Operator Theory
DOI 10.1007/978-3-0348-0692-3_14-1

Monogenic Signal Theory

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Abstract

This survey is intended as a mathematical overview of the monogenic signal theory, its current developments as well as of its applications to various problems. Several approaches to the concept of monogenic signal are given, followed by a short discussion on its advantages and drawbacks. Among those, one highlights the definitions arising from different concepts of Hilbert transforms, as well as the concept of the monogenic signal as a solution to a boundary value problem. In the last part, some recent developments in the field are presented.

Introduction

In 1946 Gabor [9] introduced the concept of the (one-dimensional) analytic signal. In signal processing, the analytic signal (linked to a real-valued function of a real variable) facilitates many mathematical manipulations since it provides a complex extension of the 1D signal via the well-known Hilbert transform. The basic idea is that the negative frequency components of the Fourier transform of a real-valued function are superfluous, due to the Hermitian symmetry of its spectrum. Hence, these negative frequency components can be discarded with no loss of information, providing one is willing to deal with a complex-valued function instead. The complex extension of the signal provided by the Hilbert transform turns out to be a useful tool for extracting interesting intrinsic features and attributes of the real univariate signal from which it originates. Mathematically, the analytic signal can be understood as a boundary value of a complex analytic function. Since in image processing one deals with two-dimensional signals a higher-dimensional counterpart of the analytic signal is required. Such generalizations were constructed by several persons in the last 20 years (see Hahn [12], Bülow [2], Larkin et al. [16], Felsberg and Sommer [8] and others). One of the most interesting of these generalizations is the so-called monogenic signal. Since it corresponds to a boundary value of a monogenic function it inherits many of its properties, chief of which is the rotation invariance. Originally it was introduced by two different approaches at almost the same time. Larkin et al. [16] introduced the so-called spiral phase quadrature filter while Felsberg and Sommer [8] introduced the monogenic signal, both based on the application of Riesz transforms as substitutes for the 1D-Hilbert transform. One must also add that Larkin, Bone and Oldfield, and Felsberg and Sommer worked independently of each other. At the end of their

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DOI 10.1007/978-3-0348-0692-3_14-1

paper, Larkin et al. made a note in which they comment on the merits of both approaches and on the coinage of term *monogenic signal* by Felsberg and Sommer, after discussions with S. Gull, A. Lasenby, and C. Doran.

Since then, it has become a useful tool for extracting intrinsically 1D-features from images. There are in fact several reasons for this. First and foremost, since it is quaternion-valued it provides a meaningful concept of a phase function, which is divided into two parts: a *phase direction* and a *phase function*. While the phase direction points in the same direction as the gradient the derivative of the phase function (sometimes also called *instantaneous frequency*) gives a measure of the change in said direction. Additionally, the abovementioned rotational invariance allows the so-called steerability of the quaternionic Hilbert transform.

This survey tries to give an idea about the monogenic signal from a mathematical point of view. Here, one describes its link to quaternionic and Clifford analysis together with some of its properties which are important for applications, such as the frame property or the link with the Radon transform. Furthermore, one will shortly describe in the end modern adaptations.

The literature on monogenic signals is rather vast, particularly from the application point of view [4–6, 10, 15, 22–24, 30]. This also means that one is unable to cite everything or everybody from this field a fact for which one apologizes in advance. Furthermore, the text is written with the abovementioned mathematical connections in mind, which also means that it will be rather short on how certain object can be interpreted/discussed in image processing, optics, and other practical fields. One believes that this part is easier to find in the literature and also relatively easy to grasp when the general scheme is understood. Special thanks to Bernstein, Bouchot, Reinhardt, and Heise for allowing the use of some of their images from [1].

