it is clear that

$$\|\Delta \chi_v\| / \|\chi_v\| \ge L(v),$$

where χ_v is the edgewise linear characteristic function of v, i.e. 1 on v, 0 on other vertices, and edgewise linear. Hence the first inequality. For the second we have for a function of finite type

$$(\Delta f, f) \le |(Af, f)| + (Df, f) \le 2(Df, f) \le 2L_{\mathrm{sup}}(f, f).$$

Since Δ is symmetric on functions of finite type, and since the set of such functions is dense in $L^2_{\mathrm{Dir}}(\mathcal{G}, cv)$, we conclude the second inequality.

□

2.8 Co-area Formulas

When an integral involves $|\nabla f|$, this term can be removed by integrating over the level surfaces of f. The simplest case of this is the one dimensional case, which intuitively expresses the first-year calculus formula: $df = f'(x)dx$. The general case is referred to as the *co-area* formula. We will use it in our Federer-Fleming theorems.

Recall that for each $e \in E$ we have an associated weight, a_e (that determines $\mathcal{E}$ restricted to e).

Definition 2.24 *For $x \in \mathcal{G} \setminus V$ we define the* area *of x to be a_e where e is the edge containing x. For $F \subset \mathcal{G} \setminus V$ we define its* area*, $\mathcal{A}(F)$, to be the sum of the areas of all non-vertex points in F.*

So if $\Omega \subset \mathcal{G}$, it makes sense to speak of $\mathcal{A}(\partial\Omega)$ provided that $\partial\Omega$ contains no vertices. Later in this subsection we define $\mathcal{A}(\partial\Omega)$ even when $\partial\Omega$ contains vertices.

For any $f \in C^0(\mathcal{G})$ we set

$$\Omega(t) = \Omega_f(t) = \{x \in \mathcal{G} | f(x) > t\}.$$

For $f \in \mathrm{Lip}_{\mathrm{fn}}(\mathcal{G})$ we have that for almost all t, $\partial\Omega(t)$ is a finite set of points (see [Mor88], page 24).

Proposition 2.25 (The Co-Area Formula) *Let $f \in \mathrm{Lip}_{\mathrm{fn}}(\mathcal{G})$ and for any t let $\mathcal{A}_t$ be the restriction of $\mathcal{A}$ to $\partial\Omega_f(t)$. Then we have $|\nabla f|\, d\mathcal{E} = d\mathcal{A}_t\, dt$, in the sense that for any $\phi \in C^0(\mathcal{G})$ we have*

$$\int \phi\, |\nabla f|\, d\mathcal{E} = \int \left(\int \phi\, d\mathcal{A}_t \right) dt$$

(where the integral in t is taken to be a Lebesgue integral). In particular:

$$\int |\nabla f|\, d\mathcal{E} = \int \int d\mathcal{A}_t\, dt = \int \mathcal{A}\Big(\partial\Omega(t)\Big)\, dt.$$

Since $\partial\Omega(t)$ contains a vertex for only finitely many t, it is irrelevant how we define its area in the proposition. Versions of this proposition have appeared implicitly in many places, and this proposition appears explicitly in [BH97].

Proof It suffices to prove this theorem when we integrate over any edge. It this case this is just the standard co-area formula in one dimension (see [Fed69] or [Mor88] for a proof).

$\square$

We remark that the one dimensional co-area formula we use is easy to prove in virtually all our applications. Indeed, this formula is equivalent to saying that if $f \in \mathrm{Lip}[0,1]$ and $a < b$ are reals, then

$$\int_{\{x\,:\,a<f(x)<b\}} |\nabla f|\, dx = \int_a^b \Big(\#\text{ of } f^{-1}(t)\Big)\, dt \tag{13}$$

In its applications to calculus on graphs, f will typically be piecewise differentiable and f' will change signs a finite number of times; then equation 13 follows immediately from $df = f'(x)dx$ (with $t = f$ in the formula). However, to prove the above co-area formula for *any* $f \in \mathrm{Lip}[0,1]$ requires a more subtle argument.

We mention that the co-area formula also holds when our functions have a finite number of discontinuities. In this case we interpret $\int \phi |\nabla f| \, d\mathcal{E}$ for such f to mean that we add $\phi(x)a_e$ times the jump in f at x for each point of discontinuity of f, where $e \in x$, and if x is a vertex then we add one such contribution for each e meeting x using the jump along e of f at x. (This understanding agrees with how we define $\int |\nabla f| \, d\mathcal{E}$ as $\mathcal{M}(f)$ in section 4.) However, to make such a formula valid we must adopt the definition below.

Definition 2.26 *For any open $\Omega \subset \mathcal{G}$, we say v* belongs to the e closure of Ω, *written $v \in \overline{\Omega|_e}$, if the closure of Ω intersected with the edge interior of e contains v. We define*

$$\mathcal{A}(\partial\Omega) = \mathcal{A}(\partial\Omega \setminus V) + \sum_{v \in \partial\Omega \cap V} \left(\sum_{e \text{ with } v \in \overline{\Omega|_e}} a_e \right)$$

Clearly $\mathcal{A}(\partial\Omega)$ is just $\int |\nabla \chi_\Omega| \, d\mathcal{E}$ in the above sense; as we shall see, it is also $\mathcal{M}(\chi_\Omega)$ in the sense of section 4.

Proposition 2.27 *The above co-area formula holds for any finitely supported locally Lipschitz function with at most a finite number of discontinuities, given the above definition of area and understanding of $\int \phi |\nabla f| \, d\mathcal{E}$.*

Proof For each jump we glue in an interval of arbitrary length, and let us redefine f on each interval to vary linearly between the values of its two limits at the jump. Now apply the old co-area formula to the new graph and new f.

$\square$

We finish by remarking that it is easy to see that

$$\mathcal{A}(\partial\Omega) = \lim_{h \to 0} \frac{\mathcal{E}(\Omega) - \mathcal{E}(\Omega_h)}{h},$$

where Ω_h denotes the set of points in Ω whose distance to $\partial\Omega$ is more than h. If we let Ω^h be the set of points in $\mathcal{G}$ whose distance to Ω is less than h (including all points in Ω), then [BH97] give a general co-area formula where the "area" would be given by

$$\lim_{h \to 0} \frac{\mathcal{E}(\Omega^h) - \mathcal{E}(\Omega)}{h}.$$

This definition disagrees with ours only when $\partial\Omega$ contains vertices, and this disagreement is important to a co-area formula only when the function has a discontinuity at one or more vertices.

2.9 Countability

We close with some remarks about the countability of our graphs. Nowhere do we assume that our graphs have countable vertex sets, edges sets, or in particular vertex degrees. However, for the sake of intuition, in most applications it is fairly safe to assume that the vertex and edge sets and vertex degrees are countable. For example, if $\|f\|_q, \|\nabla f\|_p$ are finite for some p, q, then clearly f vanishes at any vertex of uncountable degree; such vertices may exist, but they tend to be forced "boundary points" (so neighbours of such points can't "interact" through that point). Also, notice that if each vertex has countable degree, then each connected component of $\mathcal{G}$ has only countably many vertices and edges.

Notice that there are many important graphs with some countably infinite vertex degrees, such as those in [AFKM86]. However, for those graphs the lengths of the edges incident upon a vertex tend to infinity essentially as a geometric series, and so the Laplacians, for example, are still finite.

3 Preliminaries for Gradient Inequalities

By a *gradient inequality* we mean a lower bound on $\|\nabla f\|_p$ in terms of norms on f and constants that may depend on $\mathcal{G}$. In this section we give some preliminary concepts needed to state and prove the gradient inequalities appearing in the rest of this paper.

3.1 Edgewise linearity

Proposition 3.1 *Consider those $f \in C^1_{\mathrm{fn}}(\mathcal{G})$ taking on prescribed values at all vertices. For $p > 1$, $\|\nabla f\|_p$ is minimized exactly when f is edgewise linear, and for $p = 1$ when f is monotone along each edge.*

This proposition tells us that in proving a gradient inequality we can restrict ourselves to edgewise linear functions.

Proof Clearly for an edge $e = \{u, v\}$ we have

$$\int_e |\nabla f| \, d\mathcal{E} \geq |f(u) - f(v)| a_e, \tag{14}$$

with equality iff f is monotone. This proves the statement for $p = 1$ (summing over e). For $p > 1$ we apply Jensen's or Hölder's inequality to equation 14 to conclude that the integral of $|\nabla f|^p$ over e is minimized precisely when $|\nabla f|$ is constant, i.e. f is edgewise linear.

$\square$

3.2 Closed Graphs: Finite Graphs Without Boundary

When constant functions lie in $C^1_{\mathrm{Dir}}(\mathcal{G})$, there are no interesting gradient inequalities that hold over all of $C^1_{\mathrm{Dir}}(\mathcal{G})$. This is the case for finite graphs without boundary (and compact manifolds without boundary).

Definition 3.2 *A* closed *graph is a finite graph without boundary.*

We form interesting inequalities by working "modulo constant functions," in a sense to made precise below.

Our remarks are very general, and there is no loss in generality in working with an arbitrary measure, μ, on a space M whose total measure is finite and non-zero.

For any $r > 0$ and $X \in \mathbf{R}^n$ let $\{X\}^r$ denote $|X|^{r-1}X$ (interpreted as 0 if $X = 0$).

Proposition 3.3 *Let $1 < p \leq \infty$. For $f \in L^p(M, \mu)$ there is a unique $a \in \mathbf{R}$ such that $\|f - a\|_p < \|f - b\|_p$ for all $b \in \mathbf{R}$ with $b \neq a$. Furthermore, a is the unique solution to*

$$\int \{f - a\}^{p-1} \, d\mu = 0$$

for $p < \infty$, and for $p = \infty$ we have that a is the average of the essential supremum and essential infimum of f.

Proof For $p = \infty$ this is clear, so assume $p < \infty$. It is easily checked that the function $g(t) = \|f - t\|_p^p$, for $t \in \mathbf{R}$, is differentiable, and that

$$g'(t) = p \int \{f - t\}^{p-1} \, d\mu.$$

It follows that $g'(t)$ is continuous, strictly increasing, and tends to $\pm\infty$ as $t \to \pm\infty$.

$\square$

Definition 3.4 *f is said to be p-balanced if $\|f\|_p \le \|f-a\|_p$ for any $a \in \mathbf{R}$.*

The above proposition has a $p=1$ version, although a is not unique. This version is linked to the important notion of being *split*.

Definition 3.5 *A measurable f is said to be* split *if $\mu(\{x|f(x)>0\})$ and $\mu(\{x|f(x)<0\})$ are both $\le \mu(M)/2$.*

Proposition 3.6 *For any $f \in L^1(M)$ let $I \subset \mathbf{R}$ be the set on which $g(t) = \|f-t\|_1$ achieves its minimum, and let $J \subset \mathbf{R}$ be the set of t such that $f-t$ is split. Then I and J are nonempty compact intervals, and $I \subset J$.*

Proof $g(t) \ge \|t\|_1 - \|f\|_1 = |t|\mu(M) - \|f\|_1$. It follows that $g(t) \to \infty$ as $t \to \pm\infty$. Since $g(t)$ is continuous (indeed, $|g(t)-g(s)| \le |t-s|\mu(M)$ by the triangle ineqality), and convex (using the triangle inequality), it follows that I is a nonempty compact interval. Clearly J is a nonempty compact interval. Next, if $t \notin J$, then we claim $t \notin I$; indeed, assume that $\mu(\{x|f(x)>t\}) > \mu(M)/2$. Then there is a t_0 for which $\mu(\{x|f(x)>t\}) > \mu(\{x|f(x)>t_0\}) \ge \mu(M)/2$, and it is easy to see that $g(t_0) < g(t)$ so that $t \notin I$. Similarly if $\mu(\{x|f(x)<t\}) > \mu(M)/2$ we similarly conclude $t \notin I$.

$\square$

Proposition 3.7 *For any $1 \le p \le \infty$ and $f \in L^p = L^p(M,\mu)$ we have*

$$\inf_{a\in\mathbf{R}} \|f-a\|_p = \sup_{\|g\|_{p'}=1,\ \int g=0} \int fg\,d\mu. \tag{15}$$

Proof Let a minimize $\|f-t\|_p$ as a function of t. For any g as above we have

$$\int fg\,d\mu = \int (f-a)g\,d\mu \le \|f-a\|_p\|g\|_{p'} = \|f-a\|_p. \tag{16}$$

Thus equation 15 holds with $=$ replaced by $\ge$, and it remains to prove $\le$.

First consider the case $1 < p < \infty$. Equality will hold in equation 16 provided that (1) $|g|^{p'}$ is a constant times $|f-a|^p$, and (2) g and $f-a$ have the same sign. But (1) just means that $|g|$ is proportional to $|f-a|^{p/p'} = |f-a|^{p-1}$; so taking g proportional to $\{f-a\}^{p-1}$ in such a way that $\|g\|_{p'} = 1$, proposition 3.3 shows that $\int g = 0$ and thus we conclude the proposition.

For $p = 1, \infty$ we need special arguments to construct g as above. Consider $p = 1$. Take A, B respectively to be the sets where $f-a$ is positive and negative, and set

$$g(x) = \begin{cases} 1 & \text{if } x \in A, \\ -1 & \text{if } x \in B, \\ c & \text{otherwise,} \end{cases}$$

where c is any number between -1 and 1 if $\mu(A) = \mu(B) = \mu(M)/2$, and otherwise

$$c = \frac{\mu(A) - \mu(B)}{\mu(M) - \mu(A) - \mu(B)}$$

(since $f-a$ is split, we have $\mu(M) - \mu(A) - \mu(B) > 0$ unless $\mu(A) = \mu(B) = \mu(M)/2$). If $\mu(A) = \mu(B) = \mu(M)/2$, then clearly $\int g\, d\mu = 0$ and $\|g\|_\infty = 1$, and $\int fg\, d\mu = \|f-a\|_1$. If not, then $\mu(M) - 2\mu(B) > 0$ since $f-a$ is split, and so

$$c + 1 = \frac{\mu(M) - 2\mu(B)}{\mu(M) - \mu(A) - \mu(B)} > 0.$$

Similarly $c - 1 < 0$, and we conclude $-1 < c < 1$. Thus $\|g\|_\infty = 1$, and it is easy to check that $\int g\, d\mu = 0$ and $\int fg\, d\mu = \|f-a\|_1$.

Next consider $p = \infty$. There is no analogue of g in this case, but we claim there is a sequence, g_ϵ, defined for small $\epsilon > 0$, such that $\int fg_\epsilon\, d\mu \to \|f-a\|_\infty$. Namely, for any $\epsilon > 0$ let A_ϵ be the set where $f + \epsilon >$ than the essential supremum of f, and B_ϵ the set where $f - \epsilon$ is small than the essential infimum. Set

$$g_\epsilon = \frac{1}{2\mu(A_\epsilon)}\chi_{A_\epsilon} - \frac{1}{2\mu(B_\epsilon)}\chi_{B_\epsilon}.$$

It is easy to see that $\|g_\epsilon\|_1 = 1$, that $\int g_\epsilon\, d\mu = 0$, and that $\int fg_\epsilon\, d\mu$ is within ϵ of $\|f-a\|_\infty$.

