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CHARACTERIZATIONS OF THE FRIEDRICHS EXTENSIONS OF SINGULAR STURM–LIOUVILLE EXPRESSIONS*

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Abstract. A method is presented to characterize selfadjoint realizations of a singular Sturm–Liouville differential expression on a finite interval, where the singularities are of limit-circle type.

Key words. Sturm–Liouville differential operators, singularities of limit-circle type, selfadjoint realizations, Friedrichs extension

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1. Introduction. In this article we present a new method for defining selfadjoint realizations of a certain class of singular Sturm–Liouville differential expressions

$$\tau = -\frac{d}{dt}p(t)\frac{d}{dt}+q(t) \tag{1}$$

on a finite interval (a,b). We assume throughout that p and q are measurable and real-valued functions on (a,b) which satisfy the minimal conditions

$$p^{-1}, q \in L^1_{\text{loc}}(a,b). \tag{2}$$

Moreover, we assume that p is positive,

$$p(t)>0 \quad \text{a.e. on } (a,b). \tag{3}$$

Thus, τ is a quasi-differential expression in the sense of Naimark [1, §V.1]. A function $\bar{y}$ is said to be a solution of the equation $\tau y=0$ if (i) $\bar{y}$ is absolutely continuous on (a,b), (ii) $p\bar{y}'$ is equal a.e. on (a,b) to an absolutely continuous function (which, with a slight abuse of notation, we denote by the symbol $p\bar{y}'$), and (iii) the identity $-(p\bar{y}')'(t)+q(t)\bar{y}(t)=0$ holds a.e. on (a,b).

The right endpoint b is said to be a regular endpoint for τ if

$$p^{-1}, q \in L^1(c,b) \quad \text{for some } c\in(a,b). \tag{4}$$

Similarly, the left endpoint a is regular if

$$p^{-1}, q \in L^1(a,c) \quad \text{for some } c\in(a,b). \tag{5}$$

If both endpoints are regular, then the differential expression τ is called regular; otherwise, it is called singular. Note that, for τ to be regular, neither p^{-1} nor q need to be bounded on (a,b).

All solutions $\bar{y}$ of a regular Sturm–Liouville equation $\tau y=0$ are continuous on $[a,b]$, and the same property holds for the function $p\bar{y}'$. Hence, boundary values can be assigned to these functions. The characterization of those boundary conditions which

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give rise to selfadjoint realizations of a regular differential expression in the Hilbert space $L^2(a,b)$ is well known and can be found, for example, in the monographs by Akhiezer and Glazman [2, Appendix II] and Naimark [1, §5.18].

The study of singular differential expressions is considerably more difficult. The solutions of a singular Sturm–Liouville equation $\tau y=0$ generally exhibit singularities near the endpoints, so one cannot assign boundary values there. Weyl [3] has developed a theory for the construction of selfadjoint realizations of singular differential expressions, which is based on a distinction between singularities of limit-circle type and those of limit-point type. The characterizations are, however, not concrete and therefore difficult to apply. The same remarks can be made for the theory developed by Titchmarsh [4].

In this article we present a new method for characterizing selfadjoint realizations of singular Sturm–Liouville differential expressions τ of the form (1), where q is bounded and the singularity at either endpoint is of limit-circle type. We limit the discussion to the case of one singular endpoint; the extension of the method to cases where both endpoints are singular is straightforward. Specifically, we assume that the coefficients p and q satisfy, in addition to (2), (3), and (4), the conditions

$$\int_t^b p^{-1}(s)\,ds = O((t-a)^{-\gamma}) \quad \text{as } t\downarrow a, \quad \gamma\in\left(0,\frac{1}{2}\right), \tag{6}$$

$$q\in L^\infty(a,b). \tag{7}$$

Thus, b is a regular endpoint and a is a singular endpoint for τ. For bounded potentials q, the condition (6) is both necessary and sufficient for the singularity at a to be of limit-circle type.

