Proceedings of the 18th World Congress
The International Federation of Automatic Control
Milano (Italy) August 28 - September 2, 2011

On hyperbolic wavelets ⋆

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Abstract: A hyperbolic wavelet concept that can be used to describe, represent, and identify signals belonging to the space of functions H^2 on the unit disc is constructed. The wavelet is derived as the voice-transform belonging to the unitary representation of the Blaschke group upon H^2 on the disc. An alternative for discretization is also proposed and an efficient algorithm is constructed to compute the wavelet coefficients.

Keywords: Signal analysis; Nonparametric identification; Transforms; Discretization.

1. INTRODUCTION

Representations of discrete-time signals in the frequency domain are used in many fields, e.g. in detection of phenomena and changes in systems, system identification, and control design. These representations result in complex analytic functions defined on the unit disc. An important class of these functions is the Hardy space H^2 that can be considered as the space of signals carrying finite energy. The identification of H^2 signals are usually based upon physical measurements that can be interpreted as discrete points on the boundary, i.e. the unit circle. Convenient methods for system identification can be obtained in the case when an orthogonal basis of the H^2 space is used. Besides the standard trigonometric basis, further orthogonal bases can be generated starting from rational functions, this concept leads to the generalized orthogonal bases (GOBs) – see e.g. Heuberger et al. (2005). The GOBs can perfectly be used in system identification in the case when *a priori* information is available with respect to the locations of system poles. Inaccurate knowledge of the pole-positions results in infinite series representations, while approximate knowledge hopefully results in fast decay of the representation coefficients.

Wavelet-type constructions can be a promising opportunity to obtain representations in the space H^2 without making definite presumptions on the system poles; instead tracing the system poles with some type of *dilation* and *translation*, similar operators that are used in the conventional wavelet theory applied in the function space L^2 (Daubechies, 1988).

In this paper a hyperbolic wavelet transform is proposed on the conceptual base of the unitary representations of the Blaschke (hyperbolic) group on the function space $\mathrm{H}^2(\mathbb{D})$ – where $\mathbb{D}$ denotes the unit disc – with the purpose of representing signals belonging to this space, as well as using them in solving detection, system identification, and control problems. After a brief introduction into generalized orthogonal bases the concept of the wavelet-transform will be introduced, and will be extended on the basis of group theoretic principles that will lead to the notion of hyperbolic wavelets. Finally a discrete hyperbolic wavelet scheme will be constructed, which facilitates the practical computations. Other approaches of hyperbolic wavelet constructions – starting from different conceptual base – can be found in Papandreou-Suppappola et al. (1998); Luo et al. (2002).

⋆ This work is supported by the Control Engineering Research Group of HAS at Budapest University of Technology and Economics. The European Union and the European Social Fund have provided financial support to the project under the grant agreement no. TÁMOP-4.2.1./B-09/1/KMR-2010-0003.

2. SIGNAL REPRESENTATIONS IN RATIONAL ORTHOGONAL BASES

Frequency-domain description of discrete-time finite-energy signals is associated with the spectral function $X(e^{i\omega})$ (where ω denotes the circular frequency) of a sequence $(x_t, t \in \mathbb{N})$ belonging to the sequence-space ℓ^2. Let $\mathbb{D}$ and $\mathbb{T}$ denote the unit disc and the unit circle, respectively:

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \quad \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}.$$

It is known, that the spectral function $X(e^{i\omega})$ can be obtained by the transform of the sequence $(x_k, k \in \mathbb{N}) \in \ell^2$:

$$X(z) = \sum_{k=0}^{\infty} x_k z^k \quad z \in \mathbb{D}, \tag{1}$$

and X belongs to the Hardy–space $\mathrm{H}^2(\mathbb{D})$. Replacing z by z^{-1} on the righthand side of (1) we get the z–transform as it is known in the technical literature. Moreover for a.e. $\omega \in [-\pi, \pi]$ the limit function

$$X(e^{i\omega}) := \lim_{r \to 1} X(re^{i\omega}) \tag{2}$$

exists and belongs to the space $\mathrm{L}^2[-\pi, \pi]$. This function has a representation in the standard trigonometric system

$$X(e^{i\omega}) = \sum_{k=0}^{\infty} x_k e^{ik\omega} \quad (\omega \in [-\pi, \pi]), \tag{3}$$

where the series converges in $L^2[-\pi, \pi]$ norm.

