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SINGULAR SECOND-ORDER OPERATORS: THE MAXIMAL AND MINIMAL OPERATORS, AND SELFADJOINT OPERATORS IN BETWEEN*

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Abstract. Differential operators arising from the differential equation

$$-(py')' + qy = \lambda wy$$

are put in the Hilbert space setting $L^2(a, b; w)$. A new and simpler characterization of the minimal operator is exhibited.

Selfadjoint operators, which lie in between the minimal and maximal operator, are easily described in terms of conditions on boundary coefficients, which look like, and indeed are, the same as those imposed on regular problems.

Examples, drawn from mathematical physics, include the Legendre, Laguerre, Hermite, and Bessel problems.

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1. Introduction.

Many of the equations of the special functions, orthogonal polynomials, Bessel functions, and others from mathematics applications have been well known for over a century, with their primary use being in the writing of series and integral solutions of certain partial differential equations of mathematical physics. While *regular* problems, those over finite intervals with well behaved coefficients that pose no problem, those for which there are difficulties because of infinite regions or infinite or zero coefficients, and are therefore labelled as *singular*, need a more careful examination.

The problem usually first encountered with singular problems is in narrowing the choice of acceptable solutions for the various equations involved. Most of the times these problems have been studied, the methods employed to determine these acceptable solutions have been largely ad hoc with no real underlying mathematical justification. Conditions such as "boundedness," or "having a zero limit" have been used intuitively without adequate mathematical reasons.

With such a shaky foundation, the verification of other properties has also been somewhat unsatisfactory. For example, in order to assume that certain characteristic values are real, an application of the spectral resolution theorem for selfadjoint operators on a Hilbert space is essential. Many discussions show via Green's formula that the expressions in question are symmetric or hermitian, but fail to show that the operator involved is maximally extended. Unless the domain of the operator is adequately described, the verification of such a maximal extension and selfadjointness is impossible. Fortunately, in most cases there is an underlying selfadjoint operator, and so the desired, needed results are available, even if they are under the surface.

As an example, consider the expression

$$Hy = -y'' - \left(\frac{a}{x^2}\right) y, \qquad a > \frac{1}{4}$$

in $L^2(0, \infty)$. First considered by Case [6] with the constraint $y(0) = 0$, the "eigenfunctions" and "eigenvalues" seemed to involve an undetermined parameter b.

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The problem was reconsidered by Burnap et al. [5], who used a boundary term involving the solutions of

$$Hy = \pm i\, y.$$

Since there are two independent solutions of this equation that are square integrable near zero, there was a problem of which to use; again there was an undetermined parameter b.

Actually there is a different selfadjoint operator H_b for each choice of b. The condition $y(0) = 0$ is not really a condition at all, since every solution of $Hy = \lambda y$ vanishes at $x = 0$ at least as fast as $x^{\frac{1}{2}}$. (See [38].)

This was state of the art, more or less, until some fifteen years ago. Then some significant refinements began to be developed.

Within the past ten years certain crucial advancements in the algebraic formulas describing singular situations have been made, which makes possible a complete description analogous to the case of regular problems. Singular boundary conditions, described as Wronskian limits, remove the need for ad hoc constraints. Green's formula for singular problems was newly refined, making transparent what is required. All this makes it possible to describe the really interesting singular problems at an intermediate level with little work required beyond what is already available for regular problems.

A word or two concerning the history of the problem, those who contributed to its solution, and a brief description of its extensions is in order. The fundamental beginnings are found in two papers by Weyl [64], [65] in 1910, where the now famous limit-point limit-circle criterion for classifying singular points was introduced. Progress continued steadily in the 1920s with the development of functional analyses and the theory of Hilbert spaces. Of particular interest is the work of von Neumann [63] and Stone [60], culminating with the derivation of the spectral resolution of an unbounded selfadjoint operator on a Hilbert space.

Another approach was developed during the 1930s and 1940s by Titchmarsh, who applied complex analysis techniques to the development of the spectral resolution of differential operators. His work culminated in his now famous book [62].

In the late 1940s and early 1950s Levinson [45], [46] and Kodaira [35] applied Helly's theorems from statistics to the resolution problem. Subsequent contributions by Levinson and his student Coddington led to their classic textbook [8], which is still in use today.

In 1964 Atkinson [1] extended the theory to systems in a unique book, which still contains original insights and is still intriguing to researchers in the field. During the 1960s and 1970s Atkinson and his friend W. N. Everitt, both Titchmarsh students, made a number of contributions, filling in many parts of the theory [1], [2], [14]–[19].

The 1970s saw two significant contributions to the field. The idea of using a Sobolev or energy norm was introduced by Pleijel [58], [59]. A major improvement in how to write singular boundary conditions, particularly in Green's formula, was developed by Fulton [23]. Although the book by Dunford and Schwartz [12] had done similar things a few years earlier, Fulton's paper really got the ball rolling.

The 1970s also saw the introduction of indefinite inner product spaces, following the work of Bognar [4]. Krall [36] and Mingarelli and Krall [54], [55], [56] developed these spaces for the Laguerre–Legendre type, Laguerre type, and Jacobi type polynomials. There is still much to do here.

During the 1980s Hinton and Shaw [27]–[32] vigorously pursued the development of singular Hamiltonian systems, which generalize the scalar differential problems. Krall

[39]–[42] developed the spectral resolutions of such systems. Littlejohn [49], [50] considered higher-order orthogonal polynomial problems, which serve as examples for systems. He and Everitt [20] and also with Krall [21] considered various polynomial problems in left definite or energy normed spaces.

One extremely important contribution, made in 1986, is due to Kaper, Kwong, and Zettl [33]. Here are connected the classic ad hoc conditions with other constraints, especially including the appropriate limiting Wronskians.

The Russians have been heavy contributors since the 1950s. Namark's work is exhibited by his book [57]. The books of Levitan [47], [48] contribute substantially to differential systems. There are many others from the Soviet Union which space prohibits us from mentioning.

The Chinese have also recently been making substantial contributions. Cao Zhijiang [16], [11] and Sun Jiong [61] have worked on higher-order problems with middle deficiency indices as well as limit circle problems.

Most recently Hajmirzaahamad [25], [26] has extended the theory of the second-order problems associated with the generalized Laguerre and Jacobi polynomials to nonclassical situations ($\alpha < -1$ [Laguerre], $\alpha, \beta < -1$ [Jacobi]). She also developed a left definite or energy norm theory for these problems.

There are, of course, many other contributors whose work has not been cited. We apologize. Lack of space makes a really complete list impossible. It is not our intent here to present a complete historical survey, however. To do so would really require the writing of a book, and so we hope this limited, biased introduction will suffice.