$\square$

4 Federer-Fleming Theorems

In this section we prove some Federer-Fleming type theorems. Roughly speaking, these theorems say that certain functionals attain their minimum on characteristic functions.

There are many approaches to proving such theorems; our approach most closely follows that of Rothaus in [Rot81]. In the first two subsections we state the theorems. Then we give an overview of how these theorems are proved, based on a simple inequality. The later subsections give the details.

4.1 Statement of the Federer-Fleming Theorem

The classical Federer-Fleming Theorem looks as follows in graph theory. By an *admissible* $\Omega \subset \mathcal{G}$ we mean an open Ω with $\partial\Omega$ finite and Ω disjoint from $\partial\mathcal{G}$. For $1 \le \nu \le \infty$ we set

$$i_\nu(\Omega) = \mathcal{A}(\partial\Omega)/\mathcal{V}(\Omega)^{1/\nu'}, \qquad I_\nu(\mathcal{G}) = \inf_{\Omega \text{ admissible}} i_\nu(\Omega)$$

(if $\mathcal{E}(\Omega) = 0$ we take $i_\nu(\Omega) = +\infty$). Next we set

$$s_\nu(f) = \|\nabla f\|_1/\|f\|_{\nu'}, \qquad S_\nu(\mathcal{G}) = \inf_{f \subset C^1_{\mathrm{Dir}}(\mathcal{G})} s_\nu(f),$$

where we understand the norms on ∇f and f to be with respect to, respectively, $\mathcal{E}$ and $\mathcal{V}$.

We shall prove in this section the graph theoretic analogue of the Federer-Fleming theorem:

Theorem 4.1 *For any $1 \le \nu \le \infty$ we have $I_\nu(\mathcal{G}) = S_\nu(\mathcal{G})$.*

This was essentially proven[2] in [Var85, CF91].

The inequality $S \le I$ follows easily by (any reasonable way of) approximating any characteristic function of finite type by C^1_{Dir} functions. The reverse inequality is discussed in the following few subsections.

The Federer-Fleming theorem above can be viewed as a gradient inequality:

Corollary 4.2 *For any $f \in C^1_{\mathrm{Dir}}(\mathcal{G})$ we have*

$$\|\nabla f\|_1 \ge I_\nu(M)\|f\|_{\nu'}.$$

[2] Although $I = S$ was not stated in either article, the proofs of weaker statements given there are easily modified to give $I = S$; for example, in [CF91] lemma 4 involves a constant depending on ν; but this constant can be taken to be 1, as a standard inequality (equation (6.11), page 269, of [Cha93]) shows; this gives $I = S$ (or $2I = S$ with their conventions).

4.2 Federer-Fleming for Closed Graphs

The Federer-Fleming theorem above can be interesting when $\mathcal{G}$ is not finite or has a boundary. However, for finite graphs without boundary, this theorem just says that $0 = 0$, by considering $f = 1$ and $\Omega = \mathcal{G}$.

The traditional way to remedy the fact that $I_\nu = S_\nu = 0$ in the closed case is to set

$$\widetilde{i}_\nu(\Omega) = |\partial\Omega| \min(|\Omega|, |\overline{\Omega}|)^{(1/\nu)-1}, \qquad \widetilde{I}_\nu(\mathcal{G}) = \inf_{\Omega \text{ admissible}} \widetilde{i}_\nu(\Omega),$$

and

$$\widetilde{S}_\nu(\mathcal{G}) = \inf_{f \subset C^1_{\mathrm{Dir}}(\mathcal{G})} \sup_{a \in \mathbf{R}} \widetilde{s}_\nu(f - a) = \inf_{f \subset C^1_{\mathrm{Dir}}(\mathcal{G})} \frac{\|\nabla f\|_1}{\min_{a \in \mathbf{R}} \|f - a\|_{\nu'}}$$

One can prove a Federer-Fleming-type inequality, namely:

Theorem 4.3 (Closed Federer-Fleming, traditional) *If $1 \leq \nu \leq \infty$, we have $\widetilde{I}_\nu(\mathcal{G}) \leq \widetilde{S}_\nu(\mathcal{G}) \leq 2^{1/\nu} \widetilde{I}_\nu(\mathcal{G})$.*

While the above definitions of $\widetilde{I}_\nu$ and $\widetilde{S}_\nu$ are perhaps the most natural closed case versions of I_ν and S_ν, more precise inequalities are obtained with the following definitions.

Let

$$\widetilde{i}'_\nu(\Omega) = |\partial\Omega|(|\Omega|^{1-\nu} + |\Omega^{\mathrm{c}}|^{1-\nu})^{1/\nu} \qquad \widetilde{I}'_\nu(\mathcal{G}) = \inf_{\Omega \text{ admissible}} \widetilde{i}'_\nu(\Omega).$$

Clearly $\widetilde{i}_\nu(\Omega) \leq \widetilde{i}'_\nu(\Omega) \leq 2^{1/\nu} \widetilde{i}_\nu(\Omega)$, so $\widetilde{I}_\nu \leq \widetilde{I}'_\nu \leq 2^{1/\nu} \widetilde{I}_\nu$. Let

$$\widetilde{S}_{\prime,\nu} = \inf_{f \in C^1_{\mathrm{Dir}}(\mathcal{G}),\, f \mathrm{split}} s_\nu(f).$$

Our more precise version of compact Federer-Fleming is:

Theorem 4.4 *For any $1 \leq \nu \leq \infty$ we have $\widetilde{I}'_\nu = \widetilde{S}_\nu$ and $\widetilde{I} = \widetilde{S}_{\prime,\nu}$ (and clearly $\widetilde{I}_\nu \leq \widetilde{I}'_\nu \leq 2^{1/\nu} \widetilde{I}_\nu$).*

The following important corollary follows:

Corollary 4.5 *For f split we have*

$$\|\nabla f\|_1 \geq \widetilde{I}_\nu(\mathcal{G}) \|f\|_{\nu'}. \tag{17}$$

The above corollary is much easier to work in applications to Sobolev inequalities than the weaker claim that for any f we have

$$\|\nabla f\|_1 \geq \widetilde{I}_\nu(\mathcal{G})\|f\|_{\nu'}$$

for f ν'-balanced. The reason for this is that if f is split, then so is $\{f\}^\alpha$; this is not true for f balanced.

We finish by remarking that if we try to interpret what $\widetilde{I}, \widetilde{I}', \widetilde{S}, \widetilde{S'}$ would mean on a $\mathcal{G}$ with $\mathcal{V}(\mathcal{G}) = \infty$ or with a boundary, then we claim we would recover I, S. So the closed versions of the Federer-Fleming theorems proven here are reasonable analogues of the traditional Federer-Fleming theorems.

4.3 A Simple Inequality

If $M, L \in L^1[-T, T]$ are non-negative functions, then we have

$$\frac{\int_{-T}^{T} M(t)\,dt}{\int_{-T}^{T} L(t)\,dt} \geq \inf_{t\in[-T,T]} \frac{M(t)}{L(t)} \tag{18}$$

(with the convention that $a/0$ is $+\infty$ for any $a \geq 0$). In other words, the quotient of two superpositions (or averages) is at least as big as the minimum of the individual quotients. In the rest of this subsection we discuss mild variants of this very simple inequality.

If $\mathcal{F}$ is a normed vector space, for a family $\{f_t\}_{t=-T}^{t=T}$ of elements of $\mathcal{F}$ it makes sense to write "superpositions" or integrals

$$f = \int_{-T}^{T} f_t\,dt$$

as the limit (in the norm) of Riemann sums, presuming the limit exists. We say that a functional $\mathcal{H}\colon \mathcal{F} \to \mathbf{R}$ is subadditive (respectively additive) on a superposition as above if

$$\mathcal{H}(f) \leq \int_{-T}^{T} \mathcal{H}(f_t)\,dt$$

(respectively equality holds). So if $\mathcal{Q} = \mathcal{M}/\mathcal{L}$ is a quotient of two non-negative functionals, with $\mathcal{M}$ additive on the above superposition and $\mathcal{L}$ subadditive, we have

$$\mathcal{Q}(f) \geq \min_{t\in[-T,T]} \mathcal{Q}(f_t),$$

using equation 18.

Now any $f \in C^1_{\mathrm{Dir}}(\mathcal{G})$ can be written as a superposition of $\pm\chi_\Omega$, characteristic and negative characteristic functions, almost all Ω having finite boundary; here the superposition understood in some appropriate normed vector space, $\mathcal{F}$, containing $C^1_{\mathrm{Dir}}(\mathcal{G})$ and all the characteristic functions, such as $L^p(\mathcal{G})$. From the above discussion we see:

Proposition 4.6 *Let $\mathcal{Q} = \mathcal{M}/\mathcal{L}$ be a quotient of two non-negative functionals on a space, $\mathcal{F}$, as above. Assume that we can write any $f \in C^1_{\mathrm{Dir}}(\mathcal{G})$ as a superposition of plus/minus characteristic functions on which $\mathcal{M}$ and $\mathcal{L}$ respectively additive and subadditive. Then*

$$\min_{f \in C^1_{\mathrm{Dir}}(\mathcal{G})} \mathcal{Q}(f) \geq \min_{\Omega \text{ admissible}} \mathcal{Q}(\pm\chi_\Omega).$$

In fact, we will see that the above inequality holds with equality for the $\mathcal{M}$'s and $\mathcal{L}$'s of interest to us.

4.4 Our $\mathcal{M}$

In this paper we will be concerned with only one functional, $\mathcal{M}$. Namely,

$$\mathcal{M}(f) = \sup_{X \in C^1_{\mathrm{fn}}(\mathrm{T}\mathcal{G}),\ |X| \leq 1} -\int (\nabla \cdot X) f.$$

This functional is defined on $L^p(\mathcal{G})$ for any $1 \leq p \leq \infty$ and on $C^k(\mathcal{G})$ for any $k \geq 0$; however, it can take on the value $+\infty$.

Proposition 4.7 *$\mathcal{M}$ satisfies the following properties:*

1. *for $f \in C^1_{\mathrm{fn}}(\mathcal{G})$ we have $\mathcal{M}(f) = \int |\nabla f|\, d\mathcal{E}$,*
2. *for open $\Omega \subset \mathcal{G}$ with finite boundary, we have $\mathcal{M}(\pm\chi_\Omega) = \mathcal{A}(\partial\Omega)$, and*
3. *for $f \in C^1_{\mathrm{fn}}(\mathcal{G})$ with $|f| \leq T$, we have that $\mathcal{M}$ is additive with respect to the superposition*
$$f = \int_0^T \Omega_f(t)\, dt + \int_0^T -\Omega_{-f}(t)\, dt$$
(with $\Omega_f(t) = \{x \in \mathcal{G} | f(x) > t\}$ as before).

Proof By the divergence theorem we have that if $f \in C^1_{\rm fn}(\mathcal{G})$ then

$$\mathcal{M}(f) = \sup_{X \text{ as above}} \int X \cdot \nabla f \, d\mathcal{E},$$

and it follows that $\mathcal{M}(f) \leq \int |\nabla f| \, d\mathcal{E}$. The reverse inequality is done by approximating $\nabla f / |\nabla f|$. Namely, for any $\epsilon > 0$, the set where $|\nabla f| > \epsilon$ is open, and it therefore contains a finite boundary open subset, Ω_ϵ covering all but an $\mathcal{E}$-measure ϵ set of its points. Then

$$\int_{\mathcal{G} \setminus \Omega_\epsilon} |\nabla f| \, d\mathcal{E} \leq \epsilon \mathcal{E}\Big(\mathrm{Supp}(f)\Big) + \epsilon \sup |\nabla f|$$

which tends to zero as $\epsilon \to 0$. Now take $X_\epsilon = \nabla f / |\nabla f|$ on $\Omega_\epsilon \setminus V$, and otherwise anything less than one in norm in a smooth way. Then $X_\epsilon \in C^\infty_{\rm fn}(\mathrm{T}\mathcal{G})$ and

$$\int X \cdot \nabla f \, d\mathcal{E} \geq \int_{\Omega_\epsilon} |\nabla f| \, d\mathcal{E} - \int_{\mathcal{G} \setminus \Omega_\epsilon} |\nabla f| \, d\mathcal{E} = \int |\nabla f| \, d\mathcal{E} - 2 \int_{\mathcal{G} \setminus \Omega_\epsilon} |\nabla f| \, d\mathcal{E}$$

and the last expression tends to $\int |\nabla f| \, d\mathcal{E}$. Thus the reverse inequality is proven.

The second statement is proven by the divergence theorem applied to Ω, namely

$$\mathcal{M}(\chi_\Omega) = \sup_{X \text{ as above}} - \int_{\partial \Omega} \widetilde{\mathbf{n}} \cdot X.$$

This makes $\mathcal{M}(\chi_\Omega) \leq \mathcal{A}(\partial \Omega)$ clear. For the reverse inequality we take X to have $|X| = 1$ and to be of the right direction in a neighbourhood of its boundary, and anything elsewhere provided that $|X| \leq 1$ everywhere and $X \in C^1_{\rm fn}(\mathrm{T}\mathcal{G})$.

The additivity follows immediately from the co-area formula applied to $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$.

4.5 Our $\mathcal{L}$'s: Sup-linear functionals

We next describe the $\mathcal{L}$'s that we consider.

Definition 4.8 *Let $\mathcal{F}$ be a normed linear space, and let H be a collection of bounded, linear functionals on $\mathcal{F}$. We define the* sup-linear functional associated with H *to be*

$$\mathcal{H}(f) = \sup_{\ell \in H} \ell(f).$$

We now record some simple remarks:

Proposition 4.9 *The following hold:*

1. *If $0 \in H$, then $\mathcal{H}$ is non-negative.*
2. *If H is symmetric, i.e. $\ell \in H$ implies $-\ell \in H$, then $\mathcal{H}(f) = \mathcal{H}(-f)$.*
3. *$\mathcal{H}(f_1 + f_2) \le \mathcal{H}(f_1) + \mathcal{H}(f_2)$ for any $f_i \in \mathcal{F}$.*
4. *$\mathcal{H}(\alpha f) = \alpha \mathcal{H}(f)$ if α is any positive real.*
5. *If $f_n \to f$ in $\mathcal{F}$, then $\mathcal{H}(f) \le \liminf \mathcal{H}(f_n)$ (since the supremum of lower semicontinuous functions is lower semicontinuous). In particular, using 3 and 4, $\mathcal{H}$ is subadditive with respect to* any *superposition in $\mathcal{F}$.*
6. *If H is uniformly bounded, then $\mathcal{H}(f)$ is continuous.*

Now we give some examples of sup-linear functionals:

1. $\mathcal{M}(f)$ as defined in the previous subsection.
2. $\mathcal{L}(f)$ being the $L^p(\mathcal{G}, \mathcal{V})$ norm, since then
$$\mathcal{L}(f) = \sup_{\|g\|_{p'}=1} \int gf\, d\mathcal{V},$$
3. $$\mathcal{L}(f) = \inf_{a\in\mathbf{R}} \|f-a\|_{p,\mathcal{V}} = \sup_{\|g\|_{p'}=1,\ \int g\, d\mathcal{V}=0} \int gf\, d\mathcal{V},$$

The functionals mentioned in items 2 and 3 are continuous with respect to the $L^p(\mathcal{G}, \mathcal{V})$ norm, in view of proposition 4.9, item 6. More interesting examples are given in [Rot81].