Alternative bases can be obtained by involving parameter $a \in \mathbb{D}$ in the form $1 - \overline{a}z$ into the z-transform. It can easily be verified that the system $(1 - \overline{a}z)^{-n}$ $(z \in \mathbb{D}, n \in \mathbb{N})$ with pole $a^* = 1/\overline{a}$ outside $\mathbb{D}$ is complete in $H^2(\mathbb{D})$. Nevertheless, we have the opportunity to orthogonalize it by applying the Gram-Schmidt procedure, in such a way an orthogonal basis can be obtained. Assuming one pole or a complex conjugated pair results in the *Laguerre* and the *Kautz* systems, respectively. The procedure can be generalized to more poles, which leads to the concept of *generalized orthogonal bases*.

The Laguerre and Kautz systems and their generalizations can simply be constructed within the framework of the Takenaka-Malmquist system invented by S. Takenaka and F. Malmquist independently of each other in 1925, and was introduced into the field of system identification by Ninness and Gustafsson (1994).

To introduce the Takenaka–Malmquist system, let us first define the notions of the Blaschke function and Blaschke-product. The Blaschke function is defined in $H^2(\mathbb{D})$ as

$$B_a(z) := \frac{z - a}{1 - \overline{a}z} \quad (z \in \mathbb{C}, a \in \mathbb{D}), \tag{4}$$

where a is called the parameter of the Blaschke-function. The parameter a is identical to the zero and $a^* = 1/\overline{a}$ is the pole of B_a.

The most important feature of the Blaschke function is that $B_a : \mathbb{T} \to \mathbb{T}$ and $B_a : \mathbb{D} \to \mathbb{D}$ are bijections, as a consequence the Blaschke functions to be inner functions in the space $H^2(\mathbb{D})$.

The Blaschke-product for the parameter sequence $\mathbf{a} = (a_n, n \in \mathbb{N})$ is defined as

$$B_{\mathbf{a}|n}(z) := \prod_{j=0}^{n-1} B_{a_j}(z).$$

The system of functions

$$\phi_0(z) \doteq \frac{\sqrt{1 - |a_0|^2}}{1 - \overline{a_0}z} \quad \phi_n(z) \doteq \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a_n}z} B_{\mathbf{a}|n}(z)$$

$(n \in \mathbb{N})$ is called the infinite Takenaka–Malmquist system generated by $\mathbf{a}$. It has been shown that this system forms an orthonormal system. The necessary and sufficient condition of the completeness is

$$\sum_{n=0}^{\infty} (1 - |a_n|) = \infty.$$

It can be observed that the Blaschke product that is used to generate the successive basis elements plays similar role as the powers of z in the standard basis.

The Takenaka–Malmquist system spans a subspace of rational functions of the $H^2(\mathbb{D})$ space. This system is called *generalized* orthogonal basis (GOB), or – since it consist of rational functions – it is referred as *rational* orthogonal basis (ROB) in the literature.

A useful class of GOB is generated by periodic sequences. In this case the sequence $\mathbf{a}$ is obtained by the periodic repetition of a finite number of parameters $a_0, a_1, \ldots, a_{N-1} \in \mathbb{D}$, i.e. $a_n = a_k$ if $n = \ell N + k$ $(\ell \in \mathbb{N}, k = 0, 1, \ldots, N-1)$. The system ϕ_n generated by the periodic sequence $\mathbf{a}$ is of the form (see e.g. Soumelidis et al. (2002b))

$$\phi_n = \phi_k B^{\ell}_{\mathbf{a}|N} \quad (n = \ell N + k, \ell \in \mathbb{N}, k = 0, 1, \ldots, N-1).$$

In the particular case of $N = 1$ and $a_0 = a$ the discrete Laguerre-system

$$\phi_n(z) = \frac{\sqrt{1 - |a|^2}}{1 - \overline{a}z} B^n_a(z),$$

and if $N = 2$ and $a_0 = a$, $a_1 = \overline{a}$ the Kautz-system is given.

One of the most significant features of the representations in GOBs arises from the *a priori* information that is needed in association with the signal or system considered, namely the precognition of the poles associated with the system dynamics. It has been proved that in the case of exact knowledge, finite representation is obtained, approximate knowledge results in infinite series with fast convergence (Soumelidis et al., 2002b), i.e. low number of representation coefficients significantly differing from zero.

To apply GOP representations in practical problems requires discrete numerical algorithms. An efficient numerical algorithm to compute the representation coefficients can be obtained by using the notion of the *argument function* to generate a discretization scheme that gives the opportunity to apply Fast Fourier Transform (FFT) Soumelidis et al. (2002b)). The specific discretization scheme that is used needs non-uniformly spaced samples in the frequency domain, depending on the shape of the argument function associated with the Blaschke functions belonging to the system poles.