Hilbert Transform in Higher Dimensions

A Short Review

A fundamental problem in signal processing is the extraction of different features from a given real modulated signal $f(t) = a(t)\cos\theta(t)$. But it is a well-known fact that the simple extraction of the amplitude $a(t)$ is an ill-posed problem in the neighborhood of $\cos(\theta(t)) = 0$. The standard solution is to extend the real signal to a complex one by means of the Hilbert transform.

$$Hf(t) := \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{f(\tau)}{t-\tau} d\tau. \tag{1}$$

If the amplitude $a(t)$ of the signal is narrowband when compared to f, then the analytic signal $Z(t) = (I + iH)f(t)$ (written in terms of its real and imaginary parts) can be expressed in polar coordinates as

$$\begin{aligned} Z(t) &= f(t) + iHf(t) \\ &= a(t)\left[\cos(\theta(t)) + i\sin(\theta(t))\right] \\ &= a(t)e^{i\theta(t)} \end{aligned}$$

and the estimation of the amplitude becomes a well-posed problem, with the amplitude of the signal being given by

DOI 10.1007/978-3-0348-0692-3_14-1

$$a(t) = |Z(t)|$$

and its phase

$$\theta(t) = \arctan \frac{Hf(t)}{f(t)}$$

being the angular part.

Two key points should be remarked. First, notice that

$$Hf(t) = h * f(t), \tag{2}$$

with the convolution kernel $h(t) := \frac{1}{\pi t}$, or impulse response, being of slow decay. Moreover, for a signal $f \in L^2(\mathbb{R})$ under the action of the Fourier transform

$$\mathcal{F}f(\xi) := \int_{-\infty}^{+\infty} f(t)e^{-i\xi t}dt, \quad \xi \in \mathbb{R},$$

one has that the symbol, or Fourier multiplier,

$$\mathcal{F}(Hf)(\xi) = -i \ \mathrm{sgn}(\xi)\mathcal{F}f(\xi) = -i\frac{\xi}{|\xi|}\mathcal{F}f(\xi), \tag{3}$$

is an isometry on $L^2(\mathbb{R})$, that is to say,

$$|Hf(\xi)| = |\mathcal{F}f(\xi)|, \quad \xi \neq 0 \quad \Rightarrow \quad \|Hf\|_2 = \|f\|_2 \quad \text{(energy preservation)};$$
$$H^2 f(t) = -f(t) \quad \Rightarrow \quad H^* = -H.$$

Second, the numeric calculation of the discrete counterpart of Hilbert transform requires a truncation procedure. The scientific-computing software MATLAB has a routine (`hilbert( )` function) for the computation of a band-limited discrete-in-time analytic signal $Z = f_r + iHf_r$. The implementation of this routine uses the fast Fourier transform (FFT) in three steps:

- computation of the FFT of $f_r[n]$;
- elimination of the coefficients corresponding to the negative frequencies of the signal;
- performing inverse FFT.

On the basis of this routine is the fact that the Fourier transform of the complex signal obtained via the discrete Hilbert transform, $Z[n] = f_r[n] + i(Hf_r)[n]$, vanishes at all negative frequencies. However, as the Hilbert transform has a discontinuity at the origin, the usage of FFT introduces an aliasing phenomena due to the Gibbs effect and should be used with care.

One aims at presenting different extensions of the analytical signal to higher dimensions in the next subsections. To this end one proceeds with the formulation of the analytic signal as a boundary value problem for an analytic/holomorphic function. It is interesting to remark that engineers complain that such formulation is useless as it does not provide an explicit working formula for the Hilbert transform—a reason why one hardly sees it in recent books. Nevertheless, it becomes

DOI 10.1007/978-3-0348-0692-3_14-1

a keystone when one wants to obtain a mathematical framework which allows to understand the common ground of the generalizations of an analytic signal to higher dimensions and desires to establish its properties. It should be pointed out that the terminology "Riesz transform" is often used by the engineers to designate the Hilbert transform.