4.6 Approximating χ_Ω

In proposition 4.6 it is usually easy to see that equality holds. The reason for this is that if Ω is admissible, then one can approximate χ_Ω by a sequence of $C^1_{\mathrm{Dir}}(\mathcal{G})$ functions. We make this precise here.

So fix an admissible $\Omega \subset \mathcal{G}$. For any $\epsilon > 0$ sufficiently small, let f_ϵ be a function such that

1. f_ϵ vanishes on Ω^c,
2. $f_\epsilon = 1$ on the set of points, A_ϵ, whose distance to Ω^c is at least ϵ, and
3. along any edge segment of length ϵ joining Ω^c and A_ϵ, f_ϵ increases monotonically from 0 to 1 with $f_\epsilon \in C^1(\mathcal{G})$.

Then we have:

1. $\mathcal{M}(f_\epsilon) = |\partial\Omega|$ for ϵ sufficiently small.
2. $f_\epsilon \to \chi_\Omega$ as $\epsilon \to 0$ in $L^p(\mathcal{G}, \mathcal{V})$ for any $1 \le p < \infty$.
3. $\mathcal{L}(f_\epsilon) \to \mathcal{L}(\chi_\Omega)$ for any of the $\mathcal{L}$'s mentioned in the previous sections. (By the previous remark this holds for the $\mathcal{L}$'s with $p < \infty$, and it is easy to verify the $p = \infty$ case directly.)

Notice that the above construction works for any open Ω of finite type, even if $\partial\Omega$ is infinite. In this case items 2 and 3 are still true, and while item 1 is not true, we still have $\mathcal{M}(f_\epsilon) \to |\partial\Omega|$.

We easily conclude:

Proposition 4.10 *For $\mathcal{M}$ as described above, and any $\mathcal{L}$ mentioned in the previous section we have*

$$\min_{f \in C^1_{\mathrm{Dir}}(\mathcal{G})} \mathcal{Q}(f) = \min_{\Omega \text{ admissible}} \mathcal{Q}(\pm\chi_\Omega).$$

4.7 Proofs of the Federer-Fleming statements

The Federer-Fleming statements follow easily from the proceding discussions. For the traditional Federer-Fleming theorem, we have $S_\nu = I_\nu$ by the previous discussion, taking $\mathcal{L}(f)$ to be the $L^{\nu'}(\mathcal{G}, \mathcal{V})$ norm, seeing as then

$$\mathcal{L}(\chi_\Omega) = |\Omega|^{1/\nu'}.$$

For closed graphs, $\widetilde{I}_\nu \le \widetilde{S}_{',\nu}$ follows since for f split we have that the superposition of f as characteristic functions involves only χ_Ω with $|\Omega| \le \mathcal{V}(\mathcal{G})/2$. The reverse inequality follows by the approximation argument in the previous section, since $|\Omega| \le \mathcal{V}(\mathcal{G})/2$ implies that the f_ϵ in the previous section are split.

To see $\widetilde{I}'_\nu = \widetilde{S}_\nu$, we consider $\mathcal{L}(f) = \min_{a\in\mathbf{R}} \|f-a\|_{\nu'}$. For any open Ω of finite type we have the minimum of $\|\chi_\Omega - a\|_{\nu'}$ is attained when

$$\int \{\chi_\Omega - a\}^{\nu'-1}\, d\mathcal{V} = 0,$$

i.e. when

$$(1-a)^{\nu'-1}|\Omega| = a^{\nu'-1}|\Omega^{\mathrm{c}}|,$$

i.e. for

$$a = \frac{|\Omega^{\mathrm{c}}|^{\nu-1}}{|\Omega|^{\nu-1} + |\Omega^{\mathrm{c}}|^{\nu-1}}.$$

It follows that

$$\mathcal{L}(\chi_\Omega) = (|\Omega|^{1-\nu} + |\Omega^{\mathrm{c}}|^{1-\nu})^{1/\nu},$$

and thus $(\mathcal{M}/\mathcal{L})(\chi_\Omega) = i'_\nu(\Omega)$. Thus $\widetilde{I}'_\nu = \widetilde{S}_\nu$.

4.8 The Generalization of Rothaus

In the paper [Rot81] of Rothaus, it is noted that $\mathcal{L}$ may be generalized to

$$\mathcal{L}(f) = \mathcal{L}_1(f^+) + \mathcal{L}_2(f^-),$$

where $f^+ = \max(f,0)$, $f^- = \max(-f,0)$, and $\mathcal{L}_i$ for $i=1,2$ are functionals as before. It is also shown there that if we further restrict our functions from $C^1_{\mathrm{Dir}}(\mathcal{G})$ to those which in addition satisfy

$$\int f^+ P\, d\mathcal{V} = \int f^- Q\, d\mathcal{V},$$

where P, Q are fixed positive function on $\mathcal{G}$, then we get a similar Federer-Fleming theorem, where $\pm\chi_\Omega$ is replaced by

$$Q(B)\chi_A - P(A)\chi_B,$$

where A, B range over all admissible subsets of $\mathcal{G}$ with disjoint closures, and $Q(B), P(A)$ are the integrals $d\mathcal{V}$ of Q, P respectively over B, A.

Rothaus wrote his paper for manifolds, but all the proofs immediately carry over to the graph theory case using our framework.

4.9 Remarks on the Isoperimetric Constants

In graph theory one often defines the isoperimetric constants (i.e. our $I_\nu, \widetilde{I}_\nu, \widetilde{I}'_\nu$) in terms of subsets of vertices and the edges leaving them. Since we define our constants as infimums of certain quantities over all *admissible* sets, one might wonder if our constants agree with classical ones in graph theory. In fact, it is easy to see that they do.

Proposition 4.11 *In defining any one of $I_\nu, \widetilde{I}_\nu, \widetilde{I}'_\nu$, we may restrict the admissible sets, Ω by further requiring that (1) Ω is connected, and (2) if $x, y \in \Omega$ lie on an edge interior or its closure, then the entire edge segment from x to y lies in Ω.*

Any of $i_\nu, \widetilde{i}_\nu, \widetilde{i}'_\nu$ of an Ω as above will be determined by which vertices lie in Ω, and agree with their classical graph theory analogues.

Proof Part (1) is an observation of Yau (in analysis). Indeed, if A, B are admissible and disjoint, then clearly $\chi_A + \chi_B$ is a superposition for which our $\mathcal{M}$ is additive. It follows that any of $I_\nu, \widetilde{I}_\nu, \widetilde{I}'_\nu$ are at least as small on χ_A or χ_B as they are on the sum. Part (2) follows from the fact that if we add the edge segment from x to y to Ω, then $\mathcal{M}$ can't increase and $\mathcal{L}$ doesn't change.

$\square$

5 Dodziuk's and Alon's Versions of Cheeger's Inequality

We briefly described how Dodziuk's and Alon's versions of Cheeger's inequality can be derived in our framework, their similarities, and their differences.

Let
$$\lambda = \inf_{f \in C^1_{\mathrm{Dir}}(\mathcal{G})} \mathcal{R}(f), \quad \text{where} \quad \mathcal{R}(f) = \frac{\int |\nabla f|^2 \, d\mathcal{E}}{\int f^2 \, d\mathcal{V}}.$$
This λ can be understood as the first Laplacian eigenvalue, at least when the infimum is achieved by some f, and is of fundamental importance in spectral theory. We wish to lower bound λ.

Notice that when $\mathcal{G}$ is closed we have $\lambda = 0 = \mathcal{R}(1)$. At the end of this section we will modify our notion of λ in this case and state the analogous theorems.

5.1 The Basic Technique

To bound λ from below we notice that for $X \in C^1_{\mathrm{fn}}(\mathcal{G})$ and $f \in C^1_{\mathrm{Dir}}(\mathcal{G})$ we have, using Cauchy-Schwartz,

$$\int X \cdot \nabla(f^2)\, d\mathcal{E} = \int 2fX \cdot \nabla f\, d\mathcal{E} \le 2\|fX\|_{2,\mathcal{E}} \|\nabla f\|_{2,\mathcal{E}}.$$

Dividing by $\int f^2\, d\mathcal{V}$ yields

$$\mathcal{Q}_1(f) \le 2\mathcal{Q}_2(f)\sqrt{\mathcal{R}(f)},$$

where

$$\mathcal{Q}_1(f) = \frac{\int X \cdot \nabla(f^2)\, d\mathcal{E}}{\int f^2\, d\mathcal{V}}, \quad \text{and} \quad \mathcal{Q}_2(f) = \frac{\|fX\|_{2,\mathcal{E}}}{\|f\|_{2,\mathcal{V}}}.$$

Hence we have:

Proposition 5.1 *For any $f \in C^1_{\mathrm{Dir}}(\mathcal{G})$ and $X \in C^1_{\mathrm{fn}}(\mathcal{G})$ and let $\mathcal{Q}_i(f) = \mathcal{Q}_i(f, X)$ be as above. Then*

$$\lambda \ge \inf_f \sup_X \mathcal{Q}_1^2(f, X) \Big/ \Big(4\mathcal{Q}_2^2(f, X)\Big).$$

If we understand that for each f we have specified an X, then $\mathcal{Q}_i(f, X) = \mathcal{Q}_i(f)$ and we have, in particular,

$$\lambda \ge \inf_f \mathcal{Q}_1^2(f) \Big/ \Big(4\mathcal{Q}_2^2(f)\Big).$$

Of course, in the closed case we have $\lambda = 0$ and $\mathcal{Q}_1(f) = 0$ for f constant; what we seek in this case is a lower bound on the *second* Laplacian eigenvalue. We discuss this case in a later subsection.

5.2 Dodziuk's Lower Bound

Dodziuk takes $X = \nabla f / |\nabla f|$ when $\nabla f \ne 0$, and $X = 0$ otherwise. Then

$$\mathcal{Q}_1(f) = \frac{\int |\nabla(f^2)|\, d\mathcal{E}}{\int f^2\, d\mathcal{V}} \ge I_\infty(\mathcal{G}).$$

Furthermore

$$\mathcal{Q}_2(f) = \|f\|_{2,\mathcal{E}} / \|f\|_{2,\mathcal{V}}, \tag{19}$$

and according to equation 5 this is bounded above by $\rho_{\sup}^{1/2}$. We conclude:

Theorem 5.2 (Dodziuk) $\lambda \ge I_\infty^2 / (4\rho_{\sup})$.

5.3 Alon's Lower Bound

Rather than use $I_\infty(\mathcal{G})$ to lower bound λ, Alon uses what he calls the *magnification*, a type of "vertex expansion" that arises in many computer science settings including sorting and communication networks.

Definition 5.3 *$\mathcal{G}$ is a c-magnifier if for subset of vertices, $A \subset \mathcal{G} \setminus \partial\mathcal{G}$ we have $|\Gamma(A)| \geq (1+c)|A|$, where $\Gamma(A)$ denotes the neighbours of A, i.e. those vertices (possibly belonging to A) with an edge joining them to A.*

Alon's choice of X (in the sense of proposition 5.1) is more involved. For the moment assume $\mathcal{V}(v) = 1$ for all $v \in V$ and $a_e = 1$ for all $e \in E$.

Theorem 5.4 *For any set of vertices, $A \subset \mathcal{G} \setminus \partial\mathcal{G}$ there exists an $X \in C^1_{\mathrm{fn}}(\mathcal{G})$ such that*

1. *$|X| \leq 1$ everywhere,*
2. *$-\nabla \cdot X \geq c$ on A,*
3. *$-\nabla \cdot X \leq 0$ on A^{c}, and*
4. *we have $\rho_{\sup}(\mathcal{G}^{|X|^2}) \leq (2+c^2)\sup_e \ell_e/2$ (and we can replace $2+c^2$ with $2+\lfloor c\rfloor + (c-\lfloor c\rfloor)^2$, which is interesting for $c \geq 1$).*

Proof Form a network with vertices $\{s,t\} \cup B_1 \cup B_2$ as follows. Let B_1, B_2 be copies of A, V respectively. Form an edge of capacity $1+c$ from the source, s, to each B_1. Form an edge of capacity 1 from vertex of B_2 to the sink, t. For each $(b_1,b_2) \in B_1 \times B_2$ form an edge of capacity 1 if either $b_2 = b_1$ or $\{b_1,b_2\}$ is an edge in $\mathcal{G}$. It is not hard to see (see [Alo86]) that restricting any max flow of this network to $B_1 \times B_2$ we get an edgewise constant vector field, X, on $\mathcal{G}$ that satisfies

1. $|X| \leq 1$ everywhere,
2. for any v we have $\sum_{e=\{u,v\}\in E} X^+(v;e) \leq 1$, where $X^+(v;e)$ is the "in flow," i.e. $X^+(v;e) = \max(0, X|_e(v)\cdot n_{e,v})$, and
3. for any v we have
$$\sum_{e=\{u,v\}\in E} X^-(v;e) = \begin{cases} 1+c & \text{if } v \in A \\ 0 & \text{otherwise,} \end{cases}$$
where X^- is the "out flow," $X^-(v;e) = \max(0, -X|_e(v)\cdot n_{e,v}) = X^+(u;e)$.

Parts 1–3 of the theorem are now clear. By equations 2 and 8 we have

$$\rho_{\sup}(\mathcal{G}^{|X|^2}) = \sup_v \mathcal{V}^{-1}(v) \sum_{e \ni v} \mathcal{E}(e)|X(e)|^2/2 = \sup_v \sum_{e \ni v} \ell_e |X(e)|^2/2.$$

Part 4 now follows from the fact if $x_i \in \mathbf{R}$ with $0 \leq x_i \leq 1$ and their sum is d, then their sum of squares is at most $\lfloor d \rfloor + (d - \lfloor d \rfloor)^2$.

□

In lower bounding $\mathcal{R}(f)$ we may assume that $f \geq 0$, since we easily see $\mathcal{R}(|f|) \leq \mathcal{R}(f)$. Now given $f \in C^1_{\mathrm{Dir}}(\mathcal{G})$ with $f \geq 0$, choose A to be the support of f and let X be the vector field of Alon described above. We have

$$\mathcal{Q}_1(f) \geq \int -(\nabla \cdot X) f^2 \, d\mathcal{V} \Big/ \int f^2 \, d\mathcal{V} \geq c,$$

and, from equation 8

$$\mathcal{Q}_2(f) \leq \sqrt{\rho_{\sup}(\mathcal{G}^{|X|^2})} \leq \sqrt{[2 + \lfloor c \rfloor + (c - \lfloor c \rfloor)^2]/2}.$$

It follows that:

Theorem 5.5 (Alon) *For a c-magnifier as above, with $\mathcal{V}(v) = a_e = \ell_e = 1$ and for all v, e we have that*

$$\lambda \geq c^2/(2c^2 + 4),$$

and furthermore, what is more precise for $c > 1$,

$$\lambda \geq c^2 \Big/ \Big(4 + 2\lfloor c \rfloor + 2(c - \lfloor c \rfloor)^2\Big).$$

If not all ℓ_e's are one, the same result holds where we divide the right-hand-side by $\sup_e \ell_e$.