Rather, it is our aim there to describe only what occurs in the classical second-order case in the classical L^2 setting. This is indeed the situation encountered most often by users of the subject; it is here that there are substantial refinements in the theory that are easy to describe, and it is here where there is the most interest. We do so in what we believe to be the most efficient way, using the improvements of those listed, as well as many of those we have failed to cite. We hope the article will make the presentation of the subject easier for all.

In §2 we highlight the classical Weyl theory of "limit points" and "limit circles" and their implications concerning solutions in $L^2(a, b; w)$. Section 3 uses these square integrable solutions to produce Wronskian (and regular) boundary conditions. Section 4 examines the classical boundary value problem first discussed by Weyl [64]. Using these results §5 develops four singular Green's formulas. These are used in §6 to introduce the maximal and minimal operators. Lying between the maximal and minimal operators are the selfadjoint operators. These are determined by imposing singular boundary conditions, the subject of §7. Section 8 concludes with examples. A brief §9 contains some remarks.

We hope the reader will find the descriptions rendered herein accessible. It is our main goal to keep the material understandable with as little Hilbert space theory as possible.

2. Weyl theory highlights.

We consider the differential expression

$$\ell y = \frac{(-(py')' + qy)}{w},$$

over an interval (a, b), $-\infty \le a < b \le \infty$, where p^{-1}, q, and $w > 0$ are locally integrable within (a, b). This ensures that, given appropriate values of y and py' at any point $e \in (a, b)$, the equation $\ell y = f$ has a unique solution [22].

We make the following definition.

DEFINITION. The endpoint a is regular if a is finite, and p^{-1}, q, and w are integrable in an interval $[a, a+\epsilon)$ for some $\epsilon > 0$. Otherwise a is a singular point. Likewise the endpoint b is regular if b is finite, and p^{-1}, q, and w are integrable in an interval $(b-\epsilon, b]$ for some $\epsilon > 0$. Otherwise b is a singular point (see [22]).

Our setting is no longer the traditional space of continuous functions $C(a, b)$, but is of necessity $L^2(a, b; w)$, the Hilbert space with inner product

$$\langle f, g \rangle = \int_a^b f \bar{g} \, w \, dt.$$

A lack of study on a student's part of Hilbert spaces or, in particular, of L^2 spaces should prove no great burden when it is explained that the Lebesgue theory is needed primarily for completeness. Most results do not require or require only marginally the full force of Lebesgue measure.

It is our goal to study selfadjoint differential operators generated by the expression ℓ in $L^2(a, b; w)$. For example, if $(a, b) = (-1, 1)$, the expression

$$\ell y = ((1-x^2)y')'$$

in $L^2(-1, 1; 1)$ is related to the Legendre polynomials. With appropriate boundary conditions, the boundary value problem associated with ℓ has the Legendre polynomials as eigenfunctions.

If $(a, b) = (0, \infty)$, the expression

$$\ell y = \frac{(-x\, e^{-x} y')'}{e^{-x}}$$

is associated with the Laguerre polynomials.

If $(a, b) = (-\infty, \infty)$, the expression

$$\ell y = -\frac{(e^{-x^2} y')'}{e^{-x^2}}$$

is associated with the Hermite polynomials.

If (a, b) is either of the intervals $(0, \infty)$ or $(0, 1)$, the expression

$$\ell y = \frac{[-(xy')' + (n^2/x)y]}{x}$$

involves Bessel functions of various kinds.

These problems are all singular at least one endpoint. At such singular points there are a number of difficulties, not occurring with regular problems, which must be overcome. For instance, how many solutions are acceptable? Is a boundary condition required? What is a boundary condition? How is the problem made selfadjoint in the Hilbert space context?

In 1910, Weyl [64] invented an ingenious technique for surmounting some of these problems. He showed that each regular problem over an interval $[e, b']$, $a < e < b' < b$, corresponds to a point on a circle. Then he found an explicit bound for the integral square of a specific solution of the homogeneous differential equation

$$(-py')' + qy = \lambda w y.$$

But, most importantly, as b' approaches the singular point b, the circle contracts to a "limit point" or "limit circle," and the bound on the integral square remains finite.

A similar application on $[a', e]$, letting a' approach a singular point a, generates another solution, square integrable on $(a, e]$. Using both allows the door to be opened on problems singular at both endpoints.

We present the highlights of Weyl's approach: Weyl begins by studying the regular problem on a finite interval and then letting the right end point slide to approach a singular end.

Therefore let us focus our attention on an interval $[e, b']$, $a < e < b' < b$. For an arbitrary, but fixed, real α. Let θ_1 and θ_2 be solutions of

$$(-py')' + qy = \lambda wy,$$

satisfying the initial conditions

$$\theta_1(e) = \cos\alpha, \qquad \theta_2(e) = \sin\alpha,$$

$$p(e)\theta_1'(e) = \sin\alpha, \qquad p(e)\theta_2'(e) = -\cos\alpha.$$

Thus θ_1 and θ_2 are linearly independent and

$$W[\theta_1, \theta_2] = p(\theta_1\theta_2' - \theta_1'\theta_2) = -1$$

for all x.

In addition to the differential expression

$$\ell y = \frac{(-py') + qy}{w}$$

and the differential equation $\ell y = \lambda y$, we impose a boundary condition at e:

$$\cos\alpha y(e) + \sin\alpha p(e)y'(e) = 0,$$

which is satisfied by θ_2.

We also impose a boundary condition at b':

$$\cos\beta y(b') + \sin\beta p(b')y'(b') = 0,$$

where β is also real, but otherwise arbitrary.

Thus the problem

$$(-py')' + qy = \lambda wy,$$

$$\cos\alpha y(e) + \sin\alpha p(e)y'(e) = 0,$$

$$\cos\beta y(b') + \sin\beta p(b')y'(b') = 0,$$

defines a regular selfadjoint boundary value problem over $[e, b']$. Solutions involve certain real values of λ only. λ cannot be complex and have a solution $y(x, \lambda)$ associated with it.

Weyl's approach to the singular problem begins by requiring $\text{Im}\lambda \neq 0$, and attempting to find a solution

$$\psi_{b'} = \theta_1 + m_{b'}\theta_2$$

of the differential equation that satisfies the b' boundary condition only. As noted, $\psi_{b'}$ and θ_2 must be linearly independent, otherwise $\psi_{b'}$ and/or θ_2 would be a solution satisfying both boundary conditions, and λ would be real.