Formulation of the Analytic Signal as a Boundary Value Problem

The mathematical formulation of the analytic signal as a boundary value problem relays in the observation that a complex-valued analytic function $F = \mathrm{Re}F + i\ \mathrm{Im}F$ in upper half complex plane $\mathbb{C}^+$ satisfies the Cauchy–Riemann equations

$$\begin{cases} \partial_x \mathrm{Re}F = \partial_y \mathrm{Im}F \\ \partial_y \mathrm{Im}F = -\partial_x \mathrm{Re}F \end{cases}$$

and, hence, it is analytic there. Conversely, given a real-valued function f the associated Riemann–Hilbert problem with respect to the complex parameter $z = x + iy$,

$$\begin{cases} \partial_{\bar{z}} F = 0, \quad z \in \mathbb{C}^+, \\ \mathrm{Re}F|_{\mathbb{R}} = f \end{cases} \tag{4}$$

provides an analytic complex-valued function F in $\mathbb{C}^+$ *which has f as its real part when restricted to the real line*. The real part $\mathrm{Re}F$ is uniquely obtainable via convolution with the Poisson kernel in the upper half plane while the imaginary part $\mathrm{Im}F$ is recoverable by Cauchy–Riemann equations so that we have $Hf(x) = \mathrm{Im}F(x, 0)$ which, is up to a constant, the desired Hilbert transform of f. However, this formulation does not provide a clear insight on the properties of the Hilbert transform.

Another solution to problem (4) is given by the Cauchy integral formula

$$Cf(z) := \frac{1}{2\pi i}\ p.v. \int_{-\infty}^{+\infty} \frac{f(\tau)}{\tau - z} d\tau. \tag{5}$$

This solution is unique up to a constant (traditionally fixed by assigning a value to $F(z_0)$ at a given point $z_0 \in \mathbb{C}^+$) but it has the advantage of further allowing the usage of the machinery of *singular integral operators*. If one considers the trace of the Cauchy operator C, i.e. its non-tangential limit of $z \in \mathbb{C}^+$ to $x = \mathrm{Re}(z) \in \mathbb{R}$

$$\mathrm{tr}Cf(x) = \lim_{\mathbb{C}^+ \ni z \overset{n.t.}{\to} x \in \Gamma} Cf(z), \quad \Gamma = \partial\mathbb{C}^+, \tag{6}$$

then one arrives at the so-called Plemelj–Sokhotzki formula

$$\mathrm{tr}Cf = \frac{1}{2}(I + iH)f := P_\Gamma f. \tag{7}$$

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Up to the factor $\frac{1}{2}$, formula (7) corresponds to the initial concept of an analytic signal as $Z = (I + iH)f$. Therefore, and based on this description, an analytic signal can be viewed as the boundary value of an analytic function in the upper half plane $\mathbb{C}^+$ and, thus, as an element of the Hardy space $H^2(\mathbb{C}^+)$.

Function Spaces

As seen before, the analytic signal can be introduced either via the action of the 1D Hilbert transform or in the form of a boundary value of a holomorphic function in the upper half plane. In order to generalize these ideas one has to recall several concepts.

First of all, the 1D Hilbert transform is defined by a convolution with a kernel distribution and hence, it commutes with translations. Moreover, its L^2-continuity is evident since the Fourier transform converts it into a multiplication operator. This already offers a hint that analytic signals in higher dimensions should be linked to singular integral operators of convolution type which enjoy similar properties.

Second of all, the approach via a boundary value problem raises the question of an appropriate definition of "holomorphy," as well as a more detailed study of the arising Cauchy integral operator and the subsequent application of the trace operator to it. In this section a particular attention will be given to the spaces of functions associated and the respective mapping properties of such operators.

Definition 1. The space $L^p(\mathbb{R}^n), 1 \leq p < \infty$, is defined as the set of all Lebesgue mensurable functions $f : \mathbb{R}^n \to \mathbb{C}$ with finite norm

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty. \tag{8}$$

The spaces $L^p(\mathbb{R}^n)$ are Banach spaces w.r.t. the above norm (8) and in the particular case of $L^2(\mathbb{R}^n)$ it is a Hilbert space with scalar product

$$\langle f, g \rangle := \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx. \tag{9}$$

Definition 2. We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ as

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-i\langle \xi, x \rangle} dx, \quad \xi \in \mathbb{R}^n. \tag{10}$$

The Fourier transform is a continuous and linear bijection on $\mathcal{S}$, with continuous inverse

$$\mathcal{F}^{-1}u(x) = \check{u}(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} u(\xi)e^{i\langle \xi, x \rangle} d\xi, \quad x \in \mathbb{R}^n.$$

The extension of the Fourier transform to $\mathcal{S}'$ is done by duality, as usual.

