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Wavelets in Clifford Analysis

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Abstract

Clifford analysis is a higher dimensional functions theory for the Dirac operator and builds a bridge between complex function theory and harmonic analysis. The construction of wavelets is done in three different ways. Firstly, a monogenic mother wavelet is obtained from monogenic extensions (Cauchy–Kovalevskaya extensions) of special functions like Hermite and Laguerre polynomials. Based on the kernel function, Cauchy wavelets are also monogenic but not square integrable in the usual sense. On the other hand, these wavelets and their kernels are connected to the Cauchy–Riemann equations in the upper half space as well as to Bergman and Hardy spaces.

Secondly, a group theoretical approach is used to construct wavelets. This approach considers pure dilations and rotations as group actions on the unit sphere. It can be generalized by using the action of the Spin group because the Spin group is a double cover of the rotation group, whereas dilations arise from Möbius transformations. Here, Clifford analysis gives the tools to construct wavelets.

Finally, an application to image processing based on monogenic wavelets is considered. Here, the starting point are scalar-valued functions and the resulting Clifford wavelets are boundary values of monogenic functions in the upper half space. One proceeds in two steps. First choose a real- or complex-valued primary wavelet and then construct from that using the Riesz transform = Hilbert transform Clifford wavelets and Clifford wavelet frames.

Introduction

Clifford analysis is a higher dimensional function theory based on the Dirac operator or Cauchy–Riemann operator and it is considered to be a refinement of harmonic analysis. Wavelets and frames have become a standard tool in mathematical research as well as in applications such as image processing, analysis of large data sets, statistics, and denoising of signals. The basic construction goes as follows. The mother wavelet $\psi(x)$ generates a family of wavelets by

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right), \quad a > 0,\ b \in \mathbb{R}.$$

These can also be interpreted as the action of the $ax+b$-group on the real line. This gives rise to the group theoretical approach to wavelets. To be invertible the mother wavelet has to fulfill the so-called admissibility condition:

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$$C_\psi := \int_{-\infty}^{\infty} \frac{|\hat{\psi}(u)|^2}{|u|}\, du < +\infty,$$

where $\hat{\psi}$ denotes the Fourier transform of ψ. The admissibility condition implies

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 0.$$

If a wavelet has more vanishing moments

$$\int_{-\infty}^{\infty} x^n\, \psi(x)\, dx = 0, \quad n = 0,\, 1,\, 2,\, \ldots,\, N,$$

a function can be represented by a lower number of wavelet coefficients. The continuous wavelet transform (CWT) is defined as

$$F(a,b) = \langle \psi_{a,b},\, f \rangle = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \overline{\psi\left(\frac{x-b}{a}\right)}\, f(x)\, dx$$

and can be inverted by

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} F(a,b)\, \psi_{a,b}(x)\, \frac{da}{a^2} db.$$

In higher dimension wavelets can be refined by actions of the rotation group in the following sense:

$$F(a,\, \underline{b},\, \theta) = \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} \overline{\psi\left(\frac{r_{-\theta}(\underline{x} - \underline{b})}{a}\right)}\, f(\underline{x})\, d\underline{x},$$

where the mother wavelet ψ is not only translated by $\underline{b} \in \mathbb{R}^m$ and dilated by $a > 0$, but also rotated by an angle $\theta \in [0, 2\pi)$.

After some basic notions of Clifford analysis the necessary definitions are introduced, this includes the Cauchy–Kovalevskaya-extension theorem, Hardy spaces, Cauchy and Hilbert operators [15], and Hilbert modules [16]. After that some examples are given of wavelets constructed by monogenic extensions of special functions. These are the radial Hermite wavelets, Hermite and Laguerre wavelets which have been studied by F. Brackx, N. De Schepper and F. Sommen [6–9]. Another approach was used by J. Cnops to construct Cauchy wavelets and apply them to the study of function spaces [13]. The property of the Cauchy kernel to be an approximate identity is discussed in [3].

A difficult problem is the construction of wavelets on manifolds. In the context of Clifford analysis using Möbius transformations and gyrogroups M. Ferreira constructed wavelets on the sphere S^n [20, 21]. A different approach based on diffusive wavelets can be found in [4, 5]. An in some sense unifying approach was developed in [18].

Clifford wavelets can be applied to monogenic signals. In image processing the monogenic signal by M. Felsberg and G. Sommer [19] is a useful tool to analyze images and to denoise them. To analyze images wavelets have to be combined with the monogenic signal or more precise with

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the Riesz or the Hilbert transform [31, 32]. The necessary mathematical foundation of Clifford wavelets and frames can be found in [24].

Basic Clifford Analysis

Clifford analysis is in depth explained in the chapter "Introductory Clifford Analysis" of the Springer reference, see also references in there. For the convenience of the reader and to emphasize specific details we recall some parts of basis Clifford analysis. Let $\mathbb{R}^{m+1}$ be the $m+1$-dimensional Euclidean space with an orthonormal basis $e_0, e_1, e_2, \dots e_m$. Consider functions defined in $\mathbb{R}^{m+1}$ and taking values in the Clifford algebra $\mathbb{R}_{0,m+1}$ or its complexification $\mathbb{C}_{m+1}$. The non-commutative multiplication in the Clifford algebra is governed by the rule:

$$e_j e_k + e_k e_j = -2\delta_{jk}, \quad j, k = 0, 1, \dots, m.$$

A basis for $\mathbb{R}_{0,m+1}$ then consists of the elements $e_A = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k}$ where $A = (\alpha_1, \alpha_2, \dots, \alpha_k) \subset \{0, 1, \dots, m\}$ such that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq m$. For $A = \emptyset$, $e_\emptyset = 1$, the identity element of the algebra. In such a way, any element $a \in \mathbb{R}_{0,m+1}$ ($a \in \mathbb{C}_{m+1}$) may be written as

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R} \quad (a_A \in \mathbb{C}).$$

There are three involutions defined on $\mathbb{R}_{0,m+1}$. They are defined for the basic elements e_A and then extended by linearity to $\mathbb{R}_{0,m+1}$. The main involution $\hat{e}_A = (-1)^k e_A, \quad |A| = k$, the reversion $e_A^* = e_{\alpha_k} e_{\alpha_{k-1}} \cdots e_{\alpha_1}$ if $e_A = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k}$, and the conjugation $\overline{e}_A = (\hat{e}_A)^* = (e_A^*)\hat{}$. In addition

$$\overline{i} = -i$$

for the imaginary unit of the complex numbers in case of $\mathbb{C}_{m+1}$.

There are important subgroups of $\mathbb{R}^{m+1}$. The Spin group will be important for the construction of wavelets on the sphere and is given in the following way. Take $s \in \mathbb{R}^{m+1}$ with $s^2 = -1$, i.e. $s \in S^m$, the unit sphere in $\mathbb{R}^{m+1}$. The Spin group for $\mathbb{R}^{m+1}$ is defined by

$$\mathrm{Spin}(m+1) = \left\{ \prod_{j=1}^{2k} s_j : k \in \mathbb{N},\ s_j \in S^m \right\}.$$

It has to be mentioned that the Spin group is a double cover of $\mathrm{SO}(m+1)$ and therefore closely connected to rotations on the sphere.

Möbius Transformations

Another important tool for the construction of wavelets on the unit sphere are Möbius transformations. The class of conformal mappings in $\mathbb{R}^2 \sim \mathbb{C}$ is very rich. An important subclass are the so-called Möbius transformations, i.e. fractional linear transformations of the form

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$$f(z) = \frac{az+b}{cz+d}, \quad z \in \mathbb{C},$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$.

In [1, 33] it has been proven that Möbius transformations in $\mathbb{R}^m$ may be fully described by using Clifford algebras. The Möbius transformations in $\mathbb{R}^m$ form a group for compositions and the orientation preserving elements form a subgroup.

Theorem 1 (Möbius Transformations). *Let g be a Möbius transformation in $\mathbb{R}^{m+1}$. Then there exists a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfying*

- $a, b, c, d \in \Gamma(m+1) \cup \{0\}$,
- $ab^*, cd^*, c^*a, d^*b \in \mathbb{R}^{m+1}$,
- $ad^* - bc^* \in \mathbb{R}\backslash\{0\}$,

and such that for $x \in \mathbb{R}^m$

$$g(x) = \frac{ax+b}{cx+d} = (ax+b)(cx+d)^{-1}.$$

Conversely, each such A determines a Möbius transform in $\mathbb{R}^{m+1}$.

The important Möbius transformations for the construction of wavelets are those who leave the unit sphere invariant, they will be characterized in the following theorem:

Theorem 2 (Möbius Transformations Leaving the Unit Sphere Invariant).