Inserting $\psi_{b'}$ into the b' boundary condition, we find

$$\cos\beta(\theta_1 + m_{b'}\theta_2) + \sin\beta p(\theta_1' + m_{b'}\theta_2') = 0$$

at $x = b'$. Let $z = \cot\beta$. Then

$$z = -\frac{p(\theta_1' + m_{b'}\theta_2')}{(\theta_1 + m_{b'}\theta_2)}$$

is real. Setting $\mathrm{Im}(z) = 0$, we find

$$W[\psi_{b'}, \overline{\psi}_{b'}] = 0.$$

Expanded, this is

$$|m_{b'}|^2 W[\theta_2, \overline{\theta}_2] + m_{b'} W[\theta_2, \overline{\theta}_1] + \overline{m}_{b'} W[\theta_1, \overline{\theta}_2] + W[\theta_1, \overline{\theta}_1] = 0,$$

with $x = b'$, a circle in the complex m plane, which we call $C_{b'}$.

Let $m_{b'} = u + i\,v$, and let

$$2i\,A = W[\theta_2, \overline{\theta}_2],$$

$$B + iC = -W[\theta_2, \overline{\theta}_1],$$

$$B - iC = W[\theta_1, \overline{\theta}_2],$$

$$2iD = W[\theta_1, \overline{\theta}_1].$$

We find the circle equation is equivalent to

$$\left(u - \frac{C}{2A}\right)^2 + \left(v - \frac{B}{2A}\right)^2 = \frac{(B^2 + C^2 - 4AD)}{4A^2}.$$

The center of the circle $C_{b'}$ is

$$\widetilde{m}_{b'} = \frac{(C + iB)}{2A} = -\frac{W[\theta_1, \overline{\theta}_2]}{W[\theta_2, \overline{\theta}_2]}.$$

The radius is

$$r = |(B^2 + C^2 - 4AD)/4A^2|^{\frac{1}{2}}.$$

A bit of arithmetic shows

$$r = |W[\theta_2, \overline{\theta}_2]|^{-1}.$$

The interior of the circle $C_{b'}$ is also important. The "equal sign" should be replaced by "$\leq$" above. This translates into

$$\frac{W[\psi_{b'}, \overline{\psi}_{b'}]}{W[\theta_2, \overline{\theta}_2]} \leq 0.$$

Why do all this? Be patient.

We next compute the Wronskians to show how the circle $C_{b'}$ affects solutions of the differential equations. If y and z are solutions of

$$(-py')' + qy = \lambda wy,$$

then manipulation of the differential equations associated with y and z result in

$$W[y,\overline{z}](b') = W[y,z](e) + 2i\ \mathrm{Im}(\lambda) \int_e^{b'} w\, y\overline{z}\, dt.$$

Hence

$$W[\theta_2,\overline{\theta}_2](b') = 0 + 2i\,\mathrm{Im}(\lambda) \int_e^{b'} w|\theta_2|^2 dt = 2iA,$$

$$W[\theta_1,\overline{\theta}_2](b') = 1 + 2i\,\mathrm{Im}(\lambda) \int_e^{b'} w\theta_1\overline{\theta}_2 dt = B - iC,$$

$$W[\theta_1,\overline{\theta}_1](b') = 0 + 2i\,\mathrm{Im}(\lambda) \int_e^{b'} w|\theta_1|^2 dt = 2iD,$$

$$W[\psi_{b'},\overline{\psi}_{b'}](b') = m_{b'} - \overline{m}_{b'} + 2i\,\mathrm{Im}(\lambda) \int_e^{b'} w|\psi_{b'}|^2 dt.$$

Thus the center of the circle $C_{b'}$ is

$$\widetilde{m}_{b'} = \frac{1 - 2i\,\mathrm{Im}(\lambda) \int_e^{b'} w\theta_1\overline{\theta}_2 dt}{2i\,\mathrm{Im}(\lambda) \int_e^{b'} w|\theta_2|^2 dt}.$$

The radius is

$$r = |2i\,\mathrm{Im}(\lambda) \int_e^{b'} w|\theta_2|^2 dt|^{-1}.$$

Furthermore $m_{b'}$ inside or on the circle is equivalent to

$$\frac{(\mathrm{Im}(m_{b'})/\mathrm{Im}(\lambda)) + \int_e^{b'} w|\psi_{b'}|^2 dt}{\int_e^{b'} w|\theta_2|^2 dt} \leq 0.$$

This is equivalent to

$$\int_e^{b'} w|\psi_{b'}|^2 dt \leq -\frac{\mathrm{Im}(m_{b'})}{\mathrm{Im}(\lambda)}.$$

Finally, let $b'' < b' < b$, and let $m_{b'}$ be inside the circle $C_{b'}$. Then

$$\int_e^{b''} w|\psi_{b'}|^2 dt \leq \int_e^{b'} w|\psi_{b'}|^2 dt \leq -\frac{\mathrm{Im}(m_{b'})}{\mathrm{Im}(\lambda)}.$$

This shows that $m_{b'}$ is also in $C_{b''}$. The circles, therefore, are nested as $b' \to b$, and contract to a *limit point* or *limit circle*. If m_b is in their intersection and $\psi_b = \theta_1 + m_b\theta_2$, then

$$\int_e^{b} w|\psi_b|^2 dt \leq -\frac{\mathrm{Im}(m_b)}{\mathrm{Im}(\lambda)}.$$

So ψ_b is square integrable over $[e, b)$.

We call the situation where the limit is a circle, the *limit circle case*. Likewise if the limit is a point, we call this the *limit point case*.

An elementary argument involving variation of parameters and Schwarz's inequality shows that the case is independent of λ. That is, once in the limit circle case for a particular choice of λ, then always in the limit circle case for *all*, even real, λ. A proof of this is found in [8, p. 225].

If the limit is a circle C_b, then the limit radius r is greater than zero. This implies that θ_2 is also square integrable. Hence all solutions are square integrable, $\text{Im}(\lambda) \neq 0$.

If the limit is a point, then the limit radius r is zero. This only occurs if θ_2 is not square integrable. In this instance only ψ_b is square integrable, $\text{Im}(\lambda) \neq 0$.