We finish this subsection by showing one way to apply Alon's technique to the case were the vertex measure is not the counting measure. So assume $a_e = 1$, but $\mathcal{V}(v)$ and ℓ_e are arbitrary. First, modify the notion of "magnifier" to mean that

$$\mathcal{V}\Big(\Gamma(A)\Big) \geq (1 + c)\mathcal{V}(A)$$

for all $A \subset V \setminus \partial\mathcal{G}$. For such a magnifier and a fixed A, construct a similar network on $\{s, t\} \cup B_1 \cup B_2$, except (1) the capacity from s to v is $\mathcal{V}(v)$, (2) the capacity from v to w with $v \in B_1$ and $w \in B_2$ is $\mathcal{V}(w)$, and (3) the capacity from w to t is $\mathcal{V}(w)$. We get a vector field, X, such that

1. $-\nabla \cdot X$ is $\geq c$ on A and ≤ 0 on A^{c},

2. for any v we have $\sum_{e \in v} X^{+}(v; e) \leq \mathcal{V}(v)$,

3. for any v we have $\sum_{e \in v} X^{-}(v; e) \leq (1+c)\mathcal{V}(v)$ and each $X^{-}(v; e)$ with $e = \{u, v\}$ is less than $\mathcal{V}(u)$.

Now we can derive various results as before, depending on the values of $\mathcal{V}(v)$ and ℓ_e.

For example, in [BHT], ℓ_e, where $e = \{u, v\}$ is taken to be $1/\Big(\mathcal{V}(u) + \mathcal{V}(v)\Big)$. We conclude

$$\mathcal{V}^{-1}(v) \sum_{e \in v} \mathcal{E}(e)[X^{+}(v; e)]^2/2 \leq \mathcal{V}^{-2}(v) \sum_{e \in v} [X^{+}(v; e)]^2/2 \leq 1/2.$$

In trying to upper bound the same sum over X^{-}, fix v and view the $\mathcal{V}(u)$'s with $e = \{v, u\}$ as variables; given that $X^{-}(v; e) \leq \mathcal{V}(u)$ for $e = \{u, v\}$, we can always assume $X^{-}(v; e) = \mathcal{V}(u)$ in maximizing the sum over X^{-} (or we get a larger sum by taking $\mathcal{V}(u)$ smaller). Hence the sum over X^{-} is bounded by

$$\mathcal{V}^{-1}(v) \sum_{e \in v} \mathcal{E}(e)[X^{-}(v; e)]^2/2 = \mathcal{V}^{-1}(v)(1/2) \sum_{e=\{v,u\}} \frac{\mathcal{V}^2(u)}{\mathcal{V}(u) + \mathcal{V}(v)} =$$

$$\mathcal{V}^{-1}(v)(1/2) \sum_{e=\{v,u\}} \mathcal{V}(u) - \mathcal{V}^{-1}(v)(1/2) \sum_{e=\{v,u\}} \frac{1}{\mathcal{V}^{-1}(u) + \mathcal{V}^{-1}(v)}.$$

Since $1/(1+t)$ is convex, and since the sum of the $\mathcal{V}(u)$'s is $\leq (1+c)\mathcal{V}(v)$, it follows that the above sum is maximized with one $\mathcal{V}(u)$ equal to $(1+c)\mathcal{V}(v)$ and the rest zero. Hence

$$\mathcal{V}^{-1}(v)(1/2) \sum_{e \in v} \mathcal{E}(e)[X^{-}(v; e)]^2 \leq \mathcal{V}^{-1}(v)(1/2)\Big(\mathcal{V}(v) + \mathcal{V}(v)(1+c)\Big)^{-1}(1+c)^2\mathcal{V}^2(v)$$

$$= (1+c)^2/(4+2c).$$

We conclude:

Proposition 5.6 *Consider a Rayleigh quotient with $\mathcal{V}(v)$ arbitrary and $\mathcal{E}(e) = \mathcal{V}(v) + \mathcal{V}(u)$ for each $e = \{u, v\}$. Then if $\mathcal{G}$ is a c-magnifier (in the sense above), then*

$$\lambda \geq c^2(2+c)/(6+6c+2c^2).$$

In [BHT] a somewhat weaker lower bound is given, although their bound is based only on a "part" of λ_2, namely what they call λ_∞.

5.4 Improvements to Cheeger's Inequality for Graphs

Mohar first gave an improvement to Dodziuk's form of Cheeger's inequality using the special nature of graphs. From our point of view there are really two improvements. First, Mohar uses

$$\int X \cdot \nabla(f^2)\, d\mathcal{E} = 2 \int \overline{f} X \cdot \nabla f\, d\mathcal{E},$$

where $\overline{f}$ is the edgewise constant function whose value on an edge is the average of f's values at the endpoints. Thus we may replace $\mathcal{Q}_2(f)$ with

$$\widetilde{\mathcal{Q}}_2(f) = \|\overline{f}\|_{2,\mathcal{E}} / \|f\|_{2,\mathcal{V}}.$$

Next we notice that if f is linear on e with $e = \{u, v\}$ and $f(u) = b$ and $f(v) = c$, then

$$\int_e f^2\, d\mathcal{E} = a_e(b^2 + 2bc + c^2)/4, \quad \text{and} \quad \int_e |\nabla f|^2\, d\mathcal{E} = a_e(b^2 - 2bc + c^2),$$

provided that the edge lengths are 1. So like equation 7 we conclude

$$\|\overline{f}\|_{2,\mathcal{E}}^2 + (1/4)\|\nabla f\|_2^2 = \int \rho f^2\, d\mathcal{V} \le \rho_{\sup} \|f\|_{2,\mathcal{V}}^2,$$

with equality if $\mathcal{G}$ is regular. We conclude that

$$\lambda \ge I_\infty^2 / [4(\rho_{\sup} - \lambda/4)],$$

in other words

$$\lambda^2 - 4\rho_{\sup}\lambda + I_\infty^2 \le 0,$$

i.e.

$$\lambda \ge 2\rho_{\sup} - \sqrt{4\rho_{\sup}^2 - I_\infty^2}.$$

Notice that for $I_\infty/\rho_{\sup}$ small, this lower bound is within $O(I_\infty^4/\rho_{\sup}^3)$ of Dodziuk's lower bound of $I_\infty^2/(4\rho_{\sup})$.

5.5 Closed graphs

We remark that the inequalities of Dodziuk, Mohar, and Alon all generalize to closed graphs, where we take λ to be the minimum of the Rayleigh quotient over all functions whose integral with respect to $\mathcal{V}$ is zero, and where I_∞ and c are isoperimetric constants for Ω with $\mathcal{V}(\Omega) \le \mathcal{V}(\mathcal{G})/2$. Indeed, λ is the eigenvalue of an eigenfunction, f, orthogonal to the first eigenfunction (the constant function), and by the nodal region theory of [Fri93], f's restriction to either nodal region is the first eigenfunction of that nodal region. We can apply any previous result of this subsection to the nodal region with the smaller $\mathcal{V}$ measure, to conclude the analogous theorem for a closed graph.

6 The Laplacian and the Heat Kernel

One can study the Laplacian by way of the heat equation. A basic tool in understanding the heat equation is the (or a) heat kernel, which we define here. We are only interested in the "minimal non-negative" heat kernel. However, spectral theory can be used to construct a heat kernel that comes with many nice bounds; these bounds also apply to the minimal non-negative heat kernel. So we shall first construct the "spectral theory" heat kernel. We shall show that the two aforementioned heat kernels are the same in many interesting cases; we don't know if they are always the same.

6.1 The Dirichlet Initial Value Problem and the Heat Kernel

The heat kernel is intuitively obtained by solving many instances of the "heat equation." When the heat equation has a unique solution, then there is a unique heat kernel. When the heat equation solution is not unique, there may be more than one heat kernel. Here we describe what is meant by solving the heat equation, and illustrate cases where the heat equation does not have a unique solution.

Definition 6.1 *A function $u = u(x,t)\colon \mathcal{G} \times [0,T] \to \mathbf{R}$ for some $T > 0$ is said to satisfy the* heat equation *on $\mathcal{G} \times [0,T]$ if (1) u is continuous on $\mathcal{G} \times [0,T]$, (2) $u(\,\cdot\,,t)$ is edgewise linear for each t, (3) we have u_t (the partial derivative of u with respect to t) and $-\Delta u$ (the Laplacian in the variable x) exist at the point (x,t) and are equal there, for all $x \in V^{\mathrm{o}}$ and $t \in (0,T)$. We can also take $T = \infty$ in the above, replacing $[0,T]$ with $[0,\infty)$.*

Given a function f, on V^{o}, we seek a solution to the heat equation, $u(x,t)$ satisfying the two "boundary conditions"

$$u(x,0) = f(x) \quad \forall x \in V^{\mathrm{o}}$$

and

$$u(x,t) = 0 \quad \forall x \in \partial\mathcal{G},\ \forall t \in [0,T].$$

We will call this the "Dirichlet initial value problem" for the heat equation (with "initial value" f).

The following theorems give examples of non-uniqueness or uniqueness of the heat kernel, and we will prove them in appendix B (since they are not crucial to our later use of the heat kernel).

Theorem 6.2 *If $\mathcal{G}$ is finite, then any Dirichlet initial value problem has at most (in fact, exactly) one solution.*

By way of contrast we have non-uniqueness for a very "mildly" infinite graph. This example is a simple adaptation of the example in [Fri64], page 31, to graphs.

Theorem 6.3 *Let $\mathcal{G}$ be the graph whose vertices are the integers, $\mathbf{Z}$, with one edge from i to $i+1$ for all $i \in \mathbf{Z}$. Then the Dirichlet initial value problem with initial value 0 has infinitely many solutions.*

However, the infinitely many solutions referred to above are unbounded even for fixed t. So one might hope for a unique solution to the heat equation that is bounded over $\mathcal{G}$ for any fixed time, t.

Theorem 6.4 *For every $v \in V^{\circ}$, set $L(v) = \mathcal{V}(v)^{-1} \sum_{e \ni v} a_e/l_e$ and let $L_i(v)$ denote the supremum of $L(u)$ over all u with a path to v through V° of length at most i. If there is a C such that for any fixed $v \in V^{\circ}$ we have $L_j(v) \leq Cj$ for sufficiently large j, then the Dirichlet initial value problem has at most (in fact, exactly) one solution, $u(x,t)$, bounded in x for any fixed t. The same is true if there is a C such that for any fixed $v \in V^{\circ}$ we have*

$$L_0(v)L_1(v)\cdots L_j(v) \leq (Cj)^j$$

for sufficiently large j.

However, this theorem is close to being the best possible, in a sense:

Theorem 6.5 *For any $\alpha > 0$ there exists a tree, $\mathcal{G}$, such that for any fixed $v \in V^{\circ}$ we have $L_j(v) \leq 2j^{1+\alpha}$ for sufficiently large j, and such that the Dirichlet initial value problem has infinitely many solutions that are bounded on $\mathcal{G} \times [0,T]$ for any fixed T.*

Our last theorem looks like a precursor to a uniqueness theorem. However, we will use it in establishing positivity for the heat kernel, so we give its simple proof here.

Theorem 6.6 *Let $u \geq 0$ be a solution to the heat equation over $[0,T]$. Let $u(y,t_0) = 0$ with $0 < t_0 \leq T$ and $y \in V^{\circ}$. Then $u(x,t_0) = 0$ for any x that is connected to y via a (finite) path of vertices in V°.*

Proof Since $u \geq 0$ we have $u_t(y, t_0) \leq 0$ and so $\Delta u(y, t_0) \geq 0$. Hence

$$0 = \sum_{e \sim \{y,w\}} (a_e/\ell_e) u(y, t_0) \geq \sum_{e \sim \{y,w\}} (a_e/\ell_e) u(w, t_0),$$

and we conclude that $u(w, t_0) = 0$ for all $w \in V^{\mathrm{o}}$ with an edge to y. Repeatedly applying this conclusion yields the theorem.

□

6.2 Heat Kernels

Let $\delta_v = \chi_v / \mathcal{V}(v)$; δ_v is an analogue of the Dirac delta function at v.

Definition 6.7 *A* fundamental solution*, $K(x, y, t)$, for the heat equation is a function $K \colon \mathcal{G} \times \mathcal{G} \times \mathbf{R}_{\geq 0} \to \mathbf{R}$ (where $\mathbf{R}_{\geq 0}$ are the non-negative reals) such that (1) for any fixed $y \in V^{\mathrm{o}}$ we have $K(x, y, t)$, viewed as a function of x, t, is a solution to the Dirichlet initial value problem with initial value δ_y, (2) for any $y \in \partial\mathcal{G}$ we have $K(x, y, t) = 0$, and (3) $K(x, y, t)$ is edgewise linear in x and y.*

Conditions (2) and (3) imply that a fundamental solution is determined by solving the Dirichlet initial value problem at all interior vertices.

Definition 6.8 *A fundamental solution, $K(x, y, t)$, as above is said to be a* heat kernel *if (1) it is symmetric, i.e. $K(x, y, t) = K(y, x, t)$, (2) it is self-reproducing, i.e.*

$$K(x, y, t) = \int K(x, s, \tau) K(s, y, t - \tau)\, d\mathcal{V}(s) \tag{20}$$

for any $0 \leq \tau \leq t$, and (3) we may formally differentiate equation 20 in t, i.e.

$$K_t(x, y, t) = \int K(x, s, \tau) K_t(s, y, t - \tau)\, d\mathcal{V}(s).$$

Condition (3) is not usually stated in the definition of a heat kernel, but this condition will be needed in Nash's technique in section 7; it usually holds if (2) holds.

Definition 6.9 *A heat kernel, K, is* non-negative *if $K(x, y, t) \geq 0$ for all x, y, t. If there is a heat kernel, K, such that $K \leq G$ (pointwise) for any non-negative fundamental solution, G, to the heat equation, we say that K is the minimal non-negative heat kernel.*

We shall show that for locally finite graphs there always exists a minimal non-negative heat kernel. We begin by giving a natural and well-known construction of a heat kernel using spectral theory; this heat kernel is not always the minimal non-negative one, but many important bounds hold for the spectral theory heat kernel, and these bounds immediately follow for the minimal heat kernel as well.

Theorem 6.6 shows that if K is a non-negative fundamental solution and x and y are connected in $\mathcal{G}^{\mathrm{o}}$, then $K(x, y, t)$ is strictly positive for all $t > 0$.

6.3 The Heat Operator

In this section we give meaning to the expression $e^{-t\Delta}$, which we call the heat operator. We do so using the theory of quadratic forms and unbounded self-adjoint operators as in [Dav89, Dav80].

Let $\mathcal{D}$ denote the subspace of $L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$ consisting of functions of finite type; notice that $\mathcal{D}$ is dense there. Let $H^1(\mathcal{G})$ denote the closure of $\mathcal{D}$ under the norm

$$\|f\|^2_{H^1} = (f, f) + (\nabla f, \nabla f).$$

It is easy to see that the quadratic form $Q(f, g) = (\Delta f, g) = (\nabla f, \nabla g)$ defined on $\mathcal{D}$ (i.e. defined when $f, g \in \mathcal{D}$) extends to an quadratic form, $\overline{Q}$, on $H^1(\mathcal{G})$, known as the *closure* of Q. We identify $\overline{Q}$ with Q when no confusion can arise. It is easy to check that we may view $H^1(\mathcal{G})$ as a subspace of $L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$ (see the bottom of page 106 in [Dav80]).