1. *Let g be a Möbius transformation with associated matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then g leaves the unit sphere S^m invariant if and only if*

   (a) $a, b \in \Gamma(m) \cup \{0\}$ with $|a| \neq |b|$,
   (b) $c = \hat{b}$ and $d = \hat{a}$ or $c = -\hat{b}$ and $d = -\hat{a}$.

   Moreover, g leaves the open unit ball invariant if $|b| < |a|$.
2. *The set of those Möbius transformations leaving the unit sphere invariant is generated by orthogonal transformations and Möbius transformations of the type*

$$\varphi_a(x) = \frac{x-a}{ax+1},$$

where $a \in \mathbb{R}^m \cup \{0\}$ with $|a| \neq 1$. For $a = \infty$,

$$\varphi_\infty(x) = \frac{x}{|x|^2}$$

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is the inversion. Moreover φ_a leaves the open unit ball $\overset{o}{B}(1) := \{x \in \mathbb{R}^{m+1} : |x| < 1\}$ invariant if $|a| < 1$.

Remark 1. The composition of two such Möbius transformations is another Möbius transformation up to a rotation [20]:

$$(\varphi_a \circ \varphi_b)(x) = q\,\varphi_c(x)\,\overline{q}, \quad \text{with} \quad c = (1-ab)^{-1}(a+b) \quad \text{and} \quad q = \frac{1-ab}{|1-ab|}.$$

Definition 1. The symbol of the new Möbius transformation is $b \oplus a := (1-ab)^{-1}(a+b)$ and $gyr[a,b]c := \frac{1-ab}{|1-ab|}\,c\,\overline{\frac{1-ab}{|1-ab|}}$ is the action of the rotation induced by q.

The notation $b \oplus a$ is chosen because at first φ_b is applied and then φ_a . The composition of Möbius transformations is not commutative and also not associative but it gives rise to a left gyrogroup [30].

Hilbert Modules

Wavelets and transforms are defined on function spaces and preferable on Hilbert spaces. The most often used space is L^2. For Clifford-valued functions the appropriate analogue is the Hilbert module. A (right) module $\mathcal{T}$ over $\mathbb{C}_m$ is a vector space such that for each a the map

$$\begin{aligned} R(a) :\ & \mathcal{T} \to \mathcal{T}, \\ f \mapsto\ & R(a)f = fa, \end{aligned}$$

is a linear map which is well defined as a right multiplication, i.e. $R(1) = Id$ and $R(ab) = R(b)R(a)$. We only consider modules where elements of $\mathcal{T}$ are Clifford-valued functions, and where the multiplication $R(a)$ is given as

$$R(a)f(x) = f(x)a.$$

A linear map $\mathcal{T} \times \mathcal{T} \to \mathbb{C}_m$ satisfying

$$\begin{aligned} (f,g) &= \overline{(g,f)}, \\ (f,g+h) &= (f,g) + (f,h), \\ (f,ga) &= (f,g)a, \end{aligned}$$

for any f and g, h in $\mathcal{T}$ and $a \in \mathbb{C}_m$ is called a Clifford-valued inner product on $\mathcal{T}$ and denoted by $\langle .,. \rangle_{\mathcal{T}}$ [16].

Example 1 (Clifford Hilbert Module H_n). Let f be Clifford-valued and square integrable over $\mathbb{R}^{m+1}$:

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$$f(x) = \sum_A f_A e_A, \quad f_A(x) \in L^2(\mathbb{R}^{m+1}).$$

An inner product is then given by

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{R}^{m+1}} \overline{f(x)} g(x)\,dx = \int_{\mathbb{R}^{m+1}} \overline{\left(\sum_A f_A(x) e_A\right)} \left(\sum_B f_B(x) e_B\right) dx$$
$$= \sum_{A,B} \int_{\mathbb{R}^m} \overline{f_A(x)} f_B(x)\, \overline{e_A} e_B\, dx.$$

The scalar part $[\langle .,.\rangle]_0$ can be identified with the scalar product of 2^{m+1}-dimensional square integrable vectors, i.e. $f : \mathbb{R}^m \to \mathbb{C}^{2^{m+1}}$ and each component belongs to $L^2(\mathbb{R}^{m+1})$. This Clifford Hilbert module will be denoted by H_n.

Dirac and Cauchy–Riemann Operators

The Dirac operator is defined as the first order linear differential operator

$$\partial_x = \sum_{j=0}^{m} e_j \partial_{x_j} = e_0 \partial_{x_0} + \partial_{\underline{x}},$$

where $\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$ may be considered as the Dirac operator in $\mathbb{R}^m$. The Cauchy–Riemann operator is the first order linear differential operator

$$D_x = \partial_{x_0} + \sum_{j=1}^{m} \overline{e}_0 e_j \partial_{x_j} = \overline{e}_0 \partial_x = \partial_{x_0} + D_{\underline{x}}.$$

Definition 2 (Monogenic Functions). Let $\Omega \subseteq \mathbb{R}^{m+1}$ be open and let f be a C^1-function in Ω which is $\mathbb{C}_{m+1}$-valued. Then f is *left monogenic* or (*right monogenic*) in Ω if in Ω

$$\partial_x f = \sum_{j,A} e_j e_A \frac{\partial f_A}{\partial x_j} = 0 \quad \left(f \partial_x = \sum_{A,j} e_A e_j \frac{\partial f_A}{\partial x_j} = 0 \right) \quad \text{or} \quad D_x f = 0 \quad (f D_x = 0).$$

The connection between monogenic and harmonic functions is due to the fact that $\Delta_x = -\partial_x^2 = \overline{D}_x D_x$.

One way of constructing monogenic functions is by extending real-analytic functions in some open connected domain $\underline{\Omega} \subset \mathbb{R}^m$. The x_0-axis is chosen to be the real axis, which means that the variable in $\mathbb{R}^m$ is $\underline{x} = (x_1, \ldots, x_m)$. Ω will be an open connected and x_0-normal neighborhood of $\underline{\Omega}$ in $\mathbb{R}^{m+1}$, i.e. for each $x \in \Omega$, the line segment $\{x + te_0 : t \in \mathbb{R}\} \cap \Omega$ is connected and

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contains exactly one point in $\underline{\Omega}$. Let f be a real-analytic function f in $\underline{\Omega}$. The goal is to construct a function f^* such that

1. $\partial_x f^* = 0$ in Ω,
2. $f^*(x_0, \underline{x}) \mid_{x_0=0} = f^*(0, \underline{x}) = f(\underline{x})$, i.e. $f^* \mid_{\underline{\Omega}} = f$.

From (1) it follows that

$$\partial_{x_0} f^* = -\overline{e}_0 \partial_{\underline{x}} f^*.$$

Combined with (2)

$$f^*(x_0, \underline{x}) = \left(e^{-x_0 \overline{e}_0 \partial_{\underline{x}}}\right) f(\underline{x}) = \sum_{k=0}^{\infty} \frac{(-x_0)^k}{k!} (\overline{e}_0 \partial_{\underline{x}})^k f(\underline{x}).$$

the existence of f^* is guaranteed and by construction unique. Also the existence of the domain Ω can be proven. The monogenic function f^* in Ω thus obtained is called the Cauchy–Kovalevskaya extension of f.

Example 2 (Radial Hermite Polynomials). The so-called radial Hermite polynomials in $\mathbb{R}^m$ are generated by the monogenic extension G_0 of the function $\exp(-|\underline{x}|^2)$.

$$G_0(x_0, \underline{x}) = \exp\left(-\tfrac{|\underline{x}|}{2}\right) \sum_{n=0}^{\infty} \tfrac{x_0^n}{n!} H_n(\underline{x}),$$

where $H_n(\underline{x})$ are radial polynomials in $\underline{x}$, satisfying the recurrence relation

$$H_{n+1}(\underline{x}) = (\underline{x} - \partial_{\underline{x}}) H_n(\underline{x}).$$

These radial Hermite polynomials are alternatively scalar or vector valued. They are also determined by the Rodrigues' formula

$$H_n(\underline{x}) = (-1)^n \exp\left(\tfrac{|\underline{x}|^2}{2}\right) \partial_{\underline{x}}^n \exp\left(-\tfrac{|\underline{x}|^2}{2}\right).$$

The first radial Hermite polynomials are thus given by

$$H_0(\underline{x}) = 1, \quad H_1(\underline{x}) = \underline{x}, \quad H_2(\underline{x}) = \underline{x}^2 + m, \quad H_3(\underline{x}) = \underline{x}^3 + (m+2)\underline{x}.$$

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Lemma 1. *A fundamental solution for the Dirac operator is given by*

$$E(x) = \frac{1}{A_{m+1}} \frac{\overline{x}}{|x|^{m+1}} = -\frac{1}{A_{m+1}} \sum_{j=0}^{m} \frac{x_j}{|x|^{m+1}} e_j,$$

where A_{m+1} is the area of the unit sphere S^m in $\mathbb{R}^{m+1}$, i.e. $A_{m+1} = \frac{2\pi^{(m+1)/2}}{\Gamma(\frac{m+1}{2})}$.