A selfadjoint realization of τ in $L^2(a,b)$ requires the specification of two boundary conditions, one at the regular endpoint b and one at the singular endpoint a. At b we impose a condition of the usual type,

$$B_1 y(b) + B_2(py')(b) = 0, \qquad B_1^2 + B_2^2 \neq 0. \tag{8}$$

Given such a condition, there are an infinite number of conditions at a which give rise to a selfadjoint realization of τ in $L^2(a,b)$. The particular condition

$$\lim_{t\downarrow a} y(t) \text{ exists and is finite} \tag{9}$$

is known to generate a selfadjoint realization which coincides with the Friedrichs extension of the minimal operator in $L^2(a,b)$ associated with τ. Although the condition (9) is often referred to as the "natural" one, relating it to the Weyl or Titchmarsh theory of singular Sturm–Liouville problems is nontrivial.

As we will demonstrate, (9) is but one of several equivalent characterizations of the same selfadjoint realization of τ. These characterizations follow in a systematic way from a particular representation of the elements in the domain of the maximal operator defined by τ. The procedure sheds some light on the role that the particular condition (9) plays within the general framework of Weyl's theory.

2. Characterization of the Friedrichs extension. Let $\phi:(a,b)\to\mathbb{R}$ be defined by the expression

$$\phi(t) = 1 + \int_t^b p^{-1}(s)\,ds, \qquad t\in(a,b). \tag{10}$$

Then $\phi\in L^2(a,b)$, because (a,b) is finite and p satisfies (6). Furthermore, $\phi(t)\geqq 1$ for all $t\in(a,b)$ and $\phi'(t)=-p^{-1}(t)$ a.e. on (a,b).

Let M be the maximal operator associated with τ,

$$My=\tau y,\qquad y\in\operatorname{dom}M, \tag{11}$$

where $\operatorname{dom}M=\{y\in L^2(a,b): y$ and py' locally absolutely continuous on (a,b) and $\tau y\in L^2(a,b)\}$. The following lemma gives a representation of the elements of $\operatorname{dom}M$.

LEMMA 1. *For every $y\in\operatorname{dom}M$ there exist two constants c and d and an element $g\in L^2(a,b)$, such that*

$$y(t)=c\phi(t)+d+\int_a^t(\phi(t)-\phi(s))g(s)\,ds,\qquad t\in(a,b), \tag{12}$$

$$y'(t)=-cp^{-1}(t)-p^{-1}(t)\int_a^t g(s)\,ds,\qquad t\in(a,b). \tag{13}$$

Proof. Because q is bounded, $\operatorname{dom}M$ consists of those $y\in L^2(a,b)$ for which y and py' are locally absolutely continuous on (a,b) and $(py')'\in L^2(a,b)$. Hence, for every $y\in\operatorname{dom}M$ there exists a $g\in L^2(a,b)$ such that $-(py')'=g$. Integration of this identity gives the representations (12) and (13). □

Selfadjoint realizations T of τ are obtained by restricting M. The restrictions result in constraints on the element g and the constants c and d in the representation (12). The boundary condition (8) imposes one such constraint, viz.,

$$(B_1-B_2)c+B_1d=\int_a^b(B_1-B_2-B_1\phi(s))g(s)\,ds. \tag{14}$$

Another constraint is obtained by imposing a "boundary condition" at the singular endpoint. For example, the condition (9) leads to the constraint $c=0$. The following lemma explores the ramifications of this constraint.

LEMMA 2. *Let $y\in\operatorname{dom}M$. Then the following conditions are equivalent:*

(i) *y has a representation of the form* (12) *with* $c=0$;
(ii) *y is bounded on* (a,b);
(iii) $\lim_{t\downarrow a}y(t)$ *exists and is finite*;
(iv) $\lim_{t\downarrow a}(t-a)^\gamma y(t)=0$;
(v) $\lim_{t\downarrow a}(py')(t)=0$;
(vi) $\lim_{t\downarrow a}(t-a)^{-\alpha}(py')(t)=0$ *for any* $\alpha\in(0,\frac{1}{2})$;
(vii) $p^{1/2}y'\in L^2(a,b)$;
(viii) $(t-a)^{-\alpha/2}p^{1/2}y'\in L^2(a,b)$ *for any* $\alpha\in(0,\frac{1}{2})$;
(ix) $y'\in L^1(a,b)$.