Proposition 6.10 *If $f \in H^1(\mathcal{G})$ then $|f| \in H^1(\mathcal{G})$ and $Q(|f|, |f|) \leq Q(f, f)$. The same is true if $|f|$ is replaced by $g = \max\Big(0, \min(f, 1)\Big)$.*

Proof First we remark that if $h \in \mathcal{D}$ then it is easy to check that $|h| \in \mathcal{D}$ and $Q(|h|, |h|) \leq Q(h, h)$. If $f \in H^1(\mathcal{G})$ then there exist $f_n \in \mathcal{D}$ with $\|f - f_n\|_{H^1} \to 0$. Then $|f_n| \in \mathcal{D}$, and we easily see that

$$\|\, |f| - |f_n| \,\|_{H^1} \leq \|f - f_n\|_{H^1} \to 0$$

so that $|f| \in H^1(\mathcal{G})$. $Q(|f_n|, |f_n|) \leq Q(f_n, f_n)$ now shows $Q(|f|, |f|) \leq Q(f, f)$. The proof of the statement in the proposition involving g is the same.

□

Now we invoke some spectral and quadratic form theory. According to theorems 4.12 and 4.14 in [Dav80] and their proofs, Δ can be extended to self-adjoint operator, which we again call Δ, whose domain includes $\mathcal{D}$ (and lies in $H^1(\mathcal{G})$); furthermore $H^1(\mathcal{G})$ equals $\mathrm{Quad}(\Delta)$. By the spectral theorem 4.4 in [Dav80], it makes sense to speak of $e^{-t\Delta}$ for any $t \geq 0$ as a bounded operator on the closure of $\mathcal{D}$ in $L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$, which is just $L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$. We call $e^{-t\Delta}$ the "heat operator."

Using the spectral theorem it is easy to see that $e^{-\Delta t}$ has norm at most 1, and that $e^{-t\Delta}$ is strongly continuous, i.e. that $e^{-\Delta t}f$ is L^2 continuous in t for any $f \in L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$.

The proof of lemma 1.3.4 (and theorems 1.3.2 and 1.3.3) in [Dav89] show that proposition 6.10 implies that $e^{-\Delta t}$ is

1. *positivity preserving*, meaning that if $f \in L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$ satisfies $f \geq 0$ everywhere, then so does $e^{-\Delta t}f$, and
2. *a contraction* on $L^p_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$ for any $p \in [1, \infty]$, meaning that if $f \in L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V}) \cap L^p(\mathcal{G}, \mathcal{V})$ then the same is true of $e^{-\Delta t}f$ and we have $\|e^{-\Delta t}f\|_p \leq \|f\|_p$.

6.4 The Spectral Theory Heat Kernel

For $v \in V^{\mathrm{o}}$ let δ_v be the edgewise linear function that is $1/\mathcal{V}(v)$ on v and 0 on other vertices. δ_v is the analogue of the "Dirichlet delta function" in graph theory.

Definition 6.11 *The* spectral theory heat kernel *is the function* $\widetilde{K}\colon V^{\mathrm{o}} \times V^{\mathrm{o}} \times [0, \infty) \to \mathbf{R}$ *defined by*

$$\widetilde{K}(x, y, t) = (e^{-t\Delta}\delta_y, \delta_x) = (e^{-t\Delta}\delta_y)(x).$$

(Since $e^{-t\Delta}$ is a bounded operator defined on all of $L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$ and δ_y lies there, this definition makes sense.)

Keeping the notation of the last subsection, the following proposition is easy.

Proposition 6.12 $\widetilde{K}(x, y, 0) = \delta_y(x)$. *For fixed y, t we have that $\widetilde{K}(\,\cdot\,, y, t)$ lies in $L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$, and for any $f \in L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$ and x we have*

$$(e^{-t\Delta}f)(x) = \int \widetilde{K}(x, y, t) f(y)\, d\mathcal{V}(y).$$

Finally $\widetilde{K}$ is symmetric and self-reproducing. In other words, $\widetilde{K}(x,y,t) = \widetilde{K}(y,x,t)$ and

$$\widetilde{K}(x,y,t) = \int \widetilde{K}(x,s,\tau)\widetilde{K}(s,y,t-\tau)\, d\mathcal{V}(s)$$

for any $0 \le \tau \le t$.

Proof The first claim is clear. The symmetry of $\widetilde{K}$ follows from the self-adjointness of $e^{-t\Delta}$, i.e.

$$\widetilde{K}(x,y,t) = (e^{-t\Delta}\delta_y, \delta_x) = (\delta_y, e^{-t\Delta}\delta_x) = \widetilde{K}(y,x,t).$$

Since $e^{-t\Delta}$ contracts the L^2 norm, and since $\delta_y \in L^2_{\mathrm{Dir}}(\mathcal{G},\mathcal{V})$ for any $y \in V^{\mathrm{o}}$, we have $e^{-t\Delta}\delta_y \in L^2_{\mathrm{Dir}}(\mathcal{G},\mathcal{V})$ for any $t \ge 0$, and hence the claim about $K(\,\cdot\,,y,t)$ being in L^2. For any $f \in L^2_{\mathrm{Dir}}(\mathcal{G},\mathcal{V})$ and x we have

$$\int \widetilde{K}(x,s,t)f(s)\, d\mathcal{V}(s) = \int (e^{-t\Delta}\delta_x)(s)f(s)\, d\mathcal{V}(s)$$

$$= (e^{-t\Delta}\delta_x, f) = (\delta_x, e^{-t\Delta}f) = (e^{-t\Delta}f)(x).$$

For the last claim we apply the last formula with $f(s) = \widetilde{K}(s,y,t-\tau)$ to conclude

$$\int \widetilde{K}(x,s,\tau)\widetilde{K}(s,y,t-\tau)\, d\mathcal{V}(s) = \Big(e^{-\tau\Delta}\widetilde{K}(\,\cdot\,,y,t-\tau)\Big)(x)$$

$$= (e^{-\tau\Delta}e^{-(t-\tau)\Delta}\delta_y)(x) = (e^{-t\Delta}\delta_y)(x) = \widetilde{K}(x,y,t).$$

□

Proposition 6.13 *For all $x, y \in V^{\mathrm{o}}$ and $t > 0$ we have:*

$$\Delta_x \widetilde{K}(x,y,t) = -\widetilde{K}_t(x,y,t) = (\Delta e^{-t\Delta}\delta_x, \delta_y)$$

where Δ_x denotes the Laplacian in the x variable.

Similarly $\widetilde{K}_{tt}(x,y,t)$ exists and satisfies a similar formula as above; the proof below easily generalizes.

Proof The spectral theorem and Taylor's theorem easily imply that the operator $e^{-t\Delta}$ is differentiable, with limits of Newton quotients taken with respect to the L^2 operator norm, and has derivative $\Delta e^{-t\Delta}$. It follows that

$$-\widetilde{K}_t(x,y,t) = -\frac{\partial}{\partial t}(e^{-t\Delta}\delta_x, \delta_y) = (\Delta e^{-t\Delta}\delta_x, \delta_y).$$

Also

$$\Delta_x \widetilde{K}(x,y,t) = \Delta_x (e^{-t\Delta}\delta_y)(x) = (\Delta e^{-t\Delta}\delta_y)(x) = (\Delta e^{-t\Delta}\delta_y, \delta_x).$$

$\square$

The above proposition and the same type of calculation used to show that $\widetilde{K}$ is self-reproducing also shows:

Proposition 6.14 *For any $x, y \in V^{\circ}$ and $0 \leq \tau \leq t$ we have*

$$\widetilde{K}_t(x,y,t) = \int \widetilde{K}(x,s,\tau)\widetilde{K}_t(s,y,t-\tau)\, d\mathcal{V}(s).$$

In other words, the self-reproducing property of $\widetilde{K}$ can be partially differentiated with respect to time (i.e. t) within the integration.

Now we outline a proof of the following result, needed here only for subsection 6.1.

Theorem 6.15 *Let $f \in L^{\infty}_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$. Then*

$$u(x,t) = \int \widetilde{K}(x,y,t) f(y)\, d\mathcal{V}(y)$$

solves the Dirichlet initial value problem with initial value f (the L^{∞} contractive property of $e^{-t\Delta}$ shows that $\widetilde{K}(x,\,\cdot\,,t)$ is in L^1, and so the above integral exists).

We remark that the same theorem is true with L^{∞} replaced by L^p for any $1 \leq p \leq \infty$, and all aspects of the proof below are the same or easier.

Proof Clearly $u(x,0) = f(x)$.

We may assume $\mathcal{G}$ is connected, and since it is locally finite we can enumerate its vertices $V = \{v_1, v_2, \ldots\}$. For each n consider the function f_n which agrees with f on $v_1, \ldots, v_n$ and is zero on other vertices. Set

$$u_n(x,t) = \int \widetilde{K}(x,y,t) f_n(y)\, d\mathcal{V}(y).$$

Since the above integral represents a finite sum, we see that each u_n satisfies the heat equation.

Note that $|f|\widetilde{K}(x,\,\cdot\,,t) = |f|e^{-t\Delta}\delta_x$ is in L^1 and from the Lebesgue dominated convergence theorem it follows that for fixed x, $u_n(x,t) \to u(x,t)$ as $n \to \infty$.

From now on, we fix $x \in V^{\circ}$ and $t > 0$.

We first show that $u(x,t)$ is continuous in t. It suffices to notice that

$$u_n(x,t+h) - u_n(x,t) = h\frac{\partial u_n}{\partial t}(x,t_n) \tag{21}$$

for some t_n between t and $t+h$.

The right-hand-side term can be bounded by noting that

$$(u_n)_t(x,s) = (-\Delta e^{-s\Delta} f_n, \delta_x) = (e^{-s\Delta} f_n, -\Delta\delta_x).$$

and

$$|(u_n)_t(x,s)| \le \|e^{-s\Delta} f_n\|_\infty \|\Delta\delta_x\|_1 \le \|f\|_\infty \|\Delta\delta_x\|_1.$$

$\mathcal{G}$ is locally finite, therefore $\|\Delta^2\delta_x\|_1$ is finite. Taking $n \to \infty$ in equation 21 and using the pointwise convergence of u_n we conclude

$$|u(x,t+h) - u(x,t)| \le h\|f\|_\infty \|\Delta\delta_x\|_1.$$

Taking $h \to 0$ we conclude that $u(x,t)$ is continuous in t.

It can be shown that $\Delta u = -u_t$ by a similar proof technique, where we start by noting that

$$\Delta u_n = -(u_n)_t = -\frac{u_n(x,t+h) - u_n(x,t)}{h} + (h/2)(u_n)_{tt}(x,\widetilde{t}_n) \tag{22}$$

for some t_n between t and $t+h$.

$\square$

6.5 The Miminal Heat Kernel

The minimal non-negative heat kernel, K, of a locally finite graph, $\mathcal{G}$, can be constructed as the limit of the heat kernels of any sequence of finite graphs that "exhaust" $\mathcal{G}$. We prove this here and give the properties that follow. Bounds on $\widetilde{K}$ of the last subsection will apply to K. We will show that $\widetilde{K} = K$ in a number of interesting cases, but we don't know if this is generally true.

As usual, let V° be the set of interior vertices of $\mathcal{G}$. For $A \subset V^\circ$, let $K_A = K_A(x,y,t)$ be the spectral heat kernel on the graph $\mathcal{G}_A$, which is the same as $\mathcal{G}$ except that the set of boundary vertices is all vertices not in A[3]. Since $\mathcal{G}_A$ has only finitely many interior vertices

[3]In particular, $K(x,y,t)$ is defined for all $x,y \in \mathcal{G}$, identifying the geometric realization of $\mathcal{G}$ with that of $\mathcal{G}_A$. Also $K_A(x,y,t)$ vanishes if x lies on an edge whose endpoints don't lie in A.

and edges (an interior edge is one with at least one endpoint in the interior), there is a finite orthonormal basis of eigenfunctions for $L^2_{\mathrm{Dir}}(\mathcal{G}_A, \mathcal{V})$, $\phi_1, \phi_2, \ldots$, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots$, and we clearly have

$$K_A(x, y, t) = (e^{-t\Delta_A}\delta_x, \delta_y) = \sum_i e^{-t\lambda_i}\phi_i(x)\phi_i(y),$$

where Δ_A is the Laplacian associated to $\mathcal{G}_A$; more explicitly, $\Delta_A = \pi_A \Delta \pi_A$, where Δ is the Laplacian on $\mathcal{G}$ and π_A is the projection on functions on $\mathcal{G}$ that sends f to 0 on boundary vertices and $f(v)$ on interior vertices v (and is extended to be edgewise linear).

We remark that when no confusion can arise, we will sometimes drop the A from Δ_A.

Consider any sequence $A_1, A_2, \ldots$ such that (1) $A_i \subset V^{\mathrm{o}}$, (2) each A_i is finite, (3) $A_i \subset A_{i+1}$, and (4) for all $v \in V^{\mathrm{o}}$ there is an i with $v \in A_i$. We will say that the sequence A_i is a *increasing finite set exhaustion of* $\mathcal{G}$.

Theorem 6.16 *For any increasing finite set exhaustion, A_i, of $\mathcal{G}$, and any fixed x, y, t we have $K_{A_i}(x, y, t)$ is non-decreasing in i, tending to a limit $K(x, y, t)$. K is independent of the set exhaustion, and $K \leq G$ for any non-negative fundamental solution to the heat equation.*

Proof We shall need the following very easy maximum principle.

Lemma 6.17 *Let u, v be two solutions to the heat equation in $\mathcal{G} \times [0, T]$ such that $u \leq v$ on the "boundary," namely $\mathcal{G} \times \{0\}$ union $\partial\mathcal{G} \times [0, T]$. If $\mathcal{G}$ is finite then $u \leq v$ on all of $\mathcal{G} \times [0, T]$.*

Proof If $u > v$ at (x, t) then $u(x, t) > v(x, t) + \epsilon t$ for some positive $\epsilon > 0$. Since $\mathcal{G}$ is finite, $w(x, t) = u(x, t) - v(x, t) - t\epsilon$ attains a maximum, $M > 0$, over $\mathcal{G} \times [0, T]$ somewhere, say at (x_0, t_0). However, $\Delta w + w_t = -\epsilon < 0$, and it follows that either w_t or Δw is negative at (x_0, t_0). Since $t_0 > 0$ and w is maximized at (x_0, t_0), w_t cannot be negative there. But the same argument as in theorem 6.6 shows that if Δw is negative there, then $w(x, t_0)$ is greater than $w(x_0, t_0)$ for some neighbour, x, of x_0, which is again impossible.

□

Continuing with the proof of the theorem, the maximum principle applied to $\mathcal{G}_{A_i}$ shows that for any x, y, t we have $K_{A_i}(x, y, t)$ is non-decreasing in i and is $\leq G(x, y, t)$ for any non-negative fundamental solution to the heat equation. Also the maximum principle shows $K_A(x, y, t) \leq K_B(x, y, t)$ provided that $A \subset B$, which completes the proof.