We need to perform the same calculations at $x = a$. We impose at a', $a < a' < e$, the boundary condition

$$\cos\gamma y(a') + \sin\gamma p(a')y'(a') = 0,$$

where γ is real, and ask that

$$\psi_{a'} = \theta_1 + m_{a'}\theta_2$$

satisfy it. The only changes come when $\int_e^{a'}$ is replaced by $-\int_{a'}^e$ in the expressions that replace the Wronskians. Thus the center of the circle $C_{a'}$ is

$$\widetilde{m}_{a'} = \frac{1 + 2i\,\text{Im}(\lambda)\int_{a'}^e w\theta_1\overline{\theta}_2 dt}{2i\,\text{Im}(\lambda)\int_{a'}^e w|\theta_2|^2 dt},$$

and the radius is

$$r = \left|2i\,\text{Im}(\lambda)\int_{a'}^e w\right|\theta_2|^2 dt|^{-1}.$$

$m_{a'}$ inside the circle is equivalent to

$$\frac{-(\text{Im}(m_{a'})/\text{Im}(\lambda)) + \int_{a'}^e w|\psi_{a'}|^2 dt}{\int_{a'}^e w|\theta_2|^2 dt} \leq 0,$$

which is equivalent to

$$\int_{a'}^e w|\psi_{a'}|^2 dt \leq \frac{\text{Im}(m_{a'})}{\text{Im}(\lambda)}.$$

The contraction argument is unchanged, leading to nesting circles $C_{a'}$ as $a' \to a$. If m_a is in their intersection, then $\psi_a = \theta_1 + m_a\theta_2$ satisfies

$$\int_a^e w|\psi_a|^2 dt \leq \frac{\text{Im}(m_a)}{\text{Im}(\lambda)}.$$

So ψ_a is square integrable over $(a, e]$. Note that the sign of $\text{Im}(m_a)$ is the opposite of $\text{Im}(m_b)$. Thus m_a and m_b are not the same.

The other statements concerning limit circle and limit point cases and invariance with respect to λ still hold. Note that a regular end can be thought of as a limit circle case.

We shall use these L^2 solutions ψ_a and ψ_b as building blocks in what follows.

3. Singular boundary values. Regular boundary values at singular endpoints simply will not do. There are examples, for instance, where all solutions approach zero at the end. (Use Euler's equation.) Fixing the value of the dependent variable is futile in general.

Other constraints, such as boundedness, are likewise of little use in general. There are times, however, when such conditions are equivalent to what we propose here as a general answer. What is needed is something that generalizes regular boundary values, generates them at regular ends, *but still holds at singular ends*. The answer is to use Wronskian limits.

Let y be such that y and ℓy are in $L^2(e,b;w)$, and let $\ell y = f$. Let θ be a solution of

$$(-py')' + qy = \lambda wy,$$

which is in $L^2(e,b;w)$. Then

$$\int_e^{b'} [\theta(\ell y) - y(\ell\theta)]w\,dt = W[y,\theta](b') - W[y,\theta](e).$$

Since $\ell\theta = \lambda\theta$ is also square integrable, the left side has a limit as $b' \to b$. At e the Wronskian is fixed, and so the $\lim_{b'\to b} W[y,\theta](b')$ exists.

Further, at a regular end b, let θ be a solution with initial values

$$\theta(b) = -\beta, \qquad p(b)\theta'(b) = \alpha.$$

Then

$$\begin{aligned} W[y,\theta](b) &= y(b)p(b)\theta'(b) - p(b)y'(b)\theta(b) \\ &= \alpha y(b) + \beta p(b)y'(b), \end{aligned}$$

a general regular boundary value.

The converse to the argument above is also valid: If $\lim_{b'\to b} W[y,\theta](b')$ is finite for all y, then the solution θ is square integrable. Singular boundary values are generated by square integrable solutions of the homogeneous differential equation.

There is more to be said. In the limit point case all such Wronskian limits are of the form $\lim_{b'\to b} W[y,\psi_b](b')$. That is, no θ_2 Wronskians. Such limits are *always* zero, and so in the sense of Dunford and Schwartz are not only boundary values, but also annihilators.

In the limit circle case the boundary value $\lim_{b'\to b} W[y,\theta](b')$ exists for all solutions, even if λ is real. Zero is a very convenient choice of λ in this case.

In either limit point or limit circle case, if z and ℓz are in $L^2(e,b;w)$, even if z is not a solution to the homogeneous equation, then $\lim_{b'\to b} W[y,z](b')$ exists. In the limit point case it is always zero. In the limit circle case, it is a linear combination of *solution generated* boundary values, and so is not a new constraint.

Solution generated boundary values (with $\lambda = 0$ in the limit circle case) suffice.

As $x \to a$, the other singular end, we use Wronskian limits as $a' \to a$.

As stated in the introduction, Kaper, Kwong, and Zettl [33] have connected these Wronskian conditions with the classic ad hoc assumptions, used in many applications. Let b be the boundary point in question. It is assumed that for some point e between a and b, p^{-1} and q are integrable over (e,b), that $p > 0$, that

$$\int_e^t p^{-1}(s)ds = 0((t-b)^{-\gamma})$$

as $t \to b$, $0 < \gamma < \frac{1}{2}$, and finally that q is bounded on (e, b). Under these conditions the following are equivalent whenever y and ℓy are in $L^2(e, b)$.

(1) y is bounded on (e, b);
(2) $\lim_{t\to b} y(t)$ exists and is finite;
(3) $\lim_{t\to b}(t-b)^\gamma y(t) = 0$;
(4) $\lim_{t\to b}(py')(t) = 0$;
(5) $\lim_{t\to b}(t-b)^{-\alpha}(py')(t) = 0, \quad 0 < \alpha < \frac{1}{2}$;
(6) $p^{\frac{1}{2}} y'$ is in $L^2(e, b)$;
(7) $(t-b)^{-\alpha/2} p^{\frac{1}{2}} y'$ is in $L^2(e, b), 0 < \alpha < \frac{1}{2}$;
(8) y' is in $L^2(e, b)$.

The Wronskian condition is represented by 4.

$$\lim_{t\to b} py' = \lim_{t\to b} W[y, 1].$$

In virtually all the cases arising from applications, the function 1 is in $L^2(e, b)$.

4. The classical Weyl boundary value problem. In order to verify some of the statements of the preceding section, we turn our attention to a particularly useful boundary value problem, considered first by Weyl [64] in the same paper in which he developed the limit circle, limit point argument. We choose λ, fixed, with $\mathrm{Im}(\lambda) \neq 0$, and solve

$$(-py')' + qy - \lambda wy = wf,$$

$$\lim_{x\to b} W[y, \psi_b](x) = 0,$$

$$\lim_{x\to a} W[y, \psi_a](x) = 0.$$

Variation of parameters quickly yields

$$y = \psi_b \left[\int_e^x \frac{\psi_a f}{m_a - m_b} w d\xi + \beta \right] - \psi_a \left[\int_e^x \frac{\psi_b f}{m_a - m_b} w d\xi + \alpha \right].$$

Evaluation of the Wronskian limits yields

$$y = \psi_b \int_a^x \frac{\psi_a f}{m_a - m_b} w d\xi + \psi_a \int_x^b \frac{\psi_b f}{m_a - m_b} w d\xi$$

(see [51]).