The fact that GOB representation can efficiently be used when a priori knowledge is available of the system dynamics can also be considered as a drawback of the method. GOB methods require the application of some kind of non-parametric estimation method of the system poles. There is a great variety of identification methods – conventional and modern ones – operating in the frequency domain that can be used. In many cases approximate knowledge on system dynamics is sufficient to solve the particular problem, e.g. in the case of change detection. If the approximate knowledge is not enough the refinement of the pole locations should be performed; an algorithm to realize it has been proposed in Soumelidis et al. (2002a).

The demand for finding non-parametric forms in connection with the GOB representations that can be used without the need of a priori knowledge suggests to direct the theory – by analogies with signal processing methods in time- and joint time-frequency domain – toward developing *wavelet*-like constructions. The generalization of the GOB representations in this direction provides the opportunity to obtain more general identification methods of the system dynamics.

3. DEVELOPING WAVELETS

The wavelet transform – first formulated in the field of geophysics Goupillaud et al. (1984), than improved by Meyer (1990), Mallat (1989), and Daubechies (1988) – gained numerous applications in several areas of science. Its success in applying it in signals and systems theory raises – among others – from its localization capabilities both in the time and frequency domain of signals. The conventional wavelets has been constructed to handle signals belonging to the $L^2(\mathbb{R})$ space. The central idea of this paper is to extend the wavelet concept to the signal space H^2 that – if some localization properties are realized – gives the opportunity of localizing system poles, hence solving problems of system identification. In this section a brief introduction will be given into the conventional wavelet theory, than the concepts will be generalized, and in this basis a new wavelet concept will be constructed working on signals that belong to the function space H^2.

3.1 The continuous wavelet transform

The continuous wavelet transform on a function $f \in L^2(\mathbb{R})$ is formed by taking translation and dilation of a function ψ named the *mother wavelet*; the integral operator with the kernel

$$\psi^{pq}(x) := \frac{\psi((x-q)/p)}{\sqrt{p}} \quad (x \in \mathbb{R}, p \in (0,\infty), q \in \mathbb{R}) \quad (5)$$

is called *wavelet transform*:

$$(\mathcal{W}_\psi f)(p,q) := \frac{1}{\sqrt{p}} \int_{\mathbb{R}} f(x)\overline{\psi}((x-q)/p)\,dx = \langle f, \psi^{pq} \rangle$$
$$(p > 0, q \in \mathbb{R}, f \in L^2(\mathbb{R}), \quad (6)$$

where $\langle \cdot, \cdot \rangle$ means the inner product of the Hilbert-space $L^2(\mathbb{R})$. A reconstruction formula also exists, under some conditions concerning ψ, the function $f \in L^2(\mathbb{R})$ can be reconstructed from its wavelet transform $\mathcal{W}_\psi f$. To ensure the existence of the wavelet transform and its inverse the mother-wavelet ψ should satisfy some type of *admissibility* conditions, expressing it qualitatively: ψ should be localized in its independent variable. See for more detailed introduction on the continuous wavelet transform in Chui (1992) and Daubechies (1992).

3.2 Generalization of the wavelet concept

To generalize the concept let us consider the trigonometric Fourier transform. By applying the operator $(U_t g)(x) = e^{itx} g(x)$ $(t, x \in \mathbb{R})$ the Fourier-transform can be expressed in the form

$$(\mathcal{F}f)(x) = \int_{-\infty}^{\infty} f(x) e^{-itx}\,dt = \langle f, U_t 1 \rangle \quad (x \in \mathbb{R}) \quad (7)$$

The Fourier-transform is considered – according to the Plancherel theory – as an operator $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. The operator $U_t : \mathbb{R} \to \mathbb{R}$ is an unitary representation of the group $(\mathbb{R}, +)$ in the Hilbert-space $H = L^2(\mathbb{R})$ (see e.g. Wawrzyńczyk (1984) for details on group representations), if it satisfies the following conditions:

$$\begin{array}{ll} (i) & U_s((U_t)) = U_{s+t} f \quad (f \in H, s, t \in \mathbb{R}) \\ (ii) & \langle U_s f, U_s g \rangle = \langle f, g \rangle \quad (s \in \mathbb{R}, f, g \in H) \\ (iii) & \text{For every } f \in H \;\; s \to U_s f \text{ is a} \\ & \text{continuous mapping from } \mathbb{R} \text{ to } H. \end{array} \quad (8)$$

The properties (8) introduced for the Fourier-transform can be generalized by considering any topological group $(G, \cdot)$ and its unitary representation $U_x : H \to H$ $(x \in G)$ by

$$(\mathcal{V}_\phi f)(x) := \langle f, U_x \phi \rangle \quad (x \in G, f, \phi \in H)$$

that is called the *voice*-transform belonging to the unitary group-representation (see for a detailed introduction in Heil and Walnut (1989)). The voice-transform is a meta-transform, it results in several transforms by applying it to several groups; some group – transform pairs can be seen as follows:

- *Heisenberg* group – *Gabor* transform,
- *affine* group – conventional *wavelet* transform,
- *Blaschke* group – *hyperbolic wavelet* transform.