$\square$

Proposition 6.18 *K constructed above is a heat kernel.*

Proof Clearly $K(\,\cdot\,, y, 0) = \delta_y$. We wish to show that K is continuous in t for $t \geq 0$ and differentiable in $t > 0$ and satisfies the heat equation there. We integrate the heat equation that K_{A_i} satisfies (for i sufficiently large as a function of x) to conclude

$$-\int_0^t \Delta_x K_{A_i}(x,y,s) = K_{A_i}(x,y,t) - K_{A_i}(x,y,0).$$

Clearly we can pass this equation to the $i \to \infty$ limit to conclude the same for K instead of K_{A_i}. L^∞ bounds on $\widetilde{K}$ imply the same for K, and we first conclude that K is continuous in time, and then differentiable for $t > 0$ and satisfying the heat equation there.

Since the K_{A_i} are symmetric, so is K_A. Similarly we conclude that K is self-reproducing, using the bounded convergence theorem; indeed, the integrand in

$$\int K_{A_i}(x,s,\tau) K_{A_i}(s,y,t-\tau)\, d\mathcal{V}(s) \tag{23}$$

is non-negative and bounded by

$$\widetilde{K}(x,s,\tau)\widetilde{K}(s,y,t-\tau)\, d\mathcal{V}(s),$$

whose integral is bounded. Hence the $i \to \infty$ limit of equation 23 is the integral with K replacing K_{A_i}. It easily follows that K is self-reproducing. Finally, to see that the self-reproducing equation can be differentiated in t, we write

$$(K_{A_i})_t(x,y,t) = -\Delta_y K_{A_i}(x,y,t),$$

and

$$\int K_{A_i}(x,s,\tau)(K_{A_i})_t(s,y,t-\tau)\, d\mathcal{V}(s) = \int K_{A_i}(x,s,\tau)\Delta_y K_{A_i}(s,y,t-\tau)\, d\mathcal{V}(s).$$

We know that the two left-hand-sides are equal. The first left-hand-side clearly tends to $-\Delta_y K(x,y,t)$ as $i \to \infty$. For the second left-hand-side, we use the bounded convergence theorem, using an L^1 bound on $K_{A_i}(x,s,t)$ (coming from $\widetilde{K}$) and an L^∞ bound on $\Delta_y K_{A_i}(s,y,t-\tau)$ via

$$|\Delta_y K_{A_i}(s,y,t-\tau)| = |(\delta_s, e^{-(t-\tau)\Delta}\Delta\delta_y)| \leq \|e^{-(t-\tau)\Delta}\Delta\delta_y\|_\infty \leq \|\Delta\delta_y\|_\infty,$$

the last quantity being bounded for fixed y. We conclude that

$$-\Delta K(x,y,t) = \int K(x,s,\tau)\Delta_y K(s,y,t-\tau)\, d\mathcal{V}(s).$$

Hence the self-reproducing equation can be differentiated in t, and hence K is a heat kernel.

$\square$

Next we ask when is $K = \widetilde{K}$. An easy proposition is:

Proposition 6.19 *Let the Dirichlet initial value problem always have at most one solution $u(x,t)$ for $x \in \mathcal{G}$ and all $t \geq 0$ that is uniformly bounded. Then $K = \widetilde{K}$.*

Notice that the hypothesis of this proposition is satisfied, according to theorem 6.4, if Δ is bounded, or more generally $L_i(v)$ grows linearly in i for v fixed.

Proof For y fixed, $\widetilde{K}(x,y,t)$ is a solution to a Dirichlet initial value problem, bounded (by $1/\mathcal{V}(y)$) in x and t by the L^∞ contractivity operator $e^{-t\Delta}$. $K(x,y,t)$ solves the same Dirichlet initial value problem, and the same bound holds for $K(x,y,t)$, since $0 \leq K \leq \widetilde{K}$.

$\square$

We give a different criterion for K to equal $\widetilde{K}$.

Proposition 6.20 *Let $\mathcal{V}(v) \geq C$ for some constant $C > 0$. Then $K = \widetilde{K}$.*

The maximum principle shows that $\widetilde{K}(x,y,t) \leq K_{A_i}(x,y,t) + \eta(t,y,A_i)$, where

$$\eta(t,y,A) = \sup_{z \in \partial\mathcal{G}_A,\, 0 \leq \tau \leq t} \widetilde{K}(z,y,\tau).$$

It remains to see that $\eta(t,y,A_i) \to 0$ as $i \to \infty$ for any fixed t, y.

Fix a value of y and t. The fact that $\eta(t,y,A_i) \to 0$ as $i \to \infty$ follows from the L^1 boundedness of $e^{s\Delta}\delta_y$ and its continuity in s. Indeed, for every $\tau_0 \in [0,t]$ we have

$$\sum_x \widetilde{K}(x,y,\tau_0)\mathcal{V}(x)$$

is finite. So for any $\epsilon > 0$ there is some i_0 for which

$$\sum_{x \notin A_{i_0}} \widetilde{K}(x,y,\tau_0)\mathcal{V}(x) < C\epsilon,$$

and hence $\widetilde{K}(x,y,\tau_0) < \epsilon$ for any $x \notin A_{i_0}$. By the L^2 continuity of $e^{s\Delta}\delta_y$, there is an $\alpha > 0$ with $\|\widetilde{K}(.,y,\tau) - \widetilde{K}(.,y,\tau_0)\|_2^2 < \frac{\epsilon^2}{C}$ for all τ with $|\tau - \tau_0| < \alpha$ (and $\tau \geq 0$) and therefore $|\widetilde{K}(x,y,\tau) - K(x,y,\tau_0)| < \epsilon$ for all x. Hence $\widetilde{K}(x,y,\tau) < 2\epsilon$ for all $x \notin A_{i_0}$ and $|\tau - \tau_0| < \alpha$. The topological compactness of $[0,t]$ implies that it can be covered with a finite set of intervals $\{\tau \mid |\tau - \tau_0| < \alpha\}$ as above, and we conclude $\eta(t,y,A_i) < 2\epsilon$ for i greater than the largest of the finite set of i_0's that correspond to the finite covering intervals of $[0,t]$. Since $\epsilon > 0$ was arbitrary, we conclude that $\eta(t,y,A_i) \to 0$ as $i \to \infty$.

$\square$

In the sections to follow we will use the fact that if we want to prove an upper bound on $K(x,y,t)$ for fixed x,y,t, it suffices to do so for all $K_A(x,y,t)$ with A finite.

7 L^p Estimates

The Federer-Fleming theorems give us lower bounds on the L^1 norm of ∇f. Other "gradient estimates," namely L^p estimates for $p > 1$, i.e. lower bounds on $\|\nabla f\|_p$, follow readily from the L^1 estimates. Such bounds include Sobolev and Nash inequalities, Trudinger inequalities, and resulting heat kernel and eigenvalue estimates.

7.1 L^p Estimates for $p > 1$

It is easy to use the L^1 gradient estimates to obtain L^p estimates in the non-closed situation. The main new observation here is that often the analogous inequality in the closed situation follows easily using "split" functions; previous works (e.g. [CL81, CY95]) use balanced functions, which makes the estimating more difficult and weaker by a small multiplicative constant.

Proposition 7.1 *Let $F\colon \mathbf{R} \to \mathbf{R}$ be a differentiable function which preserves sign (i.e. $F(x)$ is positive, zero, or negative according to whether or not x is). Let $1 \le p \le \infty$, and further assume that $(F')^{p'}$ is convex. For any $\phi \in C^1_{\mathrm{Dir}}(\mathcal{G})$ we have*

$$I_\nu \|F(\phi)\|_{\nu'} \le \rho_{\sup}^{1/p'} \|\nabla\phi\|_p \|F'(\phi)\|_{p'}.$$

Similarly, in the closed case we have that for any split $\phi \in C^1_{\mathrm{Dir}}(\mathcal{G})$ we have the same as above with $\widetilde{I}_\nu$ replacing I_ν.

Proof The left-hand side comes from applying corollary 4.2 or equation 17 to $F(\phi)$, yielding

$$I_\nu \|F(\phi)\|_{\nu'} \le \left\| \nabla \Big(F(\phi) \Big) \right\|_1 ,$$

and $\widetilde{I}_\nu$ replacing I_ν in the closed case. Next we apply Hölder's inequality to $\nabla F(\phi) = F'(\phi)\nabla\phi$, i.e.

$$\|\nabla F(\phi)\|_1 \le \|F'(\phi)\|_{p',\mathcal{E}} \|\nabla\phi\|_p .$$

Finally equation 5 tells us that the $\mathcal{E}$ in the norm of $F'(\phi)$ can be replaced by $\mathcal{V}$ at a cost of introducing the multiplicative factor $\rho_{\text{sup}}^{1/p'}$.

$\square$

Corollary 7.2 *For any $f \in C^1_{\text{Dir}}(\mathcal{G})$ and any $\nu > p \ge 1$ we have*

$$\|\nabla\phi\|_p \ge c_{\nu,p} \|\phi\|_{p\nu/(\nu-p)} \qquad \text{where} \quad c_{\nu,p} = I_\nu \rho_{\text{sup}}^{-(p-1)/p} (\nu - p)/[p(\nu-1)].$$

The same is true in the closed case if we add a tilde to I_ν and further require f to be split.

Proof We apply the above proposition with $F(x) = \{x\}^r$ where $r = p(\nu-1)/(\nu-p)$.

$\square$

Theorem 7.3 *For $\phi \in C^1_{\text{Dir}}(\mathcal{G})$, we have for any $\nu > 2$*

$$\|\nabla f\|_2 \ge (I_\nu \rho_{\text{sup}}^{-1/2}/2) \|f\|_2^{1+(2/\nu)} \|f\|_1^{-2/\nu}.$$

The same is true in the closed case if we add a tilde to I_ν and further require ϕ to be split.

Proof Applying proposition 7.1 to $\phi = f$ with $F(x) = \{x\}^2$ and $p = q = 2$ gives

$$2\rho_{\text{sup}}^{1/2} \|f\|_2 \|\nabla f\|_2 \ge I_\nu \|f\|_{2\nu'}^2 .$$

By Hölder's inequality we have $\|f^2\|_1 \le \|f^\gamma\|_r \|f^\delta\|_{r'}$ provided that $\gamma + \delta = 2$. Take γ, δ, r, r' so that in addition we have[4] $\gamma r = 1$ and $\delta r' = 2\nu'$, we have

$$\|f\|_2^{2(2\nu'-1)/\nu'} \le \|f\|_1^{(2\nu'-2)/\nu'} \|f\|_{2\nu'}^2 .$$

Combining the two above displayed formulas yields the theorem.

[4]i.e. take $\gamma = 2(\nu'-2)/(\nu'-1)$, $\delta = 2\nu'/(\nu'-1)$, and $r = (\nu'-1)/(\nu'-2)$.

□

We remark that in the above proof we could impose $\delta r' = 1$ and $\gamma r = 2\nu/(\nu-2)$ and apply corollary 7.2 with $p = 2$. This is often done in the literature, and yields a similar result but gives the weaker constant of $c_{2,\nu}$ replacing $I_\nu \rho_{\rm sup}^{-1/2}/2$.

Corollary 7.4 *If $f \in C^1_{\rm Dir}(\mathcal{G})$ with $\int f\, d\mathcal{V} = 0$ and $\mathcal{G}$ closed, then for any $\nu > 2$ we have*

$$\|\nabla f\|_2 \geq (\widetilde{I}_\nu \rho_{\rm sup}^{-1/2}/2)\|f\|_2^{1+(2/\nu)}\|f\|_1^{-2/\nu}.$$

Proof Let a minimize $\|f-t\|_1$ as a function of t. Let $\hat{f} = f - a$. Then $\nabla f = \nabla \hat{f}$, $\|\hat{f}\|_1 \leq \|f\|_1$, and $\|\hat{f}\|_2 \geq \|f\|_2$, and the corollary follows from applying the previous theorem to $\hat{f}$, since $\hat{f}$ is split.

□

The above theorem and corollary are known as a *Nash inequality*, because it is the main inequality necessary in Nash's method. What is new here is our simple proof of this inequality in the closed case, corollary 7.4.

Corollary 7.2 is part of the Sobolev embedding theorem, and $c_{\nu,p}$ is called a Sobolev constant. The optimal value of $c_{\nu,p}$ is quite interesting, and the value in the above corollary is certainly not optimal for $\mathbf{R}^n$ (and $\nu = n$) (see [GT83], page 158). However, we do know that the $c_{\nu,p} = 0$ in the $\nu = p$ case, i.e. there is no $\|\nabla\phi\|_p$ lower bound in terms of $\|\phi\|_\infty$ and I_p.

Proposition 7.5 *Let $c^*_{\nu,p}$ be a constant depending only on ν and p such that corollary 7.2 holds for all M, ϕ with $c_{\nu,p} = I_\nu \rho_{\rm sup}^{-1/p'} c^*_{\nu,p}$. Then for any fixed p there is a $C > 0$ such that $c^*_{\nu,p} \leq C(\nu - p)^{(1/p)-1}$ for all ν sufficiently close to and greater than p.*

Proof We use a function ϕ, growing logarithmically near a point, just as in analysis. The details are in appendix A.

□

Other common pieces of the Sobolev embedding theorem, in the analysis case, are the $\nu < p$ embedding theorems, stating $\|\nabla\phi\|_p \geq c_{\nu,p}\|\phi\|_\infty$ for $c_{\nu,p}$ depending only on $p, \nu, I_\nu, \mathcal{V}(\mathcal{G})$.

Proposition 7.6 *For any split function, ϕ, and any $p > \nu \geq 1$ there is a constant $c_{\nu,p} > 0$ such that*

$$\|\nabla\phi\|_p \geq c_{\nu,p}\mathcal{V}(\mathcal{G})^{(1/p)-(1/\nu)} I_\nu \rho_{\sup}^{-1/p'} \|\phi\|_\infty$$

No lower bound on $\|\nabla\phi\|_p$ is possible based on bounds on I_ν and $\|\phi\|_\infty$ for arbitrary $\mathcal{G}$ (i.e. $\mathcal{V}(\mathcal{G})$ infinite).

Proof This is standard. We string together a number of applications of proposition 7.1 to $F(x) = \{x\}^\gamma$ for various γ, as in [GT83]. The details are in appendix A.

Finally, one has the "critical case" of the Sobolev inequalities, namely $p = \nu$, where, as we've said, $c_{\nu,\nu} = 0$. However, one can replace this with various inequalities of exponential type, also known as Trudinger inequalities (the first appears in [Tru67]). An example would be:

Proposition 7.7 *For any split function, ϕ, we have that for any $\gamma < 1$*

$$\int \left(\exp(\gamma\widetilde{\phi})\right)^{\nu'} d\mathcal{E} \leq \mathcal{V}(\mathcal{G})(1-\gamma)^{-\nu'} \quad \textit{where} \quad \widetilde{\phi} = \phi I_\nu \rho_{\sup}^{-1/\nu'} / \|\nabla\phi\|_\nu.$$

Proof By proposition 7.1 with $F(x) = \{x\}^r$ we have $\|\widetilde{\phi}^r\|_{\nu'} \leq r\|\widetilde{\phi}^{r-1}\|_{\nu'}$. It follows that for integers $r \geq 0$ we have $\|\widetilde{\phi}^r\|_{\nu'} \leq r!\|1\|_{\nu'}$, and so $\|\exp(\gamma\widetilde{\phi})\|_{\nu'} \leq \|1\|_{\nu'}/(1-\gamma)$; the proposition easily follows.