If we define

$$G(\lambda, x, \xi) = \frac{\psi_b(x,\lambda)\psi_a(\xi,\lambda)}{m_a - m_b}, \qquad a < \xi < x < b,$$

$$= \frac{\psi_b(\xi,\lambda)\psi_a(x,\lambda)}{m_a - m_b}, \qquad a < x < \xi < b,$$

then

$$y = \int_a^b G(\lambda, x, \xi) f(\xi) w(\xi) d\xi.$$

Note that G satisfies

$$G(\lambda, x, \xi) = G(\lambda, \xi, x),$$
$$\overline{G(\lambda, x, \xi)} = G(\overline{\lambda}, x, \xi).$$

We are now in a position to show that the boundary value problem is selfadjoint in the setting $L^2(a, b; w)$. That is, we can show that the differential operator

$$Ly = \frac{(-py')' + qy}{w},$$

whose domain in $L^2(a, b; w)$ is restricted by the Wronskian boundary conditions, is self-adjoint.

Let $(L - \lambda)y = f$. Then

$$(L - \lambda)^{-1} f = y = \int_a^b G(\lambda, x, \xi) f(\xi) w(\xi) d\xi.$$

Therefore, if g is in the range of $(L-\lambda)^*$, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(a, b; w)$, we have

$$\begin{aligned} \langle (L - \lambda)^{-1} f, g \rangle &= \int_a^b \left[\int_a^b G(\lambda, x, \xi) f(\xi) w(\xi) d\xi \right] \overline{g}(x) w(x) dx \\ &= \int_a^b f(\xi) \left[\int_a^b G(\overline{\lambda}, x, \xi) g(x) w(x) dx \right]^* w(\xi) d\xi \\ &= \langle f, \overline{(L - \lambda)^{-1}} g \rangle. \end{aligned}$$

This says

$$\begin{aligned} \overline{(L - \lambda)}^{-1} &= (L - \lambda)^{-1*} \\ &= (L^* - \overline{\lambda})^{-1}. \end{aligned}$$

Taking inverses and cancelling the terms λ, we find $L = L^*$.

A simple argument shows that $(L - \lambda)^{-1}$ is a bounded operator. In fact,

$$\|(L - \lambda)^{-1}\| \leq \frac{1}{|\text{Im}\lambda|}.$$

For let $(L - \lambda)y = f$; then

$$\begin{aligned} \langle y, f \rangle - \langle f, y \rangle &= \langle y, (L - \lambda) y \rangle - \langle (L - \lambda) y, y \rangle \\ &= 2i \, \text{Im}\lambda \langle y, y \rangle. \end{aligned}$$

Apply Schwarz's inequality on the left:

$$2\|y\| \, \|f\| \geq 2|\text{Im}\lambda| \, \|y\|^2.$$

Cancel $2\|y\|$ and set $y = (L-\lambda)^{-1}f$ to find

$$\left(\frac{1}{|\mathrm{Im}\lambda|}\right)\|f\| \geq \|(L-\lambda)^{-1}f\|.$$

Divide by $\|f\|$ and take the supremum.

$$\|(L-\lambda)^{-1}\| \leq \frac{1}{|\mathrm{Im}\lambda|}$$

is the result.

The solution of the Weyl problem can be used in a number of ways. In particular it can be used to resolve the difficulty of the λ dependence of the boundary conditions. It can also be used to show that if the limit point case holds at a singular end, then the boundary condition is *automatic*, satisfied by all.

First, let y and z satisfy $Ly = f,\ Lz = g$ as well as the boundary conditions at a and b. Then Green's formula shows

$$\begin{aligned} 0 &= \langle Ly, z\rangle - \langle y, Lz\rangle,\\ &= \int_a^b [((-py')' + qy)\bar{z} - y((-p\bar{z}')' + q\bar{z})]dt,\\ &= W[y,\bar{z}]_a^b. \end{aligned}$$

If y and z are modified so they vanish near a or b, then at the other end $W[y,\bar{z}] = 0$. In summary: If $\lim_{x\to a} W[y,\psi_a] = 0$ and $\lim_{x\to a} W[z,\psi_a] = 0$, then $\lim_{x\to a} W[y,\bar{z}] = 0$. If $\lim_{x\to b} W[y,\psi_b] = 0$ and $\lim_{x\to b} W[z,\psi_b] = 0$, then $\lim_{x\to b} W[y,\bar{z}] = 0$.

Second, there is a lemma due to Titchmarsh [62, p. 26], which says that if $\psi_a(x,\lambda)$ and $\psi_a(x,\lambda_0)$ are defined by the same sequence of a' boundary conditions, then

$$\lim_{x\to a} W[\psi_a(x,\lambda_0), \psi_a(x,\lambda)] = 0$$

so long as Im $\lambda_0 \neq 0$, Im $\lambda \neq 0$. Likewise, if $\psi_b(x,\lambda)$ and $\psi_b(x,\lambda_0)$ are defined by the same sequence of b' boundary conditions, then

$$\lim_{x\to b} W[\psi_b(x,\lambda_0),\ \psi_b(x,\lambda)] = 0,$$

so long as Im $\lambda_0 \neq 0$, Im $\lambda \neq 0$.

In other words, $\psi_a(x,\lambda_0)$, modified to be zero near b, satisfies both Wronskian boundary conditions. $\psi_b(x,\lambda_0)$, modified to be zero near a satisfies both boundary conditions.

If the second statements are used in the first, we conclude that if $\lim_{x\to a} W[y,\psi_a] = 0$ or $\lim_{x\to b} W[y,\psi_b] = 0$ for a specific fixed λ, then $\lim_{x\to a}[y,\psi_a] = 0$ or $\lim_{x\to b} W[y,\psi_b] = 0$ for all λ, Im $\lambda \neq 0$.

This implies that, regardless of the case, the Green's function $G(\lambda, x, \xi)$ is given by the same formula for *all* λ, Im $\lambda \neq 0$.

The independence of the boundary conditions on λ also permits us to replace z by $\bar{z}$ in the first statement to conclude that if both y and z satisfy the boundary conditions, then $\lim_{x\to a} W[y,z] = 0$ and $\lim_{x\to b} W[y,z] = 0$.