The latter one will be introduced as the main subject of the current paper.

3.3 The affine transform of $\mathbb{R}$

In the continuous wavelet transform (3.1) the mother-wavelet ψ is dilated and translated by the parameters (p, q) as it can be verified in (5). In the wavelet transform ψ is submitted to an *affine transform*. The affine transform can be formulated as follows:

$$\ell_a(x) := px + q$$
$$(x \in \mathbb{R}, a := (p,q) \in \mathbb{A} := (0,\infty) \times (-\infty, \infty)).$$

The set of affine maps $(\ell_a, a \in \mathbb{A})$ form a *group* with respect to function composition. The identity element of this group is the map

$$\ell_e(x) := x \quad (x \in \mathbb{R}, e = (1,0) \in \mathbb{A}),$$

while the inverse element of ℓ_a is the inverse function of this map,

$$\ell_a^{-1} = \ell_{a^{-1}},$$

where $a^{-1} := (1/p, -q/p)$ if $a = (p,q) \in \mathbb{A}$. Introducing a group operation on $\mathbb{A}$ by

$$a_1 \circ a_2 := (p_1 p_2, q_1 + p_1 q_2) \quad (a_j := (p_j, q_j) \in \mathbb{A}, j = 1, 2)$$

we get a group $(\mathbb{A}, \circ)$ isomorphic to the group of affine maps, i.e.

$$\ell_{a_1} \circ \ell_{a_2} = \ell_{a_1 \circ a_2}.$$

Let us consider the Hilbert-space $(H, \langle \cdot, \cdot \rangle)$, and let $\mathcal{U}$ denote the set of unitary bijections $U : H \to H$, i.e. the elements of $\mathcal{U}$ are bounded linear operators that satisfy

$$\langle Uf, Ug \rangle = \langle f, g \rangle \quad (f, g \in H),$$

that is identical to property (II) in (8). The set $\mathcal{U}$ with the composition operation $(U \circ V)f := U(Vf)$ $(f \in H)$ forms a group. The neutral element is the identity operator of H, denoted I, and the inverse element of $U \in \mathcal{U}$ is the operator U^{-1}, which is equal to the adjoint operator U^*. If the remaining properties (I) and (III) in (8) are also satisfied, the homomorphism of the group $(G, \cdot)$ onto the group $(\mathcal{U}, \circ)$ is called the unitary representation of $(G, \cdot)$ in H.

The wavelet transform can be defined on the basis of the operators

$$U_a f := \frac{1}{\sqrt{p}} f \circ \ell_{a^{-1}} \quad (a = (p,q) \in \mathbb{A}). \quad (9)$$

It is easy to see that the collection $(U_a, a \in \mathbb{A})$ is a *unitary representation* of the group $(\mathbb{A}, \circ)$ on the Hilbert–space $H := L^2(\mathbb{R})$, see for proofs in Heil and Walnut (1989). The wavelet transform can be expressed by this representation in the form

$$(\mathcal{W}_\psi f)(a) = \langle f, U_a \psi \rangle \quad (f, \psi \in H, a \in \mathbb{A}), \quad (10)$$

where $\langle \cdot, \cdot \rangle$ is the scalar product on H, as a *voice*-transform.

3.4 The Blaschke (hyperbolic) wavelet

In this section an extension of the wavelet concept will be introduced as an analogy with the affine wavelets that will give the opportunity to represent functions in the space $\mathrm{H}^2(\mathbb{D})$. The key notion that is used is the Blaschke function introduced in Section 2. Following the analogy with the affine wavelets a group theoretic approach will be elaborated.

Consider the Blaschke function based upon zero $b \in \mathbb{D}$ as defined in (4). A more convenient form is

$$B_{\mathfrak{b}}(z) = \varepsilon \frac{z-b}{1-\bar{b}z} \quad (z \in \mathbb{C}, \mathfrak{b} = (b,\varepsilon) \in \mathbb{D} \times \mathbb{T}),$$

where $\varepsilon \in \mathbb{T}$ is an arbitrary parameter. $B_{\mathfrak{b}}$ is bijection either on $\mathbb{T}$ or $\mathbb{D}$. The restriction of the Blaschke function either on the set $\mathbb{T}$ or $\mathbb{D}$ form a *group* with respect to the operation

$$(B_{\mathfrak{b}_1} \circ B_{\mathfrak{b}_2})(z) := B_{\mathfrak{b}_1}(B_{\mathfrak{b}_2}(z)).$$