E has the following properties.

1. *E is $\mathbb{R}^{m+1}$-valued and belongs to $L^1_{loc}(\mathbb{R}^{m+1})$.*
2. *E is left and right monogenic in $\mathbb{R}^{m+1}\setminus\{0\}$*
3. *and $\lim_{|x|\to\infty} E(x) = 0$.*
4. *$\partial_x E = E\partial_x = \delta(x)$, $\delta(x)$ being the classical δ-function in $\mathbb{R}^{m+1}$.*

Hardy Spaces

Hardy spaces are spaces of monogenic functions. To construct monogenic wavelets one way is to use the Hilbert (or Riesz) transform and this approach is connected with Hardy spaces in the upper and lower half space.

Definition 3 (Hardy Spaces). Let $1 < p < \infty$. Then

$$H^p(\mathbb{R}^{m+1}_\pm) = \{F \text{ monogenic in } \mathbb{R}^{m+1}_\pm : \sup_{\delta>0} \int_{\mathbb{R}^m} |F(y \pm \delta)|^p \, dy < \infty\}$$

is the Hardy space of (left) monogenic functions in $\mathbb{R}^{m+1}_\pm$. (Monogenicity being w.r.t the Cauchy–Riemann operator D_x.)

To describe functions in a Hardy space fundamental solution of the Cauchy–Riemann operator is considered. With $x = x_0 + \sum_{j=1}^{m} \overline{e}_0 e_j x_j$ the fundamental solution is

$$E(x) = \frac{1}{A_{m+1}} \frac{\overline{x}}{|x|^{m+1}}.$$

It is easily seen that E has the same properties with respect to the Cauchy–Riemann operator as in Lemma 1 the fundamental solution for the Dirac operator.

Definition 4 (Integral Operators). For $f \in L^p(\mathbb{R}^m)$ and $x \in \mathbb{R}^{m+1}\setminus\mathbb{R}^m$,

$$\mathcal{C}f(x) = \int_{\mathbb{R}^m} E(x-y) f(y) \, dy$$

is the Cauchy transform of f.
For $f \in L^p(\mathbb{R}^m)$ and a.e. $x \in \mathbb{R}^m$,

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$$\mathcal{H}f(x) = 2p.v.\int_{\Sigma} E(x-y)f(y)\,dy = 2\lim_{\epsilon\to 0+}\int_{y\in\Sigma:|x-y|>\epsilon} E(x-y)f(y)\,dy$$

is the Hilbert transform of f.

Theorem 3. *Let $f \in L^p(\mathbb{R}^m)$, $1 < p < \infty$. Then*

1. *$\mathcal{C}f \in H^p(\mathbb{R}^{m+1}_{\pm})$.*
2. *$\mathcal{C}f$ has non-tangential limits $(\mathcal{C}^{\pm})$ at almost all $x^* \in \mathbb{R}^m$.*
3. *Putting*

$$\mathbb{P}^{+}_{\mathbb{R}^m} f(x^*) = (\mathcal{C}f)^+(x^*) \quad \text{and} \quad \mathbb{P}^- f(x^*) = -(\mathcal{C}f)^-(x^*)$$

then $\mathbb{P}^{\pm}$ are bounded projections in $L^p(\mathbb{R}^m)$.

4. *(Plemelj-Sokhotzki formulae). For a.e. $x^* \in \mathbb{R}^m$,*

$$\mathbb{P}^+ f(x^*) = \tfrac{1}{2}\left(f(x^*) + \mathcal{H}f(x^*)\right) \quad \text{and} \quad \mathbb{P}^- f(x^*) = \tfrac{1}{2}\left(f(x^*) - \mathcal{H}f(x^*)\right)$$

whence

$$\mathbf{1} = \mathbb{P}^+ + \mathbb{P}^- \quad \text{and} \quad \mathcal{H} = \mathbb{P}^+ - \mathbb{P}^-.$$

In particular $\mathcal{H}$ is a bounded linear operator on $L^p(\mathbb{R}^m)$ and, putting $L^{p,\pm}(\mathbb{R}^m) = \mathbb{P}^{\pm}L^p(\Sigma)$, leads to the decomposition into Hardy spaces of boundary values of monogenic functions

$$L^p(\mathbb{R}^m) = L^{p,+}(\mathbb{R}^m) \oplus L^{p,-}(\mathbb{R}^m).$$

The boundary values of monogenic functions in frequency domain can be characterized by the Fourier transform

$$\mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}^m}\int_{\mathbb{R}^m} e^{i\langle x,\xi\rangle} f(x)\,dx.$$

Using that

$$\mathcal{F}\left(\frac{2x_j}{A_{m+1}|x|^{m+1}}\right) = -i\,\frac{\xi_j}{|\xi|}, \quad j = 1,\ldots,m,$$

and set [28]

$$\chi_{\pm}(\xi) = \tfrac{1}{2}\left(1 \pm i\,\frac{\xi}{|\xi|}\right),$$

it should be noticed that

$$\chi_{\pm}^2 = \chi_{\pm} \quad \text{and} \quad \chi_+ + \chi_- = 1,$$

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the boundary values of monogenic function in upper half space are characterized in the next theorem.

Theorem 4 (Hilbert Transform). *For $f \in L^p(\mathbb{R}^m)$ the following statements are equivalent:*

1. *The non-tangential limit of $\mathcal{C}_{\mathbb{R}^m} f$ is a.e. equal to f,*
2. *$\mathcal{H}f = f$,*
3. *$\mathcal{F}f = \chi_+ \mathcal{F}f$.*

and characterizes boundary values of monogenic functions.

Remark 2. Specifically that for a scalar-valued $f \in L^p(\mathbb{R}^m)$ the function $f + \mathcal{H}f$ satisfies $\mathcal{H}(f + \mathcal{H}f) = \mathcal{H}f + \mathcal{H}^2 f = f + \mathcal{H}f$, i.e. $f + \mathcal{H}f \in L^{p,+}$. These properties can be used to construct monogenic signals and monogenic wavelets.

Wavelets

Wavelet theory has its origin in signal analysis. The CWT is a joint time frequency transform that allows to analyze both the frequency content and the time profile at the same time. This is not possible with a Fourier transform or short-term Fourier transform. The flexibility of the wavelet transform is based on a more or less arbitrarily chosen function, the so-called admissible vector or mother wavelet, which will be dilated and translated. Therefore wavelet analysis consists in the following:

- decompose the signal into basic functions, i.e. wavelets,
- reconstruct the signals from its wavelet transform.

Therefore a wavelet transform should have finite energy and a reconstruction formula. These two properties are also shared by the Fourier transform of a signal.

The construction of wavelets can be done in a group-theoretic framework. 1D wavelets arise as coherent states of the affine group. In signal analysis literature the canonical coherent states are known as gaborettes of Gabor wavelets. This approach can be extended into higher dimensions, where an n-dimensional signal of finite energy is represented by a complex-valued function and the operations, usually applied to a signal, are obtained by combining translations, dilations, and rotations. These three operations generate the n-dimensional Euclidean group with dilations, also known as the similitude group of $\mathbb{R}^n$, SIM(n).

Several applications exist in which data to be analyzed are defined on a sphere or other manifolds. The sphere and the unit ball are of special interest not only in geophysics, astronomy but also in statistics. Group theoretical approaches can be used in these situations too.

The use of Clifford analysis is driven by the fact that Clifford analysis is a refinement of harmonic analysis and an extension of complex analysis into higher dimensions. From that point of view a refinement of wavelet analysis should be possible. The more important fact is that in image analysis a signal is not considered as a real-valued function but extended to an analytic signal which is a complex-valued function and this function are the boundary values of a holomorphic function in the upper half plane. Because an image is two dimensional a generalized analytic signal

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is used which is the so-called monogenic signal and is given by the boundary values of a monogenic function in the upper half space.