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Theorem 1. *Let $f, g \in L^2(\mathbb{R}^n)$. Then, it holds*

(i) (Parseval formula) $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$;
(ii) (Plancherel identity) $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$;
(iii) Convolution $\widehat{f * g} = \hat{f} * \hat{g}$;
(iv) $\widehat{\partial_{x_j} f}(\xi) = i\xi_j \hat{f}(\xi), \quad \widecheck{\partial_{\xi_j} u}(x) = -ix_j \check{u}(x)$.

Proof of the above results can be found in the literature, for example, in [20]. Also, the Plancherel identity implies that the Fourier transform is an isometry on $L^2(\mathbb{R}^n)$.

Definition 3 ([25]). The Hardy space $H^p(\mathbb{R}^{n+1}_+)$, $1 \leq p < \infty$, is defined as the space of vector valued functions $f : \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}$, $f = (f_0, f_1, \ldots, f_n)$, which satisfy the following Riesz system

$$\partial_{x_0} f_0 = \sum_{j=1}^{n} \partial_{x_j} f_j, \quad \partial_{x_j} f_k = \partial_{x_k} f_j, \text{ for } j \neq k, j, k = 1, \ldots, n, \tag{11}$$

in the upper half plane $\mathbb{R}^{n+1}_+ = \{(x_0, x), x_0 > 0,\ x \in \mathbb{R}^n\}$ and such that

$$\|f\|_{H^p} := \sup_{x_0>0} \int_{\mathbb{R}^n} |f(x_0, x)|^p dx < \infty. \tag{12}$$

Hence, $H^p(\mathbb{R}^{n+1}_+)$ can be also viewed as the space of the boundary values on the real axis of those vectorial functions satisfying to the Riesz system (11) and have bounded L^p-norm.

Based on these concepts one is going now to take a look at possible higher-dimensional generalizations of the analytic signal.

Hypercomplex Signals

As one knows an analytic signal is a boundary value of an analytic or holomorphic function. Any higher-dimensional extension should reflect that property. There are two accepted ways of generalizing the concept of holomorphic functions to higher dimensions. One belongs to the *theory of several complex variables* and the other is the focus of the *Clifford analysis*. As one will see, both provide useful generalizations of the analytic signal viewed as an elements of a Hardy space.

A Riemann–Hilbert Problem in $\mathbb{C}^2_+$ and the Partial Hilbert Transform

In the "upper half" space

$$\mathbb{C}^2_+ := \mathbb{C}^+ \times \mathbb{C}^+ = \{(z_1, z_2) \in \mathbb{C}^2 : z_j = x_j + iy_j,\ y_j \geq 0,\ j = 1, 2\},$$

consider the following Riemann–Hilbert problem

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$$\begin{cases} \partial_{\bar{z}_1} F = 0, \text{ on } \mathbb{C}^2_+ \\ \partial_{\bar{z}_2} F = 0, \text{ on } \mathbb{C}^2_+ \\ \mathrm{Re} F = f, \text{ in } \mathbb{R}^2 = \partial \mathbb{C}^2_+. \end{cases} \quad (13)$$

At this stage, one must point out that the domain is a poly-domain in the sense of several complex variables. Also, $\mathbb{R}^2$ corresponds to the Shilov boundary of the poly-domain $\mathbb{C}^2_+$. The solution can be expressed in the form of a Cauchy integral (c.f. [7] or [19])

$$Cf(z_1, z_2) = F(z_1, z_2) = \frac{1}{4\pi^2} p.v. \int_{\mathbb{R}^2} \frac{f(\xi_1, \xi_2)}{(\xi_1 - z_1)(\xi_2 - z_2)} d\xi_1 d\xi_2. \quad (14)$$