Define the operation induced by the function composition in the set of parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ as $B_{\mathfrak{b}_1} \circ B_{\mathfrak{b}_2} = B_{\mathfrak{b}_1 \circ \mathfrak{b}_2}$. The group $(\mathbb{B}, \circ)$ will be isomorphic with the group $((B_{\mathfrak{b}}, \mathfrak{b} \in \mathbb{B}), \circ)$. By using the notation

$$\mathfrak{b}_j := (b_j, \varepsilon_j), j \in \{1,2\} \quad \text{and} \quad \mathfrak{b} := \mathfrak{b}_1 \circ \mathfrak{b}_2,$$

$$b = \frac{b_1\bar{\varepsilon}_2 + b_2}{1 + b_1\bar{b}_2\bar{\varepsilon}_2} = B_{(-b_2,\bar{\varepsilon}_2)}(b_1\bar{\varepsilon}_2),$$

$$\varepsilon = \varepsilon_1 \frac{\varepsilon_2 + b_1\bar{b}_2}{1 + \varepsilon_2\bar{b}_1 b_2} = B_{(-b_1\bar{b}_2,\varepsilon_1)}(\varepsilon_2).$$

The neutral and inverse element of the group $(\mathbb{B}, \circ)$ are $\mathfrak{e} := (0,1)$ and $\mathfrak{b}^{-1} := (-b\varepsilon, \bar{\varepsilon})$ if $\mathfrak{b} := (b,\varepsilon) \in \mathbb{B}$ respectively. The group defined in such a way is called *Blaschke group.*

The one-parameter subgroups derived as

$$\mathbb{B}_1 := \{(r,1) : r \in [0,1)\}$$

$$\mathbb{B}_2 := \{(0,e^{it}) : t \in [0,2\pi)\}$$

play central role in the theory, since of $\mathbb{B}_1$ and $\mathbb{B}_2$ generate the group $\mathbb{B}$. Indeed, every element

$$\mathfrak{b} = (re^{i\varphi}, e^{i\vartheta}) \in \mathbb{B} \quad (r \in [0,1),\ \vartheta, \varphi \in [0,2\pi))$$

can be written in the form

$$\mathfrak{b} = (0,\varepsilon_1) \circ (r,1) \circ (0,\varepsilon_2)$$

$$(r \in [0,1),\ \varepsilon_1 = e^{i(\varphi+\vartheta)}, \varepsilon_2 = e^{-i\varphi} \in \mathbb{T}).$$

The group operation of $\mathbb{B}_1$ can be expressed by using the function

$$\operatorname{th} x := \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{e^{2x}-1}{e^{2x}+1} \quad (x \in \mathbb{R}).$$

With the notation $r_j := \operatorname{th} s_j$ $(j = 1,2)$ in the group operation

$$(r,1) := (r_1,1) \circ (r_2,1)$$

r can be expressed as

$$r = \frac{r_1 + r_2}{1 + r_1 r_2} = \frac{\operatorname{th} s_1 + \operatorname{th} s_2}{1 + \operatorname{th} s_1 \operatorname{th} s_2} = \operatorname{th}(s_1 + s_2),$$

consequently

$$r = \operatorname{th}(\operatorname{ath} r_1 + \operatorname{ath} r_2) \quad (r_1, r_2 \in (-1,1)).$$

Denote

$$\rho(z_1, z_2) := \frac{|z_1 - z_2|}{|1 - z_1\overline{z_2}|} \quad (z_1, z_2 \in \mathbb{D}).$$

It can be shown that ρ is a metric on $\mathbb{D}$ and $(\mathbb{D}, \rho)$ is a complete metric space. The Blaschke functions are isometries with respect to this metric, i.e.

$$\rho(B_{\mathfrak{b}}(z_1), B_{\mathfrak{b}}(z_2)) = \rho(z_1, z_2) \quad (z_1, z_2 \in \mathbb{D}).$$

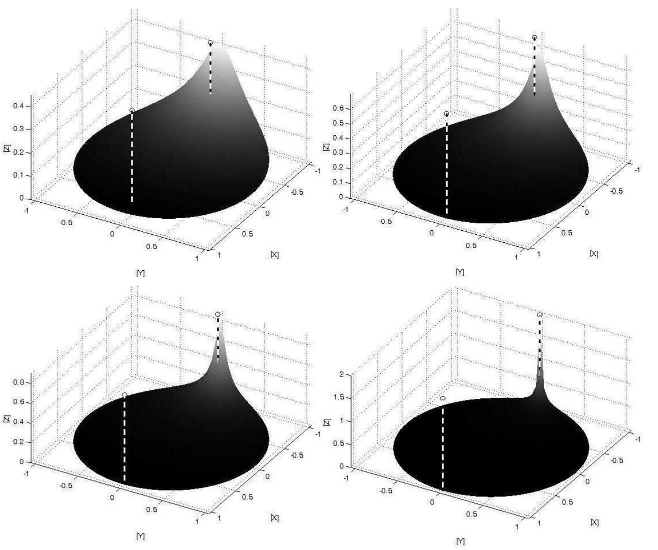

Fig. 1. Unitary group representation on function $\varphi(z) = z$ for several B.