Proof. (i)⇔(ii). Elementary estimates yield the inequalities

$$\left|\int_a^t(\phi(t)-\phi(s))g(s)\,ds\right|\leqq\phi(t)\left|\int_a^t g(s)\,ds\right|+\left|\int_a^t\phi(s)g(s)\,ds\right| \tag{15}$$
$$\leqq\left[\phi(t)(t-a)^{1/2}+\|\phi\|\right]\|g\|.$$

Because of (6), $\phi(t)(t-a)^{1/2}$ tends to zero as $t\downarrow a$, so there exists a positive constant C such that, for any $g\in L^2(a,b)$,

$$\left|\int_a^t(\phi(t)-\phi(s))g(s)\,ds\right|\leqq C\|g\|,\qquad t\in(a,b). \tag{16}$$

Every $y \in \operatorname{dom} M$ has a representation of the form (12), where the integral obeys the inequality (16). Clearly, y is bounded on (a,b) if and only if $c=0$.

(i)$\Leftrightarrow$(iii). A more careful estimate of the second term in (15) yields the inequality

$$\left|\int_a^t (\phi(t)-\phi(s))g(s)\,ds\right| \leqq \left[\phi(t)(t-a)^{1/2}+\left(\int_a^t \phi^2(s)\,ds\right)^{1/2}\right]\|g\|. \tag{17}$$

Because of (6), there exists a positive constant C such that $\phi(s) \leqq C(s-a)^{-\gamma}$ for s sufficiently close to a. Thus we find that, for any $g \in L^2(a,b)$, we have the more refined estimate

$$\left|\int_a^t (\phi(t)-\phi(s))g(s)\,ds\right| \leqq \psi(t)\|g\|, \qquad t \in (a,b), \tag{18}$$

where ψ is independent of g, ψ is bounded on (a,b) and $\psi(t)=O((t-a)^{1/2-\gamma})$ as $t \downarrow a$. Every $y \in \operatorname{dom} M$ has a representation of the form (12), where the integral tends to zero like $(t-a)^{1/2-\gamma}$ as $t \downarrow a$. Clearly, $y(t)$ tends to a finite limit as $t \downarrow a$ if and only if $c=0$.

(i)$\Leftrightarrow$(iv). The proof is similar to the proof of the previous equivalence.

(i)$\Leftrightarrow$(v). For any $g \in L^2(a,b)$ we have

$$\left|\int_a^t g(s)\,ds\right| \leqq (t-a)^{1/2}\|g\|, \qquad t \in (a,b). \tag{19}$$

Representing y as in (12), so py' is given by (13), we see that $(py')(t)$ tends to zero as $t \downarrow a$ if and only if $c=0$.

(i)$\Leftrightarrow$(vi). The equivalence follows immediately from the proof of the previous equivalence.

(i)$\Leftrightarrow$(vii). According to Lemma 1, we have for any $y \in \operatorname{dom} M$,

$$p^{1/2}(t)y'(t) = -cp^{-1/2}(t) - p^{-1/2}(t)\int_a^t g(s)\,ds, \qquad t \in (a,b), \tag{20}$$

for some constant c and some $g \in L^2(a,b)$. Now, using (19),

$$\int_a^b p^{-1}(t)\left|\int_a^t g(s)\,ds\right|^2 dt \leqq \|g\|^2 \int_a^b p^{-1}(t)(t-a)\,dt.$$

The last integral is bounded:

$$\int_a^b p^{-1}(t)(t-a)\,dt = -\int_a^b \phi'(t)(t-a)\,dt = -(b-a)+\int_a^b \phi(t)\,dt = C,$$

so

$$\int_a^b p^{-1}(t)\left|\int_a^t g(s)\,ds\right|^2 dt \leqq C\|g\|^2.$$

The second term in the right member of (20) defines therefore an element of $L^2(a,b)$. The first term, on the other hand, does not, unless $c=0$. Consequently, $p^{1/2}y' \in L^2(a,b)$ if and only if $c=0$.