Monogenic Wavelets

One way to construct monogenic wavelets is to use a monogenic function and consider it as the mother wavelet. These type of Clifford wavelets have been constructed by F. Brackx, F. Sommen, N. De Schepper. Two examples will be considered in detail, the radial Clifford–Hermite wavelets, which are a refinement of the Marr (or Mexican hat) wavelets, and the generalized Clifford–Hermite wavelets. Some other similar constructions will be briefly mentioned.

Radial Clifford–Hermite Wavelets

These wavelets are constructed in [7]. A typical wavelet is the m-dimensional Mexican hat or Marr wavelet. The mother wavelet is

$$g(\underline{x}) = -\Delta \exp(-\tfrac{|\underline{x}|^2}{2}), \quad \underline{x} \in \mathbb{R}^m,$$

where Δ denotes the Laplacian. This wavelet was originally introduced by Marr [27] and is used in image processing (5); also higher order Laplacians of the Gaussian function are used as wavelets.

The wavelet functions that will be considered are of radial type, i.e. invariant under $\mathrm{SO}(m)$. This means they take the form

$$\psi(\underline{x}) = A(|\underline{x}|) + \underline{x}\, B(|\underline{x}|),$$

where A and B scalar functions. Their Fourier transform

$$\hat{\psi}(\underline{u}) = \int_{\mathbb{R}^m} \exp(-i\,\langle \underline{x}, \underline{u}\rangle)\, \psi(\underline{x})\, d\underline{x}$$

then is of radial type too.

Theorem 5 ([6]). *For $n > 0$ the radial $L^1 \cap L^2$-functions*

$$\psi_n(\underline{x}) = \exp\left(-\tfrac{|\underline{x}|^2}{2}\right) H_n(\underline{x}) = (-1)^n \partial_{\underline{x}}^n \exp\left(-\tfrac{|\underline{x}|^2}{2}\right)$$

have vanishing mean and the Fourier transform

$$\hat{\psi}_n(\underline{u}) = (2\pi)^{m/2}(-i)^n \underline{u}^n \exp\left(-\tfrac{|\underline{u}|^2}{2}\right).$$

The wavelets fulfill an admissibility condition with admissibility constants

$$C_n = \frac{1}{A_m} \int_{\mathbb{R}^m} \frac{|\hat{\psi}_n(\underline{u})|^2}{|\underline{u}|^m}\, d\underline{u} = (2\pi)^m \frac{(n-1)!}{2},$$

A_m being the area of the unit sphere in $\mathbb{R}^m$.

Furthermore, the Clifford–Hermite wavelet ψ_n has vanishing moments up to $n-1$:

$$\int_{\mathbb{R}^m} \underline{x}^j \, \psi_n(\underline{x}) \, d\underline{x} = 0, \quad j = 0, 1, \ldots, n-1.$$

Definition 5 ([6]). The Clifford–Hermite CWT for $n > 0$

$$\mathrm{WT} : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}_+ \times \mathbb{R}^m, C_n^{-1} a^{-(m+1)})$$

is defined as

$$\mathrm{WT}(f)(a, \underline{b}) = \int_{\mathbb{R}^m} \overline{\psi_n^{a,\underline{b}}}(\underline{x}) \, f(\underline{x}) \, d\underline{x},$$

where $\psi_n^{a,\underline{b}}(\underline{x}) = \frac{1}{a^{m/2}} \psi_n\left(\frac{\underline{x}-\underline{b}}{a}\right)$ and $\psi_n(\underline{x})$ are defined in Theorem 5.

The functions $H_n(\underline{x})$ are the radial Hermite polynomials obtained by the Cauchy–Kovalevskaya extension, see Example 2.

Theorem 6 (Properties of the Clifford–Hermite CWT, [6]). *Let* $\mathrm{WT}(f)$ *be the Clifford-Wavelet transform of* $f \in L^2(\mathbb{R}^m)$*. Then*

1. *Parseval formula*

$$[\mathrm{WT}(f), \mathrm{WT}(g)] = \tfrac{1}{C_n} \int_{\mathbb{R}^m} \int_0^\infty \overline{\mathrm{WT}(f)}(a, \underline{b}) \, \mathrm{WT}(g)(a, \underline{b}) \, \frac{da}{a^{m+1}} \, d\underline{b} = \frac{1}{(2\pi)^m} \langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle.$$

2. *The Clifford–Hermite CWT maps* $L^2(\mathbb{R}^m$ *into* $L^2(\mathbb{R}_+ \times \mathbb{R}^m, C_n^{-1} a^{-(m+1)})$ *but not onto.*
3. *Reconstruction formula*

$$f(\underline{x}) = \frac{1}{C_n} \int_{\mathbb{R}^m} \int_0^\infty \psi_n^{a,\underline{b}}(\underline{x}) \, \mathrm{WT}(f)(a, \underline{b}) \frac{da}{a^{m+1}} \, d\underline{b}$$

4. *The image of the CWT is a Hilbert module with reproducing kernel given by*

$$K_n(a, \underline{b}; \alpha, \underline{\beta}) = \int_{\mathbb{R}^m} \overline{\psi_n^{\alpha,\underline{\beta}}}(\underline{x}) \psi_n^{a,\underline{b}}(\underline{x}) \, d\underline{x}.$$

Hence a function $F(a, \underline{b}) \in L^2(\mathbb{R}_+ \times \mathbb{R}^m, C_n^{-1} a^{-(m+1)})$ *is the CWT of a signal* $f(\underline{x}) \in L^2(\mathbb{R}^m)$ *iff*

$$F(a, \underline{b}) = \frac{1}{C_n} \int_{\mathbb{R}^m} \int_0^\infty \overline{K_n(a, \underline{b}; a', \underline{b}')} F(a', \underline{b}') \, \frac{da'}{(a')^{m+1}} \, d\underline{b}'.$$

The Clifford–Hermite wavelets are a refinement of the Marr wavelets because for $n = 2N$, $N > 0$:

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$$\psi_{2N}(\underline{x}) = (-1)^N \Delta_{\underline{x}} \exp\left(-\tfrac{|\underline{x}|^2}{2}\right) = \psi_N^{\text{Marr}}(\underline{x})$$

and for $n = 2N + 1$:

$$\psi_{2N+1}(\underline{x}) = -\partial_{\underline{x}} \psi_N^{\text{Marr}}(\underline{x}).$$

Generalized Clifford–Hermite Wavelets

The generalized Clifford–Hermite wavelets were constructed in [7]. Wavelets in higher dimensions are also derived by group actions, but these actions include also rotations and not only dilations and translations.

The so-called generalized Hermite polynomials are defined for $n = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$ by

$$\exp\left(\frac{|\underline{x}|^2}{2}\right)(-\partial_{\underline{x}})^n \left(\exp\left(-\frac{|\underline{x}|^2}{2}\right) P_k(\underline{x})\right) = H_{n,k}(\underline{x}) P_k(\underline{x}),$$

where $P_k(\underline{x}) \in \mathcal{M}_k$ is a monogenic homogeneous polynomial of degree k. In case of $k = 0$ the radial Hermite polynomials are obtained. The functions $H_{n,k}(\underline{x})$ are polynomials of degree n with real coefficients depending on k, satisfying

$$H_{2n+1,k}(\underline{x}) = (\underline{x} - \partial_{\underline{x}}) H_{2n,k}(\underline{x})$$

and

$$H_{2n+2,k}(\underline{x}) = (\underline{x} - \partial_{\underline{x}}) H_{2n+1,k}(\underline{x}) - 2k \frac{\underline{x}}{|\underline{x}|^2} H_{2n+1,k}(\underline{x}).$$

It should be mentioned that the set of products of the generalized Hermite polynomials and the monogenic homogeneous polynomials

$$\{H_{n,k}(\underline{x}) P_k^{(j)}(\underline{x}) : n \in \mathbb{N}, k \in \mathbb{N},\ j \le \dim \mathcal{M}_k\}$$

constitutes an orthogonal basis for $L^2\left(\mathbb{R}^m, \exp(-\frac{|\underline{x}|^2}{2})\right)$.