Finally, let us suppose that a is in the limit point case. Then if we solve the differential equation

$$(\ell - \lambda)y = f,$$

requiring only that y be in $L^2(a,b;w)$, we find

$$y = \int_a^b G(\lambda, x, \xi) f(\xi) w(\xi) d\xi + \alpha\psi_a.$$

There is no term $\beta\psi_b$. If we now compute $\lim_{x\to a} W[y, \psi_a]$, we find it is zero.

Likewise if b is in the limit point case, then $\lim_{x\to b} W[y, \psi_b] = 0$.

In these limit point cases, boundary conditions are automatically satisfied, so $\lim_{x\to a} W[y,z] = 0$ or $\lim_{x\to b} W[y,z] = 0$ for all y and z for which $\ell y = f$ and $\ell z = g$ are also in $L^2(a,b;w)$.

5. Green's formulas. The regular Green's formula, which involves the Lagrange bilinear concomitant, evaluated at a and b, must be replaced by four variations, one for each of the following cases:

(1) Limit point at a, limit point at b;
(2) Limit circle at a, limit point at b;
(3) Limit point at a, limit circle at b;
(4) Limit circle at a, limit circle at b.

We examine each case in turn.

5.1. LP at a, LP at b. Green's formula over $(a', b') \subset (a,b)$ has the form

$$\int_{a'}^{b'} [\overline{z}(\ell y) - (\overline{\ell z}) y] w\, dt = W[y, \overline{z}]_{a'}^{b'}.$$

In the previous section, under limit point conditions, we showed the right side vanishes as $a' \to a$, $b' \to b$. Thus

$$\int_a^b [z(\ell y) - (\overline{\ell z}) y] w\, dt = 0.$$

5.2. LC at a, LP at b. The difference here is that $\lim_{x\to a} W[y, \overline{z}]$ is not always zero, and in addition at $x = a$, both θ_1 and θ_2 are in $L^2(a,e;w)$ for all λ. We choose $\lambda = 0$, and assume θ_1 and θ_2 are real valued. The problem is to express $\lim_{x\to a} W[y, \overline{z}]$.

We observe the following. First,

$$W[y, \overline{z}] = (\overline{z},\ p\overline{z}') \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ py' \end{pmatrix}.$$

Second,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = - \begin{pmatrix} p\theta_1' & p\theta_2' \\ -\theta_1 & -\theta_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p\theta_1' & -\theta_1 \\ p\theta_2' & -\theta_2 \end{pmatrix},$$

since $W[\theta_1, \theta_2] = -1$. If the second is inserted in the middle of the first, we find

$$-W(y, \overline{z}) = (W\overline{[z, \theta_1]}, W\overline{[z, \theta_2]}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W[y, \theta_1] \\ W[y, \theta_2] \end{pmatrix}.$$

The importance of this substitution is that while originally z, pz', y, py' may not have individual limits at $x = a$, the Wronskians do. Hence Green's formula becomes

$$\int_a^b [\overline{z}(\ell y) - (\overline{\ell z})y]w\,dt$$
$$= \left(\lim_{x\to a} \overline{W[z,\theta_1]}, \lim_{x\to a} \overline{W[z,\theta_2]}\right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lim_{x\to a} W[y,\theta_1] \\ \lim_{x\to a} W[y,\theta_2] \end{pmatrix}.$$

5.3. LP at *a*, LC at *b*. This is virtually the same as the previous case. On the right b replaces a, and the sign changes.

$$\int_a^b [\overline{z}(\ell y) - (\overline{\ell z})y]w\,dt$$
$$= -\left(\lim_{x\to b} \overline{W[z,\theta_1]}, \lim_{x\to b} \overline{W[z,\theta_2]}\right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lim_{x\to b} W[y,\theta_1] \\ \lim_{x\to b} W[y,\theta_2] \end{pmatrix}.$$

5.4. LC at *a*, LC at *b*. This case is the "sum" of the two previous cases on the right side. Within it as a special case is the regular Green's formula. We shorten notation a bit. Let

$$B_a(y) = \begin{pmatrix} \lim_{x\to a} W[y,\theta_1] \\ \lim_{x\to a} W[y,\theta_2] \end{pmatrix}, \qquad B_b(y) = \begin{pmatrix} \lim_{x\to b} W[y,\theta_1] \\ \lim_{x\to b} W[y,\theta_2] \end{pmatrix},$$

and let $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then

$$\int_a^b [\overline{z}(\ell y) - (\overline{\ell z})y]w\,dt = (B_a(z)^*,\ B_b(z)^*) \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \begin{pmatrix} B_a(y) \\ B_b(y) \end{pmatrix},$$

where $*$ denotes conjugate transpose.

We shall use these formulas in the next sections when discussing both the maximal and minimal operators, as well as those selfadjoint operators lying in between them.

6. Maximal and minimal operators. Maximal and minimal operators are important because selfadjoint operators lie in between. That is, if L_M is the maximal operator and L_m is the minimal operator, then examining the domains we would find $L_m \subset L \subset L_M$ and $L_m \subset L^* \subset L_M$ for a large collection of operators L and L^* having the *same* form as L_M and L_m. In other words, the domains of these operators obey the set inclusion relations

$$D_m \subset D_L \subset D_M, \qquad D_m \subset D_{L^*} \subset D_M.$$

The trick is to rig D_L so $D_L = D_{L^*}$. Then L is selfadjoint.

The domain D_M consists of those elements y in $L^2(a, b; w)$ for which ℓy exists almost everywhere and is also in $L^2(a, b; w)$. It is the largest possible associated with ℓ.

By general agreement, the minimal operator L_m is the Hilbert space adjoint of L_M. $L_M^* = L_m$. There are two problems. First, what is its form? Second, what is its domain?

The first is answered quickly using a trick from the calculus of variations. Called the fundamental lemma, it shows that the form of L_m is the same as the form of L_M. In both cases $L_m y = \ell y,\ L_M y = \ell y$ [24, p. 131]. The problem lies in characterizing the domain D_m.

Green's formula unlocks the domain problem. In case 1 (LP-LP), Green's formula shows that if y is in D_M and z is in D_m, then

$$\int_a^b \overline{z}(L_M y) w\, dt = \int_a^b (\overline{L_m z}) y\, w\, dt,$$

and so no further constraint is put on D_m. Hence $D_M = D_m$, and the maximal operator is itself selfadjoint. We already knew this.