In this case the corresponding Plemelj–Sokhotzki formula is

$$\mathrm{tr} Cf(x_1, x_2) = \mathrm{tr} F(x_1, x_2) \quad (15)$$

$$= \frac{1}{4} \left[f(x_1, x_2) - \frac{1}{\pi^2} p.v. \int_{\mathbb{R}^2} \frac{f(\xi_1, \xi_2)}{(\xi_1 - x_1)(\xi_2 - x_2)} d\xi_1 d\xi_2 + i \frac{1}{\pi^2} \left(p.v. \int_{\mathbb{R}} \frac{f(\xi_1, \xi_2)}{\xi_1 - x_1} d\xi_1 + p.v. \int_{\mathbb{R}} \frac{f(\xi_1, \xi_2)}{\xi_2 - x_2} d\xi_2 \right) \right] \quad (16)$$

which, up to the factor $1/4 = (1/2)^2$, corresponds to the definition of an analytic signal in [11]. A close inspection on (16) reveals a *complex-valued Hilbert transform* in which its real part, or *total Hilbert transform*, is given by

$$H_3 f(x_1, x_2) = \frac{1}{\pi^2} p.v. \int_{\mathbb{R}^2} \frac{f(\xi_1, \xi_2)}{(\xi_1 - x_1)(\xi_2 - x_2)} d\xi_1 d\xi_2$$

while the imaginary part is decomposed into the sum of two *partial Hilbert transforms*

$$H_1 f(x_1, \cdot) = \frac{1}{\pi^2} p.v. \int_{-\infty}^{+\infty} \frac{f(\xi_1, \cdot)}{\xi_1 - x_1} d\xi_1, \quad H_2 f(\cdot, x_2) = \frac{1}{\pi^2} p.v. \int_{-\infty}^{+\infty} \frac{f(\cdot, \xi_2)}{\xi_2 - x_2} d\xi_2.$$

Notice that each partial Hilbert transform corresponds (again, up to a constant factor) to a classic 1D Hilbert transform for $f(x_1, \cdot)$, $f(\cdot, x_2)$, respectively. In Fig. 1 one observes the action of the two partial and the total Hilbert transforms onto the checkerboard image.

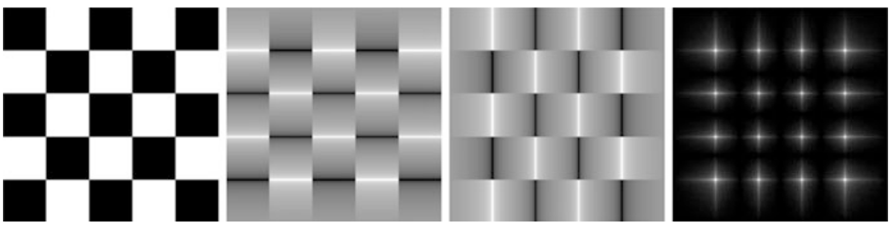

Fig. 1 Checkerboard image, together with its partial, and its total Hilbert transforms

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The behavior of the Plemelj–Sokhotzki formula (16) is more obvious in the Fourier domain. In this case the Fourier symbols of each Hilbert transform are independent of each other and one gets

$$\begin{aligned}\mathcal{F}(\mathrm{tr}Cf)(\xi_1,\xi_2) &= (1+i\ \mathrm{sgn}(\xi_1))(1+i\ \mathrm{sgn}(\xi_2))\mathcal{F}f(\xi_1,\xi_2)\\ &= [1-\mathrm{sgn}(\xi_1)\mathrm{sgn}(\xi_2)+i\ \mathrm{sgn}(\xi_1)+i\ \mathrm{sgn}(\xi_2)]\mathcal{F}f(\xi_1,\xi_2).\end{aligned}$$