The $L^1 \cap L^2$-functions

$$\psi_{n,k}(\underline{x}) = \exp\left(-\frac{|x|^2}{2}\right) H_{n,k}(\underline{x}) P_k(\underline{x})$$

have zero momentum and will be basic wavelet kernel functions if they satisfy an appropriate admissibility condition (see Theorem 7). The wavelets are defined as

$$\psi_{n,k}^{a,\underline{b},s}(\underline{x}) = \frac{1}{a^{m/2}} \overline{\psi}_{n,k}\left(\frac{\overline{s}(\underline{x} - \underline{b})s}{a}\right)\overline{s},$$

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where $a \in \mathbb{R}_+$, $\underline{b} \in \mathbb{R}^m$, $s \in \mathrm{Spin}(m)$. These functions originate from the basic wavelet function $\psi_{n,k}$ by dilation, translation, and spinor-rotation. The generalized Clifford–Hermite wavelet transform applies to functions $f \in L^2(\mathbb{R}^m)$ by

$$T_{n,k} f(a, \underline{b}, s) = \langle \psi_{n,k}^{a,\underline{b},s}(\underline{x}), f \rangle = \int_{\mathbb{R}^m} \overline{\psi_{n,k}^{a,\underline{b},s}(\underline{x})}\, f(\underline{x})\, d\underline{x}.$$

Theorem 7 ([7]).

1. *A Clifford algebra valued function $\psi \in L^2 \cap L^1(\mathbb{R}^m)$ is a basic wavelet function if it satisfies in frequency space the admissibility condition:*

$$\int_{\mathbb{R}^m} \overline{\hat{\psi}(\underline{u})}\, \hat{\psi}(\underline{u}) \frac{d\underline{u}}{|\underline{u}|^m} < \infty;$$

   this condition implies $\int_{\mathbb{R}^m} \psi(\underline{x})\, d\underline{x} = 0$.
2. *The functions*

$$\psi_{n,k}(\underline{x}) = \exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{n,k}(\underline{x}) P_k(\underline{x})$$

   satisfy the above condition provided that $\overline{P_k(\underline{x})}\, P_k(\underline{x})$ be real-valued, which, by the Funk–Hecke Theorem implies that $\overline{\hat{\psi}_{n,k}(\underline{u})}\, \hat{\psi}_{n,k}(\underline{u})$ is real-valued.
3. *The generalized Clifford–Hermite wavelet transform $T_{n,k}$ maps $L^2(\mathbb{R}^m)$ isometrically into $L^2(\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m), C_{k,n}^{-1} a^{-(m+1)}\, da\, d\underline{b}\, ds)$, where $C_{n,k}$ is the admissibility constant of $\psi_{n,k}$. But $T_{n,k}$ is no surjection!*
4. *However, from Parseval formula it follows that if $f \in L^2(\mathbb{R}^m)$ and $F_{n,k}(a, \underline{b}, s) = T_{n,k} f(\underline{x})$ then*

$$f(\underline{x}) = \frac{1}{C_{n,k}} \int_{\mathrm{Spin}\,(m)} \int_{\mathbb{R}^m} \int_0^\infty \psi_{n,k}^{a,\underline{b},s}(\underline{x}) F_{n,k}(a, \underline{b}, s) \frac{da}{a^{m+1}}\, d\underline{b}\, ds$$

   to hold weakly in $L^2(\mathbb{R}^m)$.

Clifford–Laguerre Wavelets

On the real line the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, for $\alpha > -1$, are defined by

$$L_n^{(\alpha)}(x) = x^\alpha \frac{e^x}{n!} \frac{d^n}{dx^n} \left(e^{-x} x^{n+\alpha}\right), \quad n = 0, 1, 2, \ldots.$$

They constitute an orthogonal basis for $L^2\left([0, \infty), x^\alpha e^{-x}\right)$ and satisfy the orthogonality relation

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$$\int_0^\infty L_m^{(\alpha)}(x)\, L_n^{(\alpha)}\, x^\alpha\, e^{-x}\, dx = \Gamma(1+\alpha) \binom{n+\alpha}{n} \delta_{m,n}.$$

The construction of a generalization to Clifford analysis of these classical Laguerre polynomials is based on the so-called Clifford–Heaviside functions

$$P^+ = \frac{1}{2}\left(1 + i\,\frac{\underline{x}}{|\underline{x}|}\right) \quad \text{and} \quad P^- = \frac{1}{2}\left(1 - i\,\frac{\underline{x}}{|\underline{x}|}\right).$$

These Clifford–Heaviside functions satisfy the relations

$$P^+ + P^- = 1; \quad P^+P^- = P^-P^+ = 0; \quad (P^+)^2 = P^+; \quad (P^-)^2 = P^-.$$

The Laguerre functions are based on the paravector-valued functions

$$F(\underline{x}) = e^{-|x|}\,|x|^\alpha\, P^+ \quad \text{and} \quad G(\underline{x}) = e^{-|x|}\,|x|^\alpha\, P^-; \quad \alpha \in \mathbb{R},$$

are real-analytic in the open connected domain $\mathbb{R}^m \backslash \{\underline{0}\}$ in $\mathbb{R}$. Their CK-extensions can be written as

$$F^*(x_0, \underline{x}) = e^{-|\underline{x}|} \sum_{k=0}^{\infty} \frac{x_0^k}{k!}\, |x|^{\alpha-2k} \left(L_{k,m,\alpha}^{+,+}(\underline{x}) P^+ + L_{k,m,\alpha}^{+,-}(\underline{x}) P^-\right)$$

and analogously

$$G^*(x_0, \underline{x}) = e^{-|\underline{x}|} \sum_{k=0}^{\infty} \frac{x_0^k}{k!}\, |x|^{\alpha-2k} \left(L_{k,m,\alpha}^{-,+}(\underline{x}) P^+ + L_{k,m,\alpha}^{-,-}(\underline{x}) P^-\right),$$

which exist in an open connected and x_0-normal neighborhood Ω of $\mathbb{R}^m \backslash \{\underline{0}\}$ in $\mathbb{R}^{m+1}$. By definition F^* satisfies in Ω

$$F^*(0, \underline{x}) = e^{-|x|}|x|^\alpha P^+ \quad \text{and} \quad (\partial_{x_0} + \partial_{\underline{x}}) F^*(x_0, \underline{x}) = 0.$$

From that the following recurrence relation is derived

$$\begin{aligned} L_{k+1,m,\alpha}^{+,+}(\underline{x}) P^+ + L_{k+1,m,\alpha}^{+,-}(\underline{x}) P^- = & |\underline{x}|\, \underline{x} \left(L_{k,m,\alpha}^{+,+}(\underline{x}) P^+ + L_{k,m,\alpha}^{+,-}(\underline{x}) P^-\right) \\ & + -(\alpha - 2k)\underline{x} \left(L_{k+1,m,\alpha}^{+,+}(\underline{x}) P^+ + L_{k+1,m,\alpha}^{+,-}(\underline{x}) P^-\right) \\ & + \underline{x}^2 \partial_{\underline{x}} \left(L_{k+1,m,\alpha}^{+,+}(\underline{x}) P^+ + L_{k+1,m,\alpha}^{+,-}(\underline{x}) P^-\right) \end{aligned}$$

with $\quad L_{0,m,\alpha}^{+,+}(\underline{x}) = 1 \quad$ and $\quad L_{0,m,\alpha}^{+,-}(\underline{x}) = 0.$
Note that $L_{k,m,\alpha}^{+,+}(\underline{x})$ is a polynomial of degree $2k$ in $\underline{x}$, while $L_{k,m,\alpha}^{+,-}(\underline{x})$ is a polynomial of alternative degree $2k-1$ and $2k-2$ in $\underline{x}$. Furthermore, the Clifford–Laguerre polynomials $L_{k,m,\alpha}^{+,+}(\underline{x})$ and $L_{k,m,\alpha}^{+,-}(\underline{x})$ satisfy

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$$L^{+,+}_{k,m,\alpha}(\underline{x})P^+ + L^{+,-}_{k,m,\alpha}(\underline{x})P^- = (-1)^k e^{|x|}|x|^{2k-\alpha}\,\partial^k_{\underline{x}}\left(e^{-|\underline{x}|}|x|^\alpha P^+\right).$$

Similar formulae hold for the Clifford–Laguerre polynomials generated by the CK-extension G^*.

Definition 6 ([8]). For $\alpha > -m$ and $0 < l$, let be

$$\psi_{l,m,\alpha}(\underline{x}) = \left(L^{+,+}_{k,m,\alpha}(\underline{x})P^+ + L^{+,-}_{k,m,\alpha}(\underline{x})P^-\right)|\underline{x}|^\alpha e^{-|\underline{x}|} = (-1)^l\partial^l_{\underline{x}}\left(e^{-|\underline{x}|}|\underline{x}|^{\alpha+2l}P^+\right).$$

$\psi_{l,m,\alpha} \in L^1(\mathbb{R}^m)$ and have zero momentum. For $\alpha > \frac{-m}{2}$ and $0 < l$, it is obtained that $\psi_{l,m,\alpha} \in L^2(\mathbb{R}^m)$.