This metric is connected with the Poincaré model of hyperbolic geometry. In this model the lines are the sets

$$\mathcal{L}_{\mathfrak{b}} := \{B_{\mathfrak{b}}(r) : r \in (-1,1)\} \quad (\mathfrak{b} \in \mathbb{B})$$

that are circles crossing perpendicularly the unit circle.

Based upon the relation of the Blaschke group with hyperbolic geometry it is also referred as *hyperbolic group.*

A wavelet transform on the Blaschke group can be constructed on the basis of the voice transform defined in the Hardy space $\mathrm{H}^2(\mathbb{D})$. The inner product in $\mathrm{H}^2(\mathbb{D})$ is given by

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\overline{g}(e^{it})\, dt \quad (f, g \in \mathrm{H}^2(\mathbb{D})).$$

Introduce a unitary representation of the group $(\mathbb{B}, \circ)$ on $\mathrm{H}^2(\mathbb{D})$ by using the collection of functions

$$F_{\mathfrak{b}}(z) := \frac{\sqrt{\varepsilon(1-|b|^2)}}{1-\bar{b}z} \quad (\mathfrak{b} = (b,\varepsilon) \in \mathbb{B}, z \in \overline{\mathbb{D}}),$$

where $\overline{\mathbb{D}} := \mathbb{D} \cup \mathbb{T}$. Let the operator $U_{\mathfrak{b}} : \mathrm{H}^2(\mathbb{D}) \to \mathrm{H}^2(\mathbb{D})$ be introduced for every $\mathfrak{b} = (b,\varepsilon) \in \mathbb{B}$ as

$$U_{\mathfrak{b}}\varphi := F_{\mathfrak{b}^{-1}}\varphi \circ B_{\mathfrak{b}^{-1}}.$$

This is the analogue of operator (9) belonging to the affine group. The voice transform generated by $U_{\mathfrak{b}}$ $(\mathfrak{b} \in \mathbb{B})$ is given by the following formula

$$(V_\phi f)(b) := \langle f, U_{\mathfrak{b}}\phi \rangle \quad (f, \phi \in \mathrm{H}^2(\mathbb{T})). \tag{11}$$

The collection $U_{\mathfrak{b}}$ $(\mathfrak{b} \in \mathbb{B})$ is a unitary representation of $\mathbb{B}$, i.e. the properties (i), (ii), and (iii) in (8) are satisfied, see for proofs in Pap and Schipp (2006). The properties of the unitary representation imply that the image of every complete orthonormal system (ONS) by the operators $U_{\mathfrak{b}}$ is also a complete ONS. Particularly the discrete Laguerre system in H^2 – used widely in the field of system identification – can be obtained in this way from the system $h_n(z) := z^n$ $(z \in \mathbb{T}, n \in \mathbb{N}, \mathfrak{b} = (b,1) \in \mathbb{B})$:

$$L_n^b(z) := \frac{\sqrt{1-|b|^2}}{1-\bar{b}z} B_{\mathfrak{b}}^n(z) = (U_{\mathfrak{b}}^{-1} h_n)(z). \tag{12}$$

The voice transform generated by $(U_{\mathfrak{b}}, \mathfrak{b} \in \mathbb{B})$ forms a transform related to the Blaschke group analogous to the wavelet transform on the affine group described in

Section 3. The function ϕ in (11) can be considered as the *mother wavelet*, which should be chosen in such a way that fair localization properties are assured, as well as further conditions should be satisfied arising from the particular problem to be solved and the *a priori* knowledge available of the signal or system under consideration. The operations realized by the parameters $(r, 1) \in \mathbb{B}_1$ $(r \in [0, 1))$ and $(0, \varepsilon) \in \mathbb{B}_2$ in the Blaschke functions can be associated with the translation and dilation operations respectively. Translation in the unit disc corresponds to a rotation, hence $(0, \varepsilon) \in \mathbb{B}_2$ really corresponds with the translation parameter. The correspondence between $(r, 1) \in \mathbb{B}_1$ and the dilation parameter belonging to the affine wavelet is not so evident. A dilation effect can undoubtedly be observed on the Blaschke group elements depending on the distance of their zero b from the unit circle, however it cannot be separated from the rotation effect. Figure 1 presents four instances of the unitary representation belonging to the function $\varphi(z) = z$ as an example with several group elements, i.e. Blaschke function associated with $\mathfrak{b} = (r, \delta)$ equal to $(0.8, 1)$, $(0.9, 1)$, $(0.95, 1)$, $(0.99, 1)$, respectively. As it can be observed that as parameter r increases toward the unit, the peak formed by the function gets sharper.