□

7.2 Nash's Method

Now we use the method of Nash (see [Nas58]) and its modification by Cheng and Li (see [CL81]) to obtain a diagonal heat kernel estimate.

Now form

$$G(x,y,t) = \begin{cases} K(x,y,t) & \text{if non-closed,} \\ K(x,y,t) - 1/\mathcal{V}(\mathcal{G}) & \text{if closed,} \end{cases}$$

where K is the minimal heat kernel.

Definition 7.8 *$H = H(x,y,t)$ defined for $x, y \in \mathcal{G}$ and $t \geq 0$ is* heat-like *if (0) it is edgewise linear in x, y, (1) $H(x,x,t) \geq 0$, (2) $H(x,y,t) = H(y,x,t)$, (3) $\Delta_x H(x,y,t) =$*

$\partial_t H(x,y,t)$ *, and (4) H is self-reproducing, i.e.*

$$H(x,y,t) = \int H(x,z,t')H(z,y,t-t')\,d\mathcal{V}(z)$$

for any $0 < t' < t$*, and we may partially differentiate this equation in* t *and interchange the integral and differentiation.* H *is* pre-Nash *if for any* y,t *we have* $f(x) = H(x,y,t)$ *satisfies*

$$\|\nabla f\|_2 \geq C\|f\|_2^{1+(2/\nu)}\|f\|_1^{-2/\nu} \quad \textit{and} \quad \|f\|_1 \leq \gamma \tag{24}$$

for some constants $C, \gamma > 0$*; if so then*

$$\|\nabla f\|_2 \geq C_1\|f\|_2^{1+(2/\nu)} \quad \textit{where} \quad C_1 = C\gamma^{-2/\nu}.$$

Clearly G is heat-like in the non-closed case. In the closed case we may write:

$$K(x,y,t) = \sum_i e^{-t\lambda_i}\phi_i(x)\phi_i(y)$$

where (λ_i, ϕ_i) form a complete set of eigenpairs, and notice that we may take $\lambda_1 = 0$. Then G becomes the above sum over $i > 0$, and it is easy to verify that G is heat-like.

Since for fixed y,t the integral of $K(x,y,t)$ is 1, we have that G is pre-Nash with $C = I_\nu \rho_{\rm sup}^{-1/2}/2$ and $\gamma = 1$ in the non-closed case [5]. Since $G = K - \phi_1(x)\phi_1(y)$ in the closed case, and $\phi_1(x) = \mathcal{V}^{-1/2}(\mathcal{G})$, we have G is pre-Nash in the closed case with $\gamma = 2$ and $C = \widetilde{I}_\nu \rho_{\rm sup}^{-1/2}/2$.

Theorem 7.9 (Nash) *Let* $G(x,y,t)$ *be heat-like and pre-Nash. Then we have*

$$G(x,x,t) \leq C_2 t^{-\nu/2} \qquad \textit{where} \qquad C_2 = (\nu/2)^{\nu/2}C_1^{-\nu}$$

Proof (See [Nas58].) For any fixed $x \in M$ and $t > 0$ we have

$$G(x,x,t) = \|G(x,\cdot,t/2)\|_2^2,$$

and so

$$-(\partial/\partial t)G(x,x,t) = -\int G(x,y,t/2)G_t(x,y,t/2)\,d\mathcal{V}(y)$$

$$= \int G(x,y,t/2)\Delta_y G(x,y,t/2)\,d\mathcal{V}(y)$$

[5] The left-hand inequality in equation 24 is clear if $\mathcal{G}$ is finite. In general, since K is limit of finite heat kernels of graphs with isoperimetric constants no smaller than that of $\mathcal{G}$, we may conclude the same.

$$= \|\nabla_y G(x,y,t/2)\|_2^2 \geq C_1^2 \|G(x,\cdot,t/2)\|_2^{2+(4/\nu)}.$$

Furthermore,

$$\|G(x,\cdot,t/2)\|_2^{2+(4/\nu)} = G(x,x,t)^{(2+\nu)/\nu}.$$

Hence

$$-G(x,x,t)^{-(2+\nu)/\nu}(\partial/\partial t)G(x,x,t) \geq C_1^2$$

We conclude that

$$G(x,x,t)^{-2/\nu} \geq G(x,x,0)^{-2/\nu} + (2/\nu)C_1^2 t \geq (2/\nu)C_1^2 t.$$

□

As a corollary we get an eigenvalue estimate in the closed case. Namely, we have

$$ke^{-\lambda_k t} \leq \sum_{i>0} e^{-\lambda_i t} = \int G(x,x,t)\, d\mathcal{V}(x) \leq C_2 t^{-\nu/2}\mathcal{V}(\mathcal{G})$$

for any $t > 0$. Taking $t = \nu/(2\lambda_k)$ yields

$$\lambda_k \geq C_3(k/\mathcal{V}(\mathcal{G}))^{2/\nu} \qquad \text{where} \qquad C_3 = C_2^{-2/\nu}\nu/(2e).$$

We conclude:

Corollary 7.10 *In the closed case we have for any $\nu > 2$*

$$\lambda_k \geq (k/\mathcal{V}(\mathcal{G}))^{2/\nu} C_2^{-2/\nu}\nu/(2e) = (k/\mathcal{V}(\mathcal{G}))^{2/\nu} 2^{-4/\nu}(I_\nu \rho_{\sup}^{-1/2}/2)^2/e$$

We notice that the $\nu = \infty$ bound is weaker than Cheeger's inequality by a factor of e, in that the above equation yields $\lambda_k \geq (I_\infty/2)^2/e$.

We also get a heat kernel estimate in the non-closed case as well as the closed case:

Corollary 7.11 *For any $\mathcal{G}$ and any $\nu > 2$ we have*

$$K(x,x,t) \leq C_2 t^{-\nu/2},$$

where $C_2 = (\nu/2)^{\nu/2} C_1^{-\nu}$ where $C_1 = I_\nu \rho_{\sup}^{-1/2}/2$. If $\mathcal{G}$ is closed, the same holds with K replaced by $K - 1/\mathcal{V}(\mathcal{G})$, with a tilde added to I_ν, and with C_1 multiplied by $2^{-2/\nu}$.

Proof The only new statement here is that the above inequality is also valid when $\mathcal{G}$ is infinite (i.e. the non-closed case). But in this case we simply note that the inequality holds for any $K_A(x,x,t)$ with A a finite subset of $\mathcal{G}$'s vertices, and then take the limit as A exhausts $\mathcal{G}$.

□

7.3 Other estimates

The previous results gave us heat kernel and eigenvalue estimates, when one has an isoperimetric inequality of the form $\mathcal{A}(\partial\Omega) \geq C\mathcal{V}(\Omega)^\alpha$, for some $0 < \alpha \leq 1$. We address now the issue of obtaining results of the same kind when one has an isoperimetric inequality of the form

$$\mathcal{A}(\partial\Omega) \geq \frac{\mathcal{V}(\Omega)}{\phi(\mathcal{V}(\Omega))} \tag{25}$$

where ϕ is a positive and non-decreasing function defined on $(0, \infty)$.

We will prove the following theorem (which will be a generalization of Theorem 7.9).

Theorem 7.12 *Let*

$$F(x) = \int_x^\infty \frac{(\phi(4/u))^2}{u} du.$$

and $C = \frac{1}{32\rho_{\sup}}$. *If (25) holds for any admissible* Ω *, then*

$$K(x, x, t) \leq F^{-1}(Ct).$$

Remark If we set $\phi(x) = x^{1/\nu}$, then we get Theorem 7.9 with slightly worse constants.

The proof of this theorem is obtained through the following chain of implications, where $C^1_{\mathrm{Dir}}(\Omega) = \{f \in C^1_{\mathrm{Dir}}(\mathcal{G}) \mid f \equiv 0 \text{ on } \Omega^c\}$.

$$\begin{array}{lll}
\forall \text{ admissible } \Omega, & \mathcal{A}(\partial\Omega) & \geq \dfrac{\mathcal{V}(\Omega)}{\phi(\mathcal{V}(\Omega))} & (26) \\
& \Downarrow & & \\
\forall f \in C^1_{\mathrm{Dir}}(\Omega), & \phi(\mathcal{V}(\Omega))\|\nabla f\|_{1,\mathcal{E}} & \geq \|f\|_{1,\mathcal{V}} & (27) \\
& \Downarrow & & \\
\forall f \in C^1_{\mathrm{Dir}}(\Omega), & 2\rho_{\sup}^{1/2}\phi(\mathcal{V}(\Omega))\|\nabla f\|_{2,\mathcal{E}} & \geq \|f\|_{2,\mathcal{V}} & (28) \\
& \Downarrow & & \\
\forall f \in C^1_{\mathrm{Dir}}(\mathcal{G})), & 32\rho_{\sup}\left(\phi\left(\dfrac{4\|f\|^2_{1,\mathcal{V}}}{\|f\|^2_{2,\mathcal{V}}}\right)\right)^2 \|\nabla f\|^2_{2,\mathcal{E}} & \geq \|f\|^2_{2,\mathcal{V}} & (29) \\
& \Downarrow & & \\
\forall x \in V,\ t > 0 & K(x,x,t) & \leq F^{-1}(Ct) & (30)
\end{array}$$

Proof (26) $\Rightarrow$ (27) This follows since ϕ is increasing, so we have $I_\infty(\Omega) = 1/\phi(\mathcal{V}(\Omega))$.

Proof (27) $\Rightarrow$ (28) This proof is essentially identical to that for proposition 7.1, and is done by applying (27) to f^2, then using Cauchy-Schwartz, and corollary eq:rhoconcaveineq.

Proof (28) $\Rightarrow$ (29) This is basically proposition 10.3 of [BCLSC95b] in our context. The proof goes as follows (we will omit the subscript $\mathcal{V}$ in expressions like $\|f\|_{\alpha,\mathcal{V}}$): first we notice that for $t > 0$

$$\begin{aligned}\|f\|_2^2 &= \int_{|f|\geq 2t} f^2 d\mathcal{V} + \int_{|f|\leq 2t} f^2 d\mathcal{V} \\ &\leq \int_{|f|\geq 2t} 4\left((|f|-t)^+\right)^2 d\mathcal{V} + \int_{|f|\leq 2t} f^2 d\mathcal{V} \\ &\leq 4\|(|f|-t)^+\|_2^2 + \int_{|f|\leq 2t} f^2 d\mathcal{V} \\ &\leq 4\|(|f|-t)^+\|_2^2 + 2t\|f\|_1\end{aligned}$$

We use (28) with $(|f|-t)^+$ and we obtain[6]

$$\begin{aligned}\|(|f|-t)^+\|_2^2 &\leq 4\rho_{\sup}\left(\phi(\mathcal{V}(|f|\geq t))\right)^2 \|\nabla(|f|-t)^+\|_2^2 \\ &\leq 4\rho_{\sup}\left(\phi(\mathcal{V}(|f|\geq t))\right)^2 \|\nabla f\|_2^2 \\ &\leq 4\rho_{\sup}\left(\phi(\frac{\|f\|_1}{t})\right)^2 \|\nabla f\|_2^2\end{aligned}$$

Therefore

$$\|f\|_2^2 \leq 16\rho_{\sup}\left(\phi(\frac{\|f\|_1}{t})\right)^2 \|\nabla f\|_2^2 + 2t\|f\|_1.$$

We choose t to be $\frac{\|f\|_2^2}{4\|f\|_1}$ to conclude the proof.

Proof (29) $\Rightarrow$ (30) We may assume $\mathcal{G}$ is connected. First assume that $\mathcal{G}$ is finite. By assumption, $\partial\mathcal{G}$ is non-empty (or equation 25 is immediately violated). Let $U(t) = K(x,x,t)$. As in the proof of theorem 7.9 we note that

$$-\frac{dU}{dt} = \|\nabla_y K(x,y,t/2)\|_2^2.$$

Let $f(y) = K(x,y,t/2)$. By first using the non-negativity of the heat kernel, and second its L^∞contractivity we obtain

$$\|f\|_1 = \int |K(x,y,t/2)| d\mathcal{V}(y) = \int K(x,y,t/2) d\mathcal{V}(y) \leq 1$$

[6]By approximation we have that equation 28 implies the same inequality for any Lipschitz function vanishing out of Ω. Hence we can apply this inequality to $(|f|-t)^+$.

Moreover

$$\|f\|_2^2 = \int K(x,y,t/2)^2 d\mathcal{V}(y) = K(x,x,t) = U(t).$$

We now apply (29) to f

$$32\rho_{\sup} \left(\phi\left(\frac{4\|f\|_1^2}{\|f\|_2^2} \right) \right)^2 \|\nabla f\|_2^2 \geq \|f\|_2^2$$

and obtain by using the previous remarks

$$-\frac{dU}{dt}(\phi(4/U))^2 \geq CU.$$

Dividing both sides of this inequality by U and integrating against dt from 0 to t we get

$$\int_{U(t)}^{U(0)} \frac{(\phi(4/U))^2}{U} dU \geq \int_0^t C dt$$

Therefore

$$F(U(t)) = \int_{U(t)}^{\infty} \frac{\phi(4/U))^2}{U} \geq Ct,$$

we conclude the theorem for finite $\mathcal{G}$ by using the fact that F is decreasing.

If $\mathcal{G}$ is infinite, we see that the bound applies to $K_A(x,x,t)$ for finite $A \subset V^{\mathrm{o}}$ (since clearly equation 25 also holds for all admissible Ω in $\mathcal{G}_A$), with notation as in subsection 6.5. We conclude the theorem by taking the limit as A exhausts $\mathcal{G}$.

□

A Some Sobolev Related Calculations

A.1 Logarithmic functions on radial graphs.

In this section we use $f \approx g$ to mean there exist universal constants $c_1, c_2 > 0$ such that $c_1 f \leq g \leq c_2 f$. We use $f \prec g$ to mean there is a universal constant $c_1 > 0$ such that $f \leq c_1 g$, and similarly for $f \succ g$.

A.1.1 A non-closed example

Here we give a sequence of non-closed graphs (in fact finite but with boundary) which bound the optimal constant in proposition 7.5.

Definition A.1 *The* path of length n *is the graph* $G = (V, E)$ *with* $V = \{1, \ldots, n\}$*, and an edge* $\{i, i+1\}$ *for each* $i = 1, \ldots, n-1$*. For* $\nu \geq 1$*, by the* ν-dimensional radial graph of size n, $G = G_{n,\nu}$*, we understand the path of length* n *with* n *being a boundary vertex, with the following measures,* $\mathcal{V}$ *and* $\mathcal{E}$*:* $\mathcal{E}(\{i, i+1\}) = i^{\nu-1}$*, and* $\mathcal{V}$ *is taken* natural *with respect to* $\mathcal{E}$*, meaning we take the unique measure* $\mathcal{V}$ *such that* $\mathcal{G}$ *is* 2*-regular.*

In other words, $\mathcal{V}(i) = [(i-1)^\nu + i^\nu]/2$ for $i < n$ and $\mathcal{V}(n) = (n-1)^\nu/2$. We can extend the definition above to the $n = \infty$ case, giving the (countably) infinite path and the (countably) infinite ν-dimensional radial graph in the obvious way.