Definition 7 ([8]). Let $f \in L^2(\mathbb{R}^m)$, then its Clifford–Laguerre CWT is defined by

$$T_{l,m,\alpha}f(a,\underline{b},s) = \langle\psi^{a\underline{b},s}_{l,m,\alpha}, f\rangle = \int_{\mathbb{R}^m}\overline{\psi^{a\underline{b},s}_{l,m,\alpha}(\underline{x})}\,f(\underline{x})\,d\underline{x},$$

where $\alpha > -\frac{m}{2}$ and $l > 0$,

$$\psi^{a\underline{b},s}_{l,m,\alpha}(\underline{x}) = \tfrac{1}{a^{m/2}}s\psi_{l,m,\alpha}\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right)\overline{s},$$

with $a \in \mathbb{R}_+$, $\underline{b} \in \mathbb{R}^m$ and $s \in \mathrm{Spin}\,(m)$.

Cauchy Wavelets

Cauchy wavelets had been constructed by J. Cnops [13]. The basic construction is again based on a generalization of complex analysis into higher dimensions. First, Cauchy kernel is written in an alternative way:

$$E(y,x) = E(y_0+\underline{y}, x_0+\underline{x}) = E(y-x) = \frac{1}{A_{m+1}}\frac{y-x}{|y-x|^{m+1}}$$

and the Cauchy transform can be written as

$$\mathcal{C}f(y) = \langle E(y,.), f\rangle_{L^2}.$$

Choosing the mother wavelet $\psi(\underline{x}) = E(1,\underline{x})$ and for $a > 0$ and $\underline{u} \in \mathbb{R}$ the wavelets are $\psi_{a,\underline{u}}(\underline{x}) = a^{m/2}E(\underline{u}+a,\underline{x})$ and the wavelet transformation of a function f is

$$(\mathrm{WT}f)(a,\underline{u}) = a^{m/2}(\mathcal{C}f)(\underline{u}+a).$$

By defining the radially symmetric wavelet as

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$$\psi_0(\underline{x}) = 2[E(1, \underline{x})]_0 = \frac{2}{A_{m+1}|1 - \underline{x}|^m},$$

it follows that $\mathcal{F}\psi_0(\underline{\xi}) = 2[(\mathcal{F}E)(1, \underline{\xi})]_0 = \frac{1}{\sqrt{2\pi}^m} e^{-|\underline{\xi}|}$. Therefore ψ_0 does not satisfy the admissibility condition:

$$(2\pi)^m \int_0^\infty |\mathcal{F}\psi(t)|^2 \frac{dt}{t} < \infty,$$

which means that the wavelet transformation is not a bounded operator. But this is no surprise because the Cauchy integral represents monogenic functions. But it can be proven that the Hardy spaces $H^{2,\pm}(\mathbb{R}^m)$ are invariant under the action of the wavelet transformation of the motion group $G = \mathbb{R}_0^+ \times \mathbb{R}^m$ with multiplication rule

$$(a, \underline{u})(b, \underline{v}) = (ab, \underline{u} + a\underline{v}).$$

The action of an element of G is given by

$$g(a, \underline{u})\underline{x} = a\underline{x} + \underline{u}$$

and leaves $\mathbb{R}^m$ invariant. Therefore, with the wavelet ψ_0 two (mother) wavelets $\psi_0^\pm$ can be associated which are the projections of ψ_0 onto $H^{2,\pm}$, respectively. Then

$$\phi_0^+(\underline{x}) = E(1, \underline{x}) \quad \text{and} \quad \psi_0^-(\underline{x}) = -E(-1, \underline{x}).$$

To obtain generalized Cauchy transform the starting point is the wavelet given by its Fourier transform

$$\hat{\psi}^\beta(\underline{\xi}) = (2\pi)^{-m/2} |\underline{\xi}|^\beta e^{-|\underline{\xi}|},$$

for real $\beta > 0$. It is possible to take β complex with $Re\beta > 0$. To generalize the Cauchy transform the following projections are used

$$\hat{\psi}^{\beta,\pm}(\underline{\xi}) = \frac{1}{2}\left(1 \pm i \frac{\underline{\xi}}{|\underline{\xi}|}\right) \hat{\psi}^\beta.$$

Theorem 8 (Cauchy Wavelets, [13]).

1. *The Cauchy wavelets have the explicit form*

$$a^{-\beta-m/2} \psi_{a,\underline{u}}^{\beta,+}(\underline{x}) = \frac{1}{2(2\pi)^m} \int_{\mathbb{R}^m} e^{i\langle \underline{\xi}, \underline{u}-\underline{x}\rangle} \left(1 + i \frac{\underline{\xi}}{|\underline{\xi}|}\right) |\underline{\xi}|^\beta e^{-a|\underline{\xi}|} \, d\underline{\xi} = \phi^\beta(a, \underline{u} - \underline{x}).$$

2. *The function ϕ^β is monogenic and homogeneous of degree $-\beta - m$ in the variable $a + \underline{u} - \underline{x}$, and is invariant under rotations in $\mathbb{R}^m$.*

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3. *The Cauchy wavelets $\phi(a, \underline{y})$ can be extended to monogenic functions in the domain $a > -|\underline{y}|$. In the domain $a > |\underline{y}|$ they are defined using the hypergeometric series.*
4. *If $\beta > 0$ is an integer ϕ^β is a β-th derivative of the Cauchy kernel, and can therefore be extended to $\mathbb{R}^m \backslash \{0\}$.*

The monogenic wavelet transform of a function f for the wavelet ψ^β is defined by

$$(\mathcal{C}^{\beta,+} f)(a, \underline{u}) = a^{-\beta-m/2} \int_{\mathbb{R}^m} \overline{\hat{\psi}^\beta_{a,\underline{u}}(\underline{\xi})}\, \hat{f}(\underline{\xi})\, d\underline{\xi} = a^{-\beta-m/2} \langle \hat{\psi}^\beta_{a,\underline{u}}, \hat{f} \rangle_{L^2} = a^{-\beta-m/2} \langle \psi^\beta_{a,\underline{u}}, f \rangle_{L^2}.$$

For more details see [13]. The generalized Cauchy wavelets give an insight in specific spaces of monogenic functions. Recall the Cauchy–Riemann operator $D = \partial_{x_0} + \partial_{\underline{x}}$. Let Ω be a domain in $\mathbb{R}^{m+1}$, and let α be a strictly positive, sufficiently smooth function, the so-called conformal weight. Equipped with the metric $ds^2_\alpha = ds^2_E/\alpha^2$, where ds_E is the Euclidean metric, Ω becomes a conformally flat manifold. The Cauchy–Riemann operator in this case is given by

$$D_\alpha f(x) = \alpha(x)^{(2+m)/2} D(\alpha(x)^{-m/2} f(x)).$$

A function satisfying $D_\alpha f(x) = 0$ is called monogenic for α. Since any function f, monogenic for α can be written as $\alpha^{(m-1)/2} g$, where g is Euclidean monogenic, the (local) theory of monogenic functions on conformally flat manifolds is equivalent with the (local) theory of monogenic functions on Euclidean space.
The specific case of interest here is the half space $P = \{x : x_0 > 0\}$ with the Poincaré metric $ds^2_P = ds^2_E/x^2_0$. The Cauchy–Riemann operator D_P of this manifold is given by $D_P = x_0^{(2+m)/2} D x_0^{-m/2}$.

Bergman spaces (which are Hilbert modules) B_μ are spaces of functions which are monogenic in the upper half space and which have finite norm for the inner product

$$[f, g]_\mu = \int_0^\infty a^\mu \langle f(a + \cdot), g(a + \cdot) \rangle_{L^2}\, da.$$

The resulting norm will be denoted by $[[f]]_\mu$. This norm is different from the following norm. Consider for any real μ and an arbitrary Clifford-valued function on the half space f the norm (which may be infinite)

$$||f||_\mu := \sup_{a>0} \frac{1}{a^\mu} ||f(a + \cdot)||_{L^2}.$$

Definition 8 (Function Spaces, [13]). For fixed $\mu \in \mathbb{R}$ let be

1. H_μ the space of functions satisfying $D_P f = 0$ with finite norm $||f||_\mu$,
2. M_μ the space of monogenic functions with finite norm $||f||_\mu$.

The Hardy spaces H_μ are introduced by Gilbert and Murray in [23]. They ask the question whether these Hardy spaces are trivial. Using the wavelet transform there can be given a complete answer to this question.

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Since any solution of $D_P f = 0$ can be written as $f = a^{(m-1)/2} g$, where g is monogenic, H_μ is isomorphic and isometric with the space $M_{\mu+(1-m)/2}$.