However, such complex Hilbert transform as constructed by Hahn has the disadvantage of making it difficult to distinguish between directions. Based on such considerations, Bülow developed a similar approach via quaternions (c.f. [2]) taking into account that both imaginary units can be understood as elements of the quaternionic basis with multiplication rules $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, with variables

$$z_1 = x_1 + \mathbf{i}y_1, \quad \mathbf{z}_2 = x_2 + \mathbf{j}y_2.$$

Then the above Riemann–Hilbert problem (14) can be rewritten as

$$\begin{cases}\partial_{\bar{z}_1}F = 0, \text{ on } \mathbb{C}^2_+\\ F\partial_{\bar{\mathbf{z}}_2} = 0, \text{ on } \mathbb{C}^2_+\\ \mathrm{Re}F = f \text{ in } \mathbb{R}^2,\end{cases} \tag{17}$$

with emphasis in the fact that the second equation should be understood as $\partial_{\bar{\mathbf{z}}_2}$ being applied from the right (due to the non-commutative nature of the quaternions).

Again, for an arbitrary $f \in L^2(\mathbb{R}^2)$ the solution of (17) is given in terms of a Cauchy integral

$$C_\Gamma f(z_1,\mathbf{z}_2) := F(z_1,\mathbf{z}_2) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\frac{f(\xi_1,\boldsymbol{\xi}_2)}{(\xi_1-z_1)(\boldsymbol{\xi}_2-\mathbf{z}_2)}d\xi_1 d\boldsymbol{\xi}_2. \tag{18}$$

so that one gets the Plemelj–Sokhotzki formula

$$\begin{aligned}\mathrm{tr}C_\Gamma f(x_1,x_2) &:= \mathrm{tr}F(x_1,x_2)\\ &:= P_{\Gamma,\mathbf{i}}f(x_1,x_2)P_{\Gamma,\mathbf{j}}\\ &= \frac{1}{4}(I+\mathbf{i}H_1)f(x_1,x_2)(I+\mathbf{j}H_2)\\ &= \frac{1}{4}(f+\mathbf{i}H_1f+\mathbf{j}H_2f+\mathbf{k}H_3f)(x_1,x_2),\end{aligned}$$

where $H_3 = H_1H_2$. From this representation one can separate the previous partial Hilbert transforms of Hahn. Although $\mathrm{tr}C_\Gamma f$ is now a quaternionic-valued function, it still corresponds to a boundary value of a holomorphic function in two variables with an additional advantage of allowing to distinguish between the action of the different Hilbert transforms.

For the representation in the phase-space one has to keep in mind that one has two Fourier transforms: one with respect to the complex plane generated by $\mathbf{i}$, and the second with respect to the complex plane generated by $\mathbf{j}$. Taking into account the non-commutative nature of quaternions

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($\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$) one arrives at the so-called quaternionic Fourier transform [3].

$$\mathcal{QF}f(\xi_1, \xi_2) = \int_{\mathbb{R}^2} e^{-\mathbf{i}x_1\xi_1} f(x_1, x_2) e^{-\mathbf{j}x_2\xi_2} dx_1 dx_2, \tag{19}$$

with a correspondent representation in terms of Fourier symbols

$$\mathcal{QF}(\mathrm{tr}Cf) = [1 + \mathbf{i}\,\mathrm{sgn}(\xi_1)]\mathcal{QF}f\ [1 + \mathbf{j}\,\mathrm{sgn}(\xi_2)]. \tag{20}$$

The Correspondent Riemann–Hilbert Problem in Clifford Analysis

Another approach to higher dimensions is the so-called Clifford analysis. Clifford analysis is usually regarded as a generalization to higher dimensions of the theory of holomorphic functions in the complex plane. But, more important, the structure of a Clifford algebra is such that it preserves the intrinsic geometric structure of the underlying Euclidean space. In the previous approaches, the Clifford structure was already present but it was restricted to a para-vector structure and, thus, its full potential remained unused.