In case 2 (LC-LP), with y in D_M and z in D_m, we find

$$\int_a^b \overline{z}(L_M y) w\, dt - \int_a^b (\overline{L_m z}) y\, w\, dt$$
$$= \left(\lim_{x\to a} \overline{W[z,\theta_1]}, \lim_{x\to a} \overline{W[z,\theta_2]}\right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lim_{x\to a} W[y,\theta_1] \\ \lim_{x\to a} W[y,\theta_2] \end{pmatrix} = 0.$$

Since no constraints are placed on the y terms, we are forced to conclude that

$$\lim_{x\to a} W[z,\theta_1] = 0, \qquad \lim_{x\to a} W[z,\theta_2] = 0.$$

In case 3 (LP-LC) we conclude that

$$\lim_{x\to b} W[z,\theta_1] = 0, \qquad \lim_{x\to b} W[z,\theta_2] = 0,$$

In case 4, we conclude that Wronskians of z with θ_1 and θ_2 vanish at both a and b.

These constraints are *both* necessary and sufficient, and so complete the characterization of the domain of the minimal operator.

We now turn our attention to those operators L satisfying $L_m \subset L \subset L_M$ and $L_m \subset L^* \subset L_M$, and, in particular, those for which $L = L^*$.

7. Selfadjoint operators. We noted in the previous section that when both a and b are in the limit point case, the maximal operator L_M and the minimal operator L_m agree with the Weyl operator. The result is selfadjoint.

Still left to be considered are the cases (LC-LP), (LP-LC), and (LC-LC). The similarity of cases (LC-LP) and (LP-LC) will again allow us to get a lot for free.

7.1. Boundary value problems in the case (LC-LP). Green's formula for elements y and z in D_M is

$$\int_a^b \overline{z}(L_M y) w\, dt - \int_a^b (\overline{L_M z}) y\, w\, dt$$
$$= (\lim_{x\to a} \overline{W[z,\theta_1]}, \lim_{x\to a} \overline{W[z,\theta_2]}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lim_{x\to a} W[y,\theta_1] \\ \lim_{x\to a} W[y,\theta_2] \end{pmatrix}.$$

Operators L lying between L_m and L_M are determined by having their domains restricted by boundary conditions of the form

$$\alpha \lim_{x\to a} W[y,\theta_1] + \beta \lim_{x\to a} W[y,\theta_2] = 0,$$

$|\alpha| + |\beta| \neq 0$. The adjoint L^* is also such an operator. We shall show how to find it.

We choose γ and δ so that $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is nonsingular, and then $\epsilon, \zeta, \eta, \kappa$ so

$$\begin{pmatrix} \epsilon^* & \eta^* \\ \zeta^* & \kappa^* \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If this is inserted into the right side of Green's formula, then

$$\int_a^b \overline{z}(L_M y) w\, dt - \int_a^b (\overline{L_M z}) y\, w\, dt$$
$$= \left(\epsilon \lim_{x\to a} W[z,\theta_1] + \zeta \lim_{x\to a} W[z,\theta_2]\right)^* \left(\alpha \lim_{x\to a} W[y,\theta_1] + \beta \lim_{x\to a} W[y,\theta_2]\right)$$
$$+ \left(\eta \lim_{x\to a} W[z,\theta_1] + \kappa \lim_{x\to a} W[z,\theta_2]\right)^* \left(\gamma \lim_{x\to a} W[y,\theta_1] + \delta \lim_{x\to a} W[y,\theta_2]\right).$$

Now require y to be in D_L, determined by the (α,β) boundary condition, and require z to be in D_{L^*}. The left side of the formula is zero. The first term on the right is zero. Since the (γ,δ) boundary value in y is arbitrary, we conclude that

$$\eta \lim_{x\to a} W[z,\theta_1] + \kappa \lim_{x\to a} W[z,\theta_2] = 0.$$

This is a necessary and sufficient condition for determining the domain D_{L^*}. L^* has the same form as L. Its domain is determined by a different boundary condition.

In order to determine when selfadjointness occurs, it is convenient to introduce parametric boundary conditions [51]. We have

$$\eta \lim_{x\to a} W[z,\theta_1] + \kappa W[z,\theta_2] = 0.$$

The other constraint

$$\epsilon \lim_{x\to a} W[z,\theta_1] + \zeta \lim_{x\to a} W[z,\theta_2] = \Delta,$$

where Δ is arbitrary. This is equivalent to

$$\left(\overline{\lim_{x\to a} W[z,\theta_1]}, \overline{\lim_{x\to a} W[z,\theta_2]}\right) \begin{pmatrix} \epsilon^* & \eta^* \\ \zeta^* & \kappa^* \end{pmatrix} = (\Delta^*, 0).$$

Right muliplication by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ yields

$$\lim_{x\to a} W[z,\theta_1] = -\beta^*\Delta, \qquad \lim_{x\to a} W[z,\theta_1] = \alpha^*\Delta,$$

parametric boundary conditions. If z is in D_L, then it satisfies the (α,β) boundary condition. Using the parametric forms, we find that

$$(-\alpha\beta^* + \beta\alpha^*)\Delta = 0.$$

Since Δ is arbitrary,

$$\alpha\beta^* = \beta\alpha^*.$$

This is equivalent to making α and β real. The condition is reversible, and so

$$\alpha\beta^* = \beta\alpha^*$$

is necessary and sufficient.

7.2. Boundary value problems in the case (LP-LC). Here the only changes are a minus sign on the right side of Green's formula and the replacement of $x \to a$ by $x \to b$. The results are the same. The boundary condition

$$\alpha \lim_{x\to b} W[y, \theta_1] + \beta \lim_{x\to b} W[y, \theta_2] = 0$$

is selfadjoint if and only if $\alpha\,\beta^* = \beta\,\alpha^*$.

7.3. Boundary value problems in the case (LC-LC). Green's formula, given in §5.4, is

$$\int_a^b \overline{z}(L_M y) w\, dt - \int_a^b (\overline{L_M z}) y\, w\, dt = (B_a(z)^*, B_b(z)^*) \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \begin{pmatrix} B_a(y) \\ B_b(y) \end{pmatrix}.$$

We exploit again the idea of making an insertion, in this case for $\left(\begin{smallmatrix} J & 0 \\ 0 & -J \end{smallmatrix}\right)$, find the adjoint and parametric adjoint conditions, then examine selfadjointness.