Theorem 9 ([13]).

1. *For $\mu > 0$ the space M_μ is trivial. For $\mu \leq 0$ it holds $B_{-2\mu-1} \subset M_\mu$.*
2. *Let $\beta > 0$. Then $\mathcal{C}^{\beta,+} : L^2(\mathbb{R}^m) \to B_{2\beta-1}$ is, up to a constant, an isometry.*
3. *The reproducing kernel of $B_{2\beta-1}$ is given by*

$$\frac{2^{2\beta}}{\Gamma(2\beta)\,(a_1+a_2)^{2\beta+m}}\psi^{2\beta,+}\left(\frac{\underline{u}_1-\underline{u}_2}{a_1+a_2}\right) = \frac{2^{2\beta}}{\Gamma(2\beta)}\phi^{2\beta}(a_1+a_2, \underline{u}_1-\underline{u}_2),$$

   where $\phi^{2\beta}$ is a generalized Cauchy wavelet.
4. *H_μ is not empty if $\mu \leq -1+m$.*

CWTs on the Unit Sphere

Wavelets on the Euclidean space are generated by translations, dilations, and rotations. For the sphere it is a specific problem to define an appropriate dilation. This problem has been solved by W. Freeden and coworkers [22] and by J.-P. Antoine and P. Vandergheynst [2], they used the conformal group of S^{m-1} and its Iwasawa decomposition. This approach is restricted to pure dilations on the unit sphere. The solution for the general case was given by M. Ferreira in his thesis [20]. Both approaches can be unified in a general and constructive way as wavelets on Lie groups and homogeneous spaces. This general approach was given by S. Ebert and J. Wirth in [17, 18]. In the context of Clifford analysis similar constructions can be done by defining monogenic functions via irreducible representations of the Spin group. Such constructions were already considered by M. Mitrea, V. Kisil and J. Cnops for the case of $\mathbb{R}^m$ and Hardy spaces, see [13, 14, 25, 26, 29].

Conformal Group of the Unit Ball

Theorem 10 ([1,21]). *The group $\mathcal{M}(B^m)$ of all conformal mappings of the unit ball B^m onto itself admits the matricial representation*

$$\begin{pmatrix} u & \hat{v} \\ v & \hat{u} \end{pmatrix}, \quad u, v \in \Gamma(m) \cup \{0\}, \quad uv^* \in \mathbb{R}^m, \quad |u|^2 - |v|^2 = 1.$$

Via this matrix representation $\mathcal{M}(B^m)$ can be identified with the group $\mathrm{Pin}^+(1,m)$. Because the unit ball can be identified with the right coset $\mathrm{Pin}(m)\backslash\mathcal{M}(B^m)$, it follows that $\mathcal{M}(B^m) \sim \mathrm{Pin}(m) \times B^m$. Furthermore,

$$\begin{pmatrix} u & \hat{v} \\ v & \hat{u} \end{pmatrix} = |u| \begin{pmatrix} \frac{u}{|u|} & 0 \\ 0 & \frac{\hat{u}}{|u|} \end{pmatrix} \begin{pmatrix} 1 & \frac{\overline{u}}{|u|^2}\hat{v} \\ \frac{u^*}{|u|^2}v & 1 \end{pmatrix} = \frac{1}{\sqrt{1-|a|^2}} \begin{pmatrix} w & 0 \\ 0 & \hat{w} \end{pmatrix} \begin{pmatrix} 1 & \hat{a} \\ a & 1 \end{pmatrix},$$

where $w = \frac{u}{|u|}$, $a = \frac{u^*}{|u|^2}v$, $\sqrt{1-|a|^2} = |u|^{-1}$. Due to $uv^* \in \mathbb{R}^m$ it follows $u^*v \in \mathbb{R}^m$ and therefore $a \in \mathbb{R}$. Moreover, $|u|^2 - |v|^2 = 1$ implies $|a|^2 = \frac{|u|^2}{|v|^2} < 1$.

Lemma 2. *Each $a \in B^m$ can be described as*

$$a = sre_m\overline{s},$$

where $r \in [0, 1)$ and $s = s_1 \cdots s_{m-1} \in \mathrm{Spin}(m)$, with

$$s_i = \cos\frac{\theta_i}{2} + e_{i+1}e_i \sin\frac{\theta}{2}, \quad i = 1, \ldots, m,$$

where $0 \leq \theta_1 < 2\pi, 0 \leq \theta_i < \pi, i = 2, \ldots, m-1$.

Since $a \in B^m$, $\hat{a} = \overline{a} = -a$. Therefore,

$$\begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 & -sre_m\overline{s} \\ sre_m\overline{s} & 1 \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 & -re_m \\ re_m & 1 \end{pmatrix} \begin{pmatrix} \overline{s} & 0 \\ 0 & \overline{s} \end{pmatrix}$$

and hence, the global Cartan or KAK decomposition for an arbitrary element of $\mathrm{Spin}^+(1, n)$ is

$$\begin{pmatrix} u & \hat{v} \\ v & \hat{u} \end{pmatrix} = \frac{1}{\sqrt{1-r^2}} \begin{pmatrix} ws & 0 \\ 0 & \hat{w}s \end{pmatrix} \begin{pmatrix} 1 & -re_m \\ re_m & 1 \end{pmatrix} \begin{pmatrix} \overline{s} & 0 \\ 0 & \overline{s} \end{pmatrix}$$

Specifically, the second matrix on the left-hand side corresponds to the subgroup $\mathrm{Spin}(1, 1)$ of hyperbolic rotations.

Instead of working with matrices the following transformations will be used:

- $R_s(x) := sx\overline{s}$, $\quad s \in \mathrm{Spin}(m)$, which denotes a rotation in $\mathbb{R}^m$;
- $\varphi_a(x) := (x-a)(1+ax)^{-1}$, $\quad a \in B^m$, which is a Möbius transformation.

For the construction of a spherical CWT, motions (rotations and translations) and dilations on S^{m-1} have to be defined. Translations correspond to rotations of the homogeneous space $\mathrm{Spin}(m)/\mathrm{Spin}(m-1)$ and rotations can be realized as rotations around a certain axis on the sphere. Thus, both translations and rotations can be associated with the action of $\mathrm{Spin}(m)$ on S^{m-1}. Dilations arise from Möbius transformations of type $\varphi_a(x)$. For $f \in L^2(S^{m-1}, dS)$, where dS is the usual Lebesgue surface measure on S^{m-1}, the rotation and spherical dilation operators are defined by

$$R_s f(x) = f(\overline{s}xs), \quad s \in \mathrm{Spin}(m), \quad \text{and} \quad D_a f(x) = \left(\frac{1-|a|^2}{|1-ax|^2}\right)^{\frac{m-1}{2}} f(\varphi_{-a}(x)), \quad a \in B^m,$$

where $\left(\frac{1-|a|^2}{|1-ax|^2}\right)^{\frac{m-1}{2}}$ is the Jacobian of $\varphi_{-a}(x)$ in S^{m-1}. From these operators it is possible to construct the unitary representation of the group $\mathrm{Spin}^+(1, m)$ in $L^2(S^{m-1}, dS)$

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$$U(s,a)f(x) = R_s D_a f(x) = \left(\frac{1-|a|^2}{|1-ax|^2}\right)^{\frac{m-1}{2}} f(\varphi_{-a}(\bar{s}xs)).$$

J.-P. Antoine and P. Vandergheynst solved this problem for the special case of Spin(1,1), and M. Ferreira gave the solution for the general case [20]. An entire class of sections σ can be obtained by

$$\sigma(te_m) = te_m \oplus g(t)e_{m-1} = \left(0,\ldots,0,\ \frac{g(t)(1-t^2)}{1+(g(t))^2},\ \frac{t(1+g(t)^2)}{1+(tg(t))^2}\right),$$

where $g : (-1,1) \to (-1,1)$ is called the generating function of the section. Depending on the properties of the function g there are sections that are Borel maps and also smooth sections.

Theorem 11 (Continuous Wavelet Frames on the Sphere, [21]). *Let $\psi \in L^2(S^{m-1})$ such that the family $\{R_s D_{te_m}\psi,\ s \in \mathrm{Spin}(m),\ t \in (-1,1)\}$ is a continuous frame, that is, there exist constants $0 < A \leq B < \infty$ such that*

$$A\,||f||^2 \leq \int_{\mathrm{Spin}(m)} \int_{-1}^{1} |\langle f, R_s D_{te_m}\psi\rangle|^2\, d\mu(te_m)\, d\mu(s) \leq B\,||f||^2, \quad \forall f \in L^2(S^{m-1}),$$

where the measure $d\mu(te_m)$ is equivalent to the measure $d\mu(u) = \frac{du}{u^m}$, by means of the bijection given by $t = \frac{u-1}{u+1}$, $u \in \mathbb{R}^+$ and $d\mu(s)$ the invariant measure on $\mathrm{Spin}(m)$. Then ψ is an admissible function for any continuous section W_σ and the system $\{R_s D_{\sigma(te_m)}\psi,\ s \in \mathrm{Spin}(m),\ t \in (-1,1)\}$ is also a continuous frame.