4. DISCRETIZATION OF HYPERBOLIC WAVELETS

Discretization of the hyperbolic wavelet constructions is a significant step toward finding algorithms for practical computations. In this section an alternative of the discretization is presented that is based upon the Laguerre system. Other discrete hyperbolic wavelet constructions that can result in bi-orthogonal or orthogonal systems in a sequence of nested subspaces of H^2 – i.e. can form multiresolution schemes (see e.g. Daubechies (1992))– have been proposed by Pap and Schipp (2009).

To define a discrete version of (11) let us consider a discrete subset $\mathbb{B}_0$ of $\mathbb{B}$. The system of the form

$$\phi_b := U_{\mathfrak{b}}\phi \quad (\mathfrak{b} \in \mathbb{B}_0)$$

is called *discrete hyperbolic wavelet*. An obvious selection of a mesh of discretization points in the unit disc can be done on the basis of one-parameter subgroups of the hyperbolic group that can directly be associated with the translation (rotation) and the dilation of the concerned functions. By considering the sequences

$$\mathfrak{a}^n = (r_n, 1) \in \mathbb{B}_1, \ \mathfrak{b}_n = (0, e^{i\frac{2\pi}{m_n}}) \in \mathbb{B}_2 \ (n = 1, 2, \dots)$$

where $1 \le m_n \in \mathbb{N}$, and using $\mathfrak{a}^n$ as the $\mathrm{n^{th}}$ power of $\mathfrak{a}$,

$$\mathbb{B}_0 := \{\mathfrak{c}_{m_k} := \mathfrak{a}^n \circ \mathfrak{b}_n^k : 0 \le k < m_n, n = 1, 2, \dots\} \quad (13)$$

will be given. An example of this scheme can be seen in Figure 2. Every discretization point represents an element of the hyperbolic group that is composed by two elements taken from the subgroups $\mathbb{B}_1$ and $\mathbb{B}_2$, i.e. it means consecutive compositions of two Blaschke functions one with real parameter and another with pure rotation.

The voice-transform that is performed with the unitary representation upon the one-parameter subgroup defined by real Blaschke parameter results in the classical Laguerre representation of a function of H^2. In the case when the mother wavelet φ can be expressed by a power series of the complex variable z, according to (12) the representation

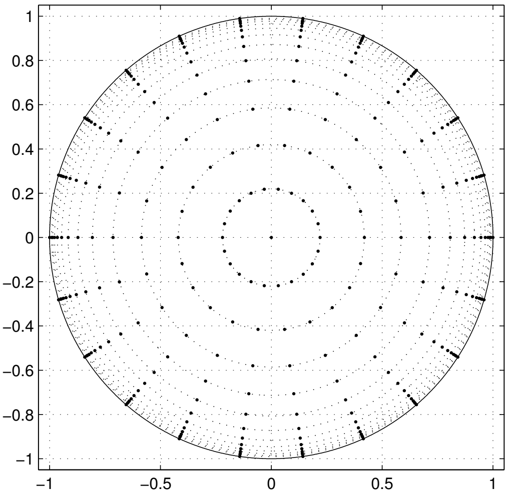

Fig. 2. A hyperbolic discretization scheme.

can be computed as the linear combination of the Laguerre coefficients of the function in every discretization point by considering the real Blaschke parameter belonging to it. The translation parameters should also be considered; rotation on the unitary representation belonging to the radial subgroup associated with the real axis of the complex plane can be replaced with a counter-rotation of the associated function. Hence the computations can be performed by applying the following procedure:

Procedure 1. Computation of the hyperbolic wavelet coefficients belonging to function $f \in \mathrm{H}^2$:

(1) Expressing the mother wavelet φ in polynomial form, i.e. in the form of a polynomial or a power series

$$\varphi(z) = \sum_k p_n z^k.$$

(2) Computing the Laguerre representation of the function f belonging to every r_n associated with $\mathfrak{a}^n \in \mathbb{B}_1$.
(3) Counter-rotating the function f in the angles associated by the subgroup elements $\mathfrak{b}_n^k \in \mathbb{B}_2$.
(4) Repeating Step 2 on the rotated function. By this way every discretization point is covered. .
(5) Computing the wavelet values in every discretization point as a linear combination of the Laguerre coefficients by using the polynomial coefficients obtained in step 1, respectively.