First we observe:

Proposition A.2 *For* $1 \leq \nu \leq n$ *(allowing* $n = \infty$*) we have* $I_\nu(G) \approx \nu^{1/\nu'}$ *for* $G = G_{n,\nu}$*.*

Proof According to Yau's remark, it suffices to consider Ω connected, i.e. $\Omega = (a, b)$ with $1 < a < b < n$, or $\Omega = [1, b)$. Let i and j be respectively the smallest integer and the biggest integer, such that $[i, j] \subset \Omega$. Note that

$$\mathcal{A}(\partial\Omega) = (1/2)\Big((i-1)^{\nu-1} + j^{\nu-1}\Big) \approx j^{\nu-1},$$

and

$$\mathcal{V}(\Omega) = \sum_{t=i}^{j} [(t-1)^{\nu-1} + t^{\nu-1}]/2 \approx \nu j^\nu,$$

and hence

$$i_\nu(\Omega) \approx \nu^{1/\nu'}.$$

$\square$

Define f_m on G as above for any $m \leq n$ via

$$f_m(i) = \begin{cases} \log(m/i) & \text{for } i \leq m, \\ 0 & \text{for } i > m. \end{cases}$$

We have

$$\|\nabla f_m\|_p^p = \sum_{i=1}^{m-1} \log^p\Big(1+(i+1)^{-1}\Big) i^{\nu-1} \approx \sum_{i=1}^{m-1} i^{\nu-p-1} \approx \begin{cases} m^{\nu-p}/(\nu-p) & \text{if } \nu > p, \\ \log m & \text{if } \nu = p, \\ 1 & \text{if } \nu < p. \end{cases}$$

Also

$$\|f_m\|_q^q = \sum_{i=1}^{m-1} \Big(\log(m/i)\Big)^q i^{\nu-1} \succ \Big(\log(m/k)\Big)^q k^\nu/\nu$$

for any $k = 1, \ldots m$. Taking $k = me^{-q/\nu}$ (assuming $me^{-q/\nu} \geq 1$) yields:

$$\|f_m\|_q \succ (q/\nu) m^{\nu/q} \nu^{-1/q}.$$

Of course $\|f_m\|_\infty = \log m$.

Taking $q = p\nu/(\nu - p)$ for $\nu \geq p$ (meaning $q = \infty$ with $\nu = p$), yields the first part of proposition 7.5. Taking $q = \infty$ for $\nu < p$ yields the last claim of proposition 7.6.

A.1.2 A Closed Example

In this section we give a sequence of closed graphs demonstrating the same bound on the optimal constant in proposition 7.5. We do so by the process of *doubling*.

Definition A.3 *Let G be a graph. By the* double *of G we mean the graph H formed from taking two copies of G, G_+, G_-, (which we sometimes call the positive and negative parts) and by identifying the boundary points of G_+ with those of G_- and declaring such points to be no longer boundary points. H comes with a natural involution ι exchanging G_+ and G_-. The $\mathcal{V}$ measure on the boundary of G is doubled in H to preserve naturality.*

Proposition A.4 *Let f be an odd function on a doubled graph, i.e. $f(\iota(x)) = -f(x)$ for all x. Then f is split and p-balanced for all p.*

Proof Clearly $\int \{f\}^{p-1}\, d\mathcal{V} = 0$, and all claims follow.

□

Definition A.5 *The n-th* doubled ν-dimensional radial graph, *$\widetilde{G}_{n,\nu}$ is the double of $G_{n,\nu}$.*

We similarly show that $I_\nu(\widetilde{G}_{n,\nu}) \approx \nu^{1/\nu'}$ (it suffices to take connected Ω with $|\Omega| \le |\widetilde{G}_{n,\nu}|/2$). Now we take $\widetilde{f}_m$ on $\widetilde{G}_{n,\nu}$ to be f_m on the positive part and $-f_m$ on the negative part. Clearly $\widetilde{f}_m$ is odd, and therefore split and p-balanced for all p. The calculations of $\|\widetilde{f}_m\|_q$ and $\|\nabla\widetilde{f}_m\|_p$ go through as for f_m.

A.2 Classical graphs and the ν-dimensional graph

In the previous subsection we found graphs demonstrating the best possible constant in proposition 7.5. But these graphs weren't classical graphs, in that their measures weren't traditional. We can modify these graphs to obtain traditional regular measures with the same results (e.g. as in proposition 7.5).

Let $V(1),\dots,V(n)$ be a non-decreasing sequence of positive rationals such that $V(n) = 2[V(n-1)-V(n-2)+\cdots]$. Let G be a classical graph as follows: its vertices are partitioned into n sets, $V_1,\dots,V_n$, with $|V_i| = kV(i)$ for an integer k making $kV(i)$ integral for all i; its edges are $E(i)$ copies of a complete graph between V_i and V_{i+1} for each i, where $E(i) = \ell[kV(i)V(i+1)]^{-1}[V(i)-V(i-1)+V(i-2)-\cdots]$, for an ℓ which makes the $E(i)$ integers. Now declare V_n to be the boundary points of G, and form the double, H. Then H is an ℓ-regular graph. Clearly for any ν we have I_ν is the same for H as it is for the double, $\widetilde{H}$, of the path of length n (with boundary point n) and which is ℓ-regular and with measure $\mathcal{V}(i) = kV(i)$. And clearly a function, f, on $\widetilde{H}$ lifts to a function on H which preserves $\|f\|_q$ and $\|\nabla f\|_p$ for any p and q.

In the above paragraph, fix a ν and an integer m, and set $V(i) = \lfloor mi^{\nu-1}\rfloor$ for $1 \le i \le n-1$, and

$$V(n) = 2[V(n-1) - V(n-2) + \cdots].$$

Then we see $V(n) = mn^{\nu-1}[1+o(1)]$, and it follows that the calculations of the previous subsection for the optimal constant in proposition 7.5 all hold here.

A.3 The $p < \nu$ Sobolev Imbedding

Here we prove proposition 7.6. By scaling $\mathcal{V}$ and $\mathcal{E}$ we may assume $\mathcal{V}(\mathcal{G}) = 1$ and $\rho_{\sup} = 1$. As in [GT83] page 156, set $\widetilde{u} = I_\nu u/\|\nabla u\|_p$. Since $|M| = 1$ we have $\|\nabla u\|_1 \le \|\nabla u\|_p$, and so $\|\widetilde{u}\|_{\nu'} \le 1$. By proposition 7.1

$$\|u^\gamma\|_{\nu'} I_\nu \le \gamma \|u^{\gamma-1}\|_{p'} \|\nabla u\|_p,$$

i.e.

$$\|\widetilde{u}^\gamma\|_{\nu'} \le \gamma \|\widetilde{u}^{\gamma-1}\|_{p'},$$

i.e.

$$\|\widetilde{u}\|_{\gamma\nu'} \le \gamma^{1/\gamma} \|\widetilde{u}\|_{p'(\gamma-1)}^{1-(1/\gamma)}, \tag{31}$$

provided that $p'(\gamma - 1) \ge 1$ (this is the convexity of $(F')^{p'}$). Set $\gamma_i = 1 + \delta + \cdots + \delta^{i+1}$ where $\delta = \nu'/p' > 1$; we have $p'(\gamma_0 - 1) = \nu'$ and $p'(\gamma_i - 1) = \nu'\gamma_{i-1}$. Applying equation 31 n times we have for any n

$$\|\widetilde{u}\|_{\gamma_n\nu'} \le \left(\gamma_0\gamma_1^{1/\delta}\cdots\gamma_n^{1/\delta^n}\right)^{\delta^n/\gamma^n}$$

using the fact that $\|\widetilde{u}\|_{\nu'} \le 1$. It follows that for any n

$$\|\widetilde{u}\|_{\gamma_n\nu'} \le c_1^{c_2}$$

where

$$c_1 = \gamma_0\gamma_1^{1/\delta}\gamma_2^{1/\delta^2}\cdots \quad \text{and} \quad c_2 = p'(\nu' - p')/(\nu')^2.$$

Taking $n \to \infty$ yields the proposition.

$\square$

B Proofs of Uniqueness and Non-uniqueness

Notice that if u_1, u_2 are two solutions to the Dirichlet initial value problem with same initial value, then $u = u_1 - u_2$ satisfies the same with initial value 0. So to prove uniqueness it suffices to prove uniqueness in this case. Similarly for non-uniqueness.

We begin by proving theorem 6.2. If $u(x,t)$ solves the Dirichlet initial value problem in $[0,T]$ with initial value 0, then assuming $u > 0$ somewhere, we have that u's maximum value, M, over $\mathcal{G} \times [0,T]$ is attained somewhere (since $\mathcal{G}$ is finite), say $u(x_0,t_0) = M$ (of course, $t_0 > 0$ and $x \in V^{\circ}$). Arguing as in theorem 6.6 we have $u_t(x_0,t_0) \ge 0$ and so $\Delta u(x_0,t_0) \le 0$ and we conclude $u(w,t_0) = M$ for all $w \in V^{\circ}$ with an edge to x_0. Similarly we conclude $u(w,t_0) = M$ for all $w \in V^{\circ}$ with a path to x_0 in V°.

Consider that

$$\frac{d}{dt}\int u^2(x,t)\,d\mathcal{V}(x) = \int 2u(x,t)u_t(x,t)\,d\mathcal{V}(x)$$

$$= -\int 2u(x,t)\Delta u(x,t)\,d\mathcal{V}(x) = -\int |\nabla u|^2\,d\mathcal{V}(x) \le 0.$$

In brief, $\int u^2\,d\mathcal{V}(x)$ is non-increasing. In the above we could have assumed that t_0 was as small as possible, i.e. that $u(x,t) < M$ for $t < t_0$ (again using the fact that $\mathcal{G}$ is finite).

Since u is differentiable in t and $\mathcal{G}$ is finite, we have that $u(x,t)$ remains positive for all x and t near t_0. But then $\int u^2(x,t)\,d\mathcal{V}(x)$ is visibly less for t slighly less than t_0 than it is for $t=t_0$, a contradiction.

□

Next we prove theorem 6.3. We proceed as in [Fri64]. It is known (see [Man35]) that for any $\delta>0$ there is a non-zero function $f\in C^\infty(\mathbf{R})$ vanishing outside of $(0,1)$ and with $|f^{(m)}(t)|\le C^m m^{(1+\delta)m}$ for a constant C. Let $\mathcal{G}$ be the graph whose vertices are the integers, $\mathbf{Z}$, and with an edge $\{i,i+1\}$ for each $i\in\mathbf{Z}$; endow $\mathcal{G}$ with the standard $\mathcal{V}$, a_e, and ℓ_e, and take the boundary to be empty. Then for $x\in\mathbf{Z}$,

$$u(x,t)=\sum_{m=0}^{\infty} f^{(m)}(t)\binom{x+m}{2m}$$

is a finite sum; extending to $x\notin\mathbf{Z}$ by linearity, it is easy to verify that u satisfies the heat equation. Since $u(x,0)=0$, this means the Dirichlet initial value problem does not have a unique solution. (We easily see that $|u(x,t)|\le (C_1|x|)^{|x|(1+\delta)+1}$.)

□

Next we prove theorem 6.4 Let $u(x,t)$ be a solution to the Dirichlet initial value problem with initial value 0 whose absolute value is bounded in $[0,T]$ by a finite constant B.

We claim, by induction, that for any integer $j\ge 0$ we have

$$|u(x,t)|\le BL_0(v)\cdots L_{j-1}(v)(2t)^j/j! \tag{32}$$

For $j=0$ this just says $|u|\le B$, which is our assumption. Now assume the assumption holds for some j. Write

$$|\Delta u(x,t)|\le L(v)(|u(x,t)|+\sup_{y\sim x}|u(y,t)|)\le$$

$$L(x)2B\sup_{y\sim x}L_0(y)\cdots L_{j-1}(y)(2t)^j/j!\le 2BL_0(x)\cdots L_j(x)(2t)^j/j!,$$

and conclude

$$|u(x,t)|=\left|\int_0^t u_s(x,s)\,ds\right|\le 2^{j+1}BL_0(x)\cdots L_j(x)\int_0^t s^j/j!\,ds.$$

We conclude equation 32 holds for $j+1$.

Now we take $j\to\infty$. We conclude that there is an $\epsilon>0$ (depending only on the C in the theorem's hypothesis on L) such that $u(x,t)=0$ for all x and $t\le\epsilon$. But then we again conclude $u(x,t)=0$ for t less than any multiple of epsilon, i.e. for all t.

□

Next we prove theorem 6.5. Let $n_i = \lfloor i^{1+\alpha} \rfloor$ for i a positive integer, so that n_i is a positive integer with n_i growing like $i^{1+\alpha}$. Let $\mathcal{G}$ be the the tree whose vertices are as follows: it has one vertex labelled "1." The vertex "1" has n_1 edges to n_1 different vertices, each labelled "2." Each vertex labelled "2" has n_2 edges to n_2 distinct vertices labelled "3," for a total of $n_1 n_2$ vertices labelled "3." Similarly there are $n_1 \cdots n_i$ vertices labelled "$i+1$," each with one edge to an "i" vertex, and with n_{i+1} edges to "$i+2$" vertices.

Consider a function f, on $\mathcal{G}$, whose value at the "i" vertices is $f(i)$. We wish to make f satisfy $\Delta f = -f$ at all vertices which for "i" vertices with $i > 1$ entails:

$$[f(i) - f(i-1)] + [f(i) - f(i+1)]n_i = -f(i),$$

or

$$f(i+1) = [(2 + n_i)f(i) - f(i-1)]/n_i,$$

and which similarly entails $f(2) = (1 + n_1)f(1)/n_1$ at the "1" vertex. So choose $f(1) = 1$, let $f(2) = (1 + n_1)f(1)/n_1$ and let $f(i)$ for $i \geq 2$ be determined recursively by the above equation. We easily see by induction that $f(i) < f(i+1)$. Also clearly

$$f(i+1) \leq (2 + n_i)f(i)/n_i,$$

and so

$$f(i) \leq (1 + 2n_{i-1}^{-1}) \cdots (1 + 2n_1^{-1}).$$

Since $\alpha > 0$ we have

$$(1 + 2n_1^{-1})(1 + 2n_2^{-1}) \cdots < \infty,$$

and so the $f(i)$ are bounded.

It follows that $u(i,t) = e^t f(i)$ satisfies the heat equation, and the L^∞ norm of u increases as a function of t. But by our heat kernel construction we know that the Dirichlet initial value problem with initial condition f has a solution $w(i,t)$ whose L^∞ norm is no more than that of f. So any multiple of $u - w$ is a non-zero solution of the heat equation with zero boundary conditions and which is bounded for any fixed t.

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