As a consequence of this theorem it can be proven (for details see [21]) that there exists admissible $\psi \in L^2(S^{m-1})$.

Definition 9 (Generalized Spherical CWT, [21]). For an arbitrary section W_σ and an admissible function $\psi \in L^2(S^{m-1})$ the generalized spherical CWT is defined as

$$W_\psi[f](s,\sigma(te_m)) = \langle R_s D_{\sigma(te_m)}\psi, f\rangle = \int_{S^{m-1}} \overline{R_s D_{\sigma(te_m)}\psi(x)}\, f(x)\, dS.$$

The wavelet transform is a mapping from $L^2(S^{m-1}, dS)$ to $L^2(\mathrm{Spin}(m) \times \sigma(te_m),\ d\mu(s)\, d\mu(te_m))$. Moreover, there exists a reconstruction formula and also a Plancherel Theorem.

These ideas using Möbius transformations and the Spin group had also been used for wavelets and frames over the unit ball [10]. Fortunately, there are also nice applications of this approach. In [11] Gabor frames and sparse recovery principles based on hypercomplex function theory had been used for the inversion of a noisy Radon transform on SO(3).

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Monogenic Wavelets and Monogenic Signals

The monogenic signal introduced by M. Felsberg and G. Sommer is a generalization of Gabor's analytic signal. Its application to image processing is based on the following considerations and mainly pursued by M. Unser, D. Van De Ville and coworkers. The reasons for that and the construction of suitable wavelets are well described in [31], where they present a functional framework for the design of tight steerable wavelet frames. In image processing the Hilbert operator in Clifford analysis is called Riesz operator because the components are built up by the Riesz potentials.

The steerable pyramid is a multi-orientation, multi-scale image decomposition that was developed by E.P. Simoncelli and others. It is a wavelet-like representation whose analysis functions are dilated versions of a single directional wavelet. Steerability refers to the property that the underlying wavelets can be rotated to any orientation by forming a suitable linear combinations of a primary directional wavelets. The Riesz transform maps a primal isotropic wavelet frame of $L^2(\mathbb{R}^m)$ into a directional wavelet whose basis functions are steerable. S. Held in [24] constructed monogenic wavelets-frames. These constructions are all based on the fact that a monogenic function can be constructed from a scalar-valued function by using the Hilbert transform, see Theorem 4. Because the Hilbert transform and their components are also called Riesz transforms these authors call the Hilbert transform Riesz transform.

The Clifford–Hilbert module H_n was already introduced in Example 1. For more information on frames see, for example, [12].

Definition 10 ([24]). Let $\{f_k\}_{k\in\mathbb{N}} \subset H_n$ be a sequence such that there exist constants $0 < A \leq B < \infty$, satisfying the frame inequality

$$A\,||f||^2 \leq \sum_{k\in\mathbb{N}} |\langle f,\ f_k\rangle|^2 \leq B\,||f||^2, \quad \forall f \in H_n.$$

Then $\{f_k\}_{k\in\mathbb{N}}$ is called a Clifford frame for H_n. A is called a lower frame bound and B is called an upper frame bound.

- A Clifford frame is called tight, iff $A = B$ is possible.
- Let T be the synthesis operator of the frame $\{f_k\}_{k\in\mathbb{N}}$. Then the Clifford frame operator is defined by

$$S : H_n \to H_n, \quad f \mapsto Sf := TT^*f = \sum_{k\in\mathbb{N}} \langle f,\ f_k\rangle f_k.$$

- A sequence $\{g_k\}_{k\in\mathbb{N}} \subset H_n$ is called a dual frame of the Clifford frame $\{f_k\}_{k\in\mathbb{N}}$ iff $\{g_k\}_{k\in\mathbb{N}}$ is a Clifford frame and the frame decomposition

$$f = \sum_{k\in\mathbb{N}} \langle f,\ f_k\rangle g_k = \sum_{k\in\mathbb{N}} \langle f,\ g_k\rangle f_k$$

is fulfilled for all $f \in H_n$.

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Definition 11 (Clifford Wavelet Transform, [24]). Let $\psi \in L^2(\mathbb{R}^n)_n$ and let D be a dilation matrix. Then

$$\{D^j T_k \psi\}_{j,k},$$

where $j \in \mathbb{Z}$, $T_k f(\underline{x}) := f(\underline{x} - \underline{k})$, $\underline{k} \in \mathbb{Z}^n$ and a dilation matrix D, i.e. a matrix D with eigenvalues greater than one, via $Df(x) = |\det(D)|^{1/2} f(Dx)$, is called a wavelet system with mother wavelet ψ. Let $f \in L^2(\mathbb{R}^n)_n$ then

$$W_\psi(f) := \{\langle f, D^j T_x \psi\rangle\}_{j,k}$$

is called the Clifford wavelet transform.

Remark 3. S. Held called it "hypercomplex" wavelet transform instead of Clifford wavelet transform.

Definition 12 (Monogenic Wavelet, [24]). Let $\psi \in L^2(\mathbb{R}^n)$ (real-valued) be a mother wavelet for $L^2(\mathbb{R}^n)$ with respect to a dilation matrix D. Then the monogenic wavelet transform system $\{D^j T_k \psi_m\}_{j,k}$ corresponding to ψ is generated by the monogenic mother wavelet

$$\psi_m = \psi + \mathcal{H}\psi = \psi + \sum_\alpha e_\alpha \mathcal{H}_\alpha \psi,$$

where $\mathcal{H}$ is the Hilbert transform = Riesz transform. An element $D^j T_k \psi$ of the monogenic wavelet system is called a monogenic wavelet. The Clifford wavelet transform W_{ψ_m} is called the monogenic wavelet transform.

The following theorem proves the ability of the Riesz = Hilbert transform to build up monogenic wavelets and frames.

Theorem 12 (Riesz Transforms of Frames, [24]). *Let $\{f_k\}_{k\in\mathbb{N}}$ be a frame for $L^2(\mathbb{R}^2, \mathbb{R})$ with frame bounds A and B. Then*

1. *$\{f_k\}_{k\in\mathbb{N}}$ is a Clifford frame for $L^2(\mathbb{R}^n, \mathbb{C})_n \sim L^2(\mathbb{R}^n, \mathbb{C}_n)$ with the same frame bounds A and B.*
2. *The Riesz transformed frame $\{\mathcal{H} f_k\}_{k\in\mathbb{N}}$ is a Clifford frame for $L^2(\mathbb{R}^n, \mathbb{C})_n$ with the same frame bounds A and B.*

This means that the Riesz transform of a frame for $L^2(\mathbb{R}^n, \mathbb{R})$ yields a Clifford frame for $L^2(\mathbb{R}^n, \mathbb{C}_n)$.

The next theorem shows that it is possible to calculate a monogenic signal from the coefficients of the monogenic wavelet.

Theorem 13 (Monogenic Wavelet Frame, [24]). *Let $\psi \in L^2(\mathbb{R}^n)$ generate a wavelet frame for $L^2(\mathbb{R}^n$ with respect to a dilation matrix D with frame bounds $0 < m \leq M < \infty$. Then the* *monogenic wavelet generates a wavelet frame for $L^{2,+}(\mathbb{R}^n)$ with frame bounds $2m$ and $2M$. Let the dilation matrix constitute a rotated dilation, i.e. $D = D_d\,\rho$, where $d > 1$ and $\rho \in \mathrm{SO}(n)$. Then the wavelet transform of a function in $L^2(\mathbb{R}^n)$ satisfies*

$$W_{\psi_m} f(t, j) = W_\psi f(t, j) + \sum_{l,k} e_l (\rho^j)_{k,l} W_\psi \mathcal{H}_k f(t, j).$$

Conclusion and Future Directions

Clifford wavelets and frames have been proven to be very useful in a lot of applications. Up to now these wavelets are constructed inside the L^2-theory. More important results should be obtained by an L^p-theory and the combination with properties of the Hilbert transform. Specifically compressed sensing for monogenic signals should make the theory further applicable to 3D problems and large data problems.

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