The realization described above can easily be realized by using the concept of the argument function associated with the Blaschke function, see Soumelidis et al. (2002b). A discrete algorithm of computing the coefficients of a Laguerre (or any other orthogonal rational) representation can be obtained by applying a non-uniform discretization scheme on the unit circle that can be obtained by the use of the inverse of the argument function. The computations can be performed by using Fast Fourier Transform (FFT) that makes the algorithm rather efficient. A summary of the algorithm can be found as follow.

The argument function belonging to the Blaschke function with parameter $a \in \mathbb{D}$ can be expressed on the unit circle as

$$B_a(e^{it}) = e^{i\beta_a(t)},$$

where β_a is called *argument function*. Laguerre representation uses a single Blaschke function, hence the argument

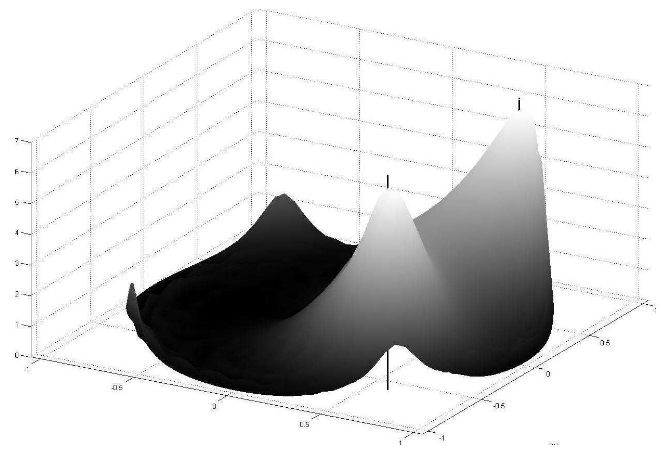

Fig. 3. The wavelet representation of a 2-pole system with multiplicity 2 – mother wavelet is $\varphi(z) = (1-z)^{-1}$.

function can easily be expressed as it can be expressed by introducing $a = re^{i\varphi}$ as

$$\beta_a(t) = 2\arctan\Big(\mu \tan \tfrac{1}{2}(t-\varphi)\Big) + \gamma$$

with $\mu = (1+r)/(1-r)$ and an arbitrary constant γ.

The inverse of the argument function $t = \beta_a^{-1}(s)$ can be expressed with the formula

$$t = 2\arctan\Big(\mu^{-1} \tan \tfrac{1}{2}(s-\gamma)\Big) + \varphi,$$

where γ can be found such a way that the function forms a bijection from the interval $[-\pi, \pi]$ to itself. The Laguerre representation coefficients can be computed by applying the orthogonality principle onto the elements of the Laguerre basis, i.e. by computing the scalar products

$$l_n = \langle f, L_n \rangle \quad n = 0, 1, 2, \ldots \tag{14}$$

By using the definition of the scalar product belonging to the Hilbert space H^2 and introducing the inverse argument function the scalar product (14) can be expressed as the Fourier-transform of the following function:

$$f_a(s) = \left[\frac{f(e^{it})(1-re^{i(t-\varphi)})}{\sqrt{1-r^2}} \right]_{t=\beta_a^{-1}(s)}$$

The proof can be obtained as a simplification of the general case belonging to Generalized Orthogonal Bases; a detailed introduction to this can be found in Soumelidis et al. (2002b). The application of the inverse argument function results in a non-uniform sampling scheme applied on the unit circle, the function f_a should be sampled according to this. The samples can be considered as spectral measurements arranged non-uniformly in the frequency axis.

An example is presented in Figure 3. The wavelet representation of a 2-pole system with the conjugated complex pole pair $p_1 = 0.8628e^{i\pi/4}$, $p_2 = 0.8628e^{-i\pi/4}$ and multiplicity 2 has been computed, by using $\varphi(z) = (1-z)^2$ as a mother wavelet. This selection of the mother wavelet seems to be perfect: two local maxima can be observed in the poles positions. The figure presents the surface of the wavelet coefficients interpolated upon the discretization points presented in Figure 2.

5. CONCLUSION

A hyperbolic wavelet concept that can be used to describe, represent and identify signals belonging to the space $\mathrm{H}^2(\mathbb{D})$ has been constructed. The are derived as the voice-transform belonging to the unitary representation of the Blaschke group upon $\mathrm{H}^2(\mathbb{D})$. An efficient algorithm has been proposed to compute the wavelet coefficients.

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