Let A an B be $m \times 2$ matrices, $0 \le m \le 4$, and let C and D be $(4-m) \times 2$ matrices such that $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is nonsingular. We further let $\tilde{A}$ and $\tilde{B}$ be $m \times 2$ matrices, and let $\tilde{C}$ and $\tilde{D}$ be $(4-m) \times 2$ matrices such that

$$\begin{pmatrix} \tilde{A}^* & \tilde{C}^* \\ \tilde{B}^* & \tilde{D}^* \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}.$$

Substituting for $\left(\begin{smallmatrix} J & 0 \\ 0 & -J \end{smallmatrix}\right)$ in Green's formula we find

$$\begin{aligned} &\int_a^b \overline{z}(L_M y) w\, dt - \int_a^b (\overline{L_M z}) y\, w\, dt \\ &= (\tilde{A}\, B_a(z) + \tilde{B}\, B_b(z))^* (A\, B_a(y) + B\, B_b(y)) \\ &\quad + (\tilde{C}\, B_a(z) + \tilde{D}\, B_b(z))^* (C\, B_a(y) + D\, B_b(y)). \end{aligned}$$

If L is a restriction of L_M with domain restricted by the boundary condition

$$A\, B_a(y) + B\, B_b(y) = 0,$$

then L^* has a domain restricted by

$$\tilde{C}\, B_a(z) + \tilde{D}\, B_b(z) = 0.$$

If $\tilde{A}\, B_a(z) + \tilde{B}\, B_b(z) = \Delta$, arbitrary, then

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \begin{pmatrix} B_a(z) \\ B_b(z) \end{pmatrix} = \begin{pmatrix} \Delta \\ 0 \end{pmatrix}.$$

This can be solved by yielding the parametric boundary conditions

$$B_a(z) = J\, A^* \Delta, \qquad B_b(z) = -J\, B^* \Delta.$$

If L is selfadjoint, then z satisfies the (A, B) boundary condition. Hence

$$A\,J\,A^* = B\,J\,B^*,$$

again a necessary and sufficient condition for L to be selfadjoint.

Note that regular boundary conditions are subsumed within this case. The condition for selfadjointness is well known in the regular case.

In summary, the operator

$$Ly = \frac{((-py')' + qy)}{w},$$

whose domain is restricted by

(1) In the case (LP-LP), nothing;

(2),(3) In the case (LC-LP) or (LP-LC) by

$$\alpha \lim W[y, \theta_1] + \beta \lim W[y, \theta_2] = 0, \qquad \alpha\beta^* = \beta\alpha^*;$$

(4) In the case (LC-LC) by

$$A\,B_a(y) + B\,B_b(y) = 0,\ A\,J\,A^* = B\,J\,B^*,$$

is selfadjoint in $L^2(a, b; w)$.

8. Examples. We limit ourselves to one in each case and refer to [51] and [52] for further examples.

8.1. The Hermite operator. The Hermite operator is given by

$$Ly = \frac{(-e^{-x^2}y')'}{e^{-x^2}},$$

set in $L^2(-\infty, \infty; e^{-x^2})$. It is in the limit point case at both $\pm\infty$, and so is selfadjoint without the need to specify boundary conditions.

8.2. The Laguerre operator. The Laguerre operator is given by

$$Ly = \frac{(-x\,e^{-x}y')'}{e^{-x}},$$

set in $L^2(0, \infty; e^{-x})$. It is limit point at ∞, and no boundary condition needs to be specified there. It is limit circle at zero. Two solutions to $(x\,e^{-x}y')' = 0$ are $\theta_1 = 1$, and $\theta_2 = -\int_1^x (e^{\xi}/\xi)d\xi$. (In §2 let $\alpha = 0$, $e = 1$.) A boundary condition at $x = 0$ takes the form

$$\alpha \lim_{x\to b} W[y, \theta_1] + \beta \lim_{x\to b} W[y, \theta_2] = 0.$$

It is selfadjoint if α and β are real. The Laguerre polynomials are found to be eigenfunctions of L if $\alpha = 1, \beta = 0$. The boundary condition in this case is

$$-\lim_{x\to 0} x\,e^{-x}y'(x) = 0.$$

8.3. The Bessel operator. Let the interval in question be $(0,1]$. The Bessel operator (of "order n") is given by

$$Ly = \frac{(-xy')' + (n^2/x)y}{x},$$

set in $L^2(0,1;x)$. We can let the "interior point" e be at $e=1$, since it is a regular point.

Solutions of $Ly = 0$ are $\theta_1 = \frac{1}{2}(x^n + x^{-n})$ and $\theta_2 = \frac{1}{n}(x^n - x^{-n})$ if $n \neq 0$. If $n = 0$, $\theta_1 = 1$ and $\theta_2 = \log x$ are solutions.

If $|n| \geq 1$, $x = 0$ is in the limit point case. No boundary condition is required. At $x = 1$, $W[y,\theta_1] = -y'(1)$, $W[y,\theta_2] = y(1)$, and so the boundary condition is

$$-\alpha y'(1) + \beta y(1) = 0.$$

If $|n| \leq 1$, both zero and one are in the limit circle case. The general boundary condition

$$A\, B_0(y) + B\, B_1(y) = 0$$

is appropriate. Bessel functions of the first kind satisfy separated boundary conditions of the form

$$\lim_{x\to 0} x(n\, x^{n-1} y(x) - x^n y'(x)) = 0, \qquad n \neq 0,$$

or

$$\lim_{x\to 0}(-x\, y'(x)) = 0, \qquad n = 0,$$

and at $x = 1$,

$$-\alpha\, y(1) + \beta\, y'(1) = 0.$$

8.4. The Legendre operator. Set in $L^2(-1,1;1)$, the Legendre operator is given by

$$Ly = (-(1-x^2)y')'.$$

The limit circle case holds at both ± 1. If the interior point $e = 0$, the solutions $\theta_1 = 1$ and $\theta_2 = \frac{1}{2}\log((1-x)/(1+x))$ generate the appropriate Wronskian boundary values as $x \to \pm 1$. In general a boundary condition would involve terms from both ± 1. We content ourselves here with noting that the Legendre polynomials satisfy

$$\lim_{x\to -1} -(1-x^2)y'(x) = 0, \qquad \lim_{x\to 1} -(1-x^2)y'(x) = 0.$$

9. Remarks. It is possible to extend these results to higher-order equations. We cite [13], [39], [40], and [61]. We note, however, that for higher-order problems the results are substantially more complicated.

The spectral resolutions or eigenfunction expansions are also substantially more involved than in the regular case. In general neither a sum (Fourier series) nor an integral (Fourier transform) is sufficient, but instead integration with respect to a measure of bounded variation is required [8]. For the specific examples mentioned, there are other more direct methods that are more efficient [7], [9], [34].

We hope the reader has found this description of singular boundary value problems illuminating. It is a fascinating subject, truly worthy of serious examination.

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