(i)$\Leftrightarrow$(viii). We use the representation (20) of $p^{1/2}y'$ and observe that

$$\int_a^b (t-a)^{-\alpha}p^{-1}(t)\left|\int_a^t g(s)\,ds\right|^2 dt \leqq \|g\|^2 \int_a^b p^{-1}(t)(t-a)^{1-\alpha}dt.$$

The last integral is still bounded if $\alpha\in(0,\frac{1}{2})$, so

$$\int_a^b (t-a)^{-\alpha} p^{-1}(t)\left|\int_a^t g(s)\,ds\right|^2 dt \leqq C\|g\|^2.$$

The expression $(t-a)^{-\alpha/2}p^{-1/2}(t)\int_a^t g(s)\,ds$ defines therefore an element of $L^2(a,b)$ as long as $\alpha\in(0,\frac{1}{2})$. On the other hand, the expression $(t-a)^{-\alpha/2}p^{-1/2}(t)$ clearly does not define an element of $L^2(a,b)$ if $\alpha\in(0,\frac{1}{2})$, so the function $t\mapsto(t-a)^{-\alpha/2}p^{1/2}(t)y'(t)$ with $\alpha\in(0,\frac{1}{2})$ belongs to $L^2(a,b)$ if and only if $c=0$.

(i)$\Leftrightarrow$(ix). According to Lemma 1 we have, for any $y\in\operatorname{dom} M$,

(21) $$y'(t) = -cp^{-1}(t) - p^{-1}(t)\int_a^t g(s)\,ds, \qquad t\in(a,b),$$

for some constant c and some $g\in L^2(a,b)$. Now,

$$\int_a^b \left|p^{-1}(t)\int_a^t g(s)\,ds\right| dt \leqq \int_a^b p^{-1}(t)\int_a^t |g(s)|\,ds\,dt$$

$$= \int_a^b |g(s)|\int_s^b p^{-1}(t)\,dt\,ds \leqq \|g\|^2\left(\int_a^b\left(\int_s^b p^{-1}(t)\,dt\right)^2 ds\right)^{1/2}.$$

The last integral is bounded, so the second term in the right member of (21) defines an element of $L^1(a,b)$. The first term does not, unless $c=0$. Hence, $y'\in L^1(a,b)$ if and only if $c=0$. □

Lemma 2 shows that the domain of the maximal operator M can be restricted in many equivalent ways. Let T be defined by

(22) $$Ty = My, \qquad y\in\operatorname{dom} T,$$

where $\operatorname{dom} T = \{y\in\operatorname{dom} M : y$ satisfies (8) and any one of the conditions (i)–(ix) of Lemma 2$\}$.

THEOREM 3. *T is selfadjoint in $L^2(a,b)$.*

Proof. Let $f,g\in\operatorname{dom} T$. Then

$$(Tf,g) = [f,g]_a^b + (f,Tg),$$

where

$$[f,g] = -(pf')\bar{g} + f(p\bar{g}').$$

The bilinear form $[\cdot,\cdot]$ vanishes at b, because f and g satisfy the boundary condition (9). It also vanishes at a, as one verifies most easily using the conditions (iii) and (v) of Lemma 2. Hence, T is symmetric.

It follows from Lemma 1 and the definition of T that M, the maximal operator, is a one-dimensional extension of T. Furthermore, M is a two-dimensional extension of the minimal operator associated with τ. Since T is symmetric, we have $T^*\supset T$. But T^* cannot be a proper extension of T, because then T^* would coincide with M; hence, $T^*=T$. □

The operator T coincides with the Friedrichs extension of the minimal operator associated with the differential expression τ. Hence, we have established several equivalent characterizations of the Friedrichs extension. In the special case of the Legendre differential operator, a proof of the equivalence of the condition (iii), (v) and (vii) of Lemma 2 can be found in Akhiezer and Glazman [2, Appendix II, §9]; the other characterizations appear to be new. The simple characterization given by the condition (i) of Lemma 2 appears to be particularly interesting.

REFERENCES

[1] M. A. NAIMARK, *Linear Differential Operators*, 2 Vols., Frederick Ungar Publ., New York, 1967, 1968.

[2] N. I. AKHIEZER AND I. M. GLAZMAN, *Theory of Linear Operators in Hilbert Space*, 2 Vols., Frederick Ungar Publ. Co., New York, 1961, 1963.

[3] H. WEYL, *Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen*, Math. Ann., 68 (1910), pp. 220–269.

[4] E. C. TITCHMARSH, *Eigenfunction Expansions Associated with Second-Order Differential Equations*, 2 Vols., 2nd Edition, Oxford Univ. Press, Cambridge, 1962.