Basic Concepts in Clifford Analysis

For $n \in \mathbb{N}$, let us consider the vector space $\mathbb{R}^n$ spanned by an orthonormal basis $\{e_j, j = 1, \ldots, n\}$. The Clifford algebra $C\ell_{0,n}$ is the free real algebra constructed over $\mathbb{R}^n$ generated by the anti-commutation relationship

$$e_j e_k + e_k e_j = -2\delta_{jk} e_0, \quad j, k = 1, \ldots, n,$$

where e_0 is the identity of $C\ell_{0,n}$, and δ_{ij} is the Kronecker symbol. A basis for $C\ell_{0,n}$ is given by

$$e_\emptyset = e_0, \quad e_A = e_{\{j_1,\ldots,j_s\}} = e_{j_1} \cdots e_{j_s},\ 1 \leqslant j_1 < \cdots < j_s \leqslant n.$$

Hence, each element $x \in C\ell_{0,n}$ may be written in the form $x = \sum_A x_A e_A$, with $x_A \in \mathbb{R}$.

One defines the Clifford conjugation as the automorphism $\bar{\cdot} : C\ell_{0,n} \to C\ell_{0,n}$ given by

$$x = \sum_A x_A e_A \mapsto \overline{x} = \sum_A x_A \overline{e}_A,$$

where

$$\overline{e}_0 = e_0, \quad \overline{e}_j = -e_j, j = 1, \cdots, n, \quad , \overline{e_A e_B} = \overline{e}_B \overline{e}_A.$$

Definition 4 (Inner Products). Let $x = \sum_A x_A e_A$, $y = \sum_B y_B e_B \in C\ell_{0,n}$. We define

(i) the scalar inner product $(\cdot, \cdot)$ on $C\ell_{0,n}$ as

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$$(x, y) := \sum_A x_A y_A. \tag{21}$$

This inner product induces a norm $\|\cdot\|$ in $C\ell_{0,n}$, where

$$\|x\|^2 := \sum_A x_A^2.$$

(ii) the Clifford-valued inner product $\langle \cdot, \cdot \rangle$ as

$$\langle x, y \rangle := x\overline{y}. \tag{22}$$

Obviously, the scalar part of (22) coincides with (21), that is

$$\mathrm{Sc}\langle x, y \rangle = (x, y), \quad x, y \in C\ell_{0,n}.$$

Moreover, if $x = x_0 + \sum_{j=1}^n x_j e_j$, then

$$\|x\|^2 = x\overline{x} = \overline{x}x = \sum_{j=0}^n x_j^2 = \langle x, x \rangle,$$

corresponds to the Euclidean norm in $\mathbb{R}^{n+1}$. Furthermore, x is invertible with inverse $x^{-1} = \frac{\overline{x}}{\|x\|^2}$.

The *even subalgebra* $C\ell_{0,n}^+$ is the subalgebra of all $x \in C\ell_{0,n}$ of the form $x = \sum_{A,|A| \text{ even}} x_A e_A$. $x \in C\ell_{0,n}$ may be written in the form $x = \sum_A x_A e_A$. The quaternionic algebra $\mathbb{H}$ is viewed as being isomorphic to $C\ell_{0,3}^+$, by the means of the relations

$$\mathbf{i} = e_1 e_2, \quad \mathbf{j} = e_2 e_3, \quad \mathbf{k} = -e_1 e_3.$$

In this case one also has for $q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$ that

$$\overline{q} = x_0 - (x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}), \quad \|q\|^2 = \sum_{j=0}^3 x_j^2, \quad q^{-1} = \frac{\overline{q}}{\|q\|^2}.$$

Functions with values in $C\ell_{0,n}$ are defined as

$$f : \Omega \subset \mathbb{R}^n \to C\ell_{0,n}, \quad x \mapsto f(x) = \sum_A f_A(x) e_A,$$

where f_A are real valued. Properties such as continuity, differentiability, and so on are ascribed to f by imposing that all its real-valued components f_A fulfill such property. For f and g one defines the scalar inner product

$$(f, g) := \int_\Omega \sum_A f_A(x) g_A(x) dx \tag{23}$$

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and the Clifford inner product

$$\langle f, g\rangle = \int_{\Omega} \overline{f(x)} g(x) dx.$$

For para-vector-valued functions one has $(f, g) = \mathrm{Sc}\langle f, g\rangle$. The above defined inner product makes $L_2(\mathbb{R}^n, C\ell_{0,n})$ a right Clifford Hilbert module with induced norm $\int_{\Omega} \|f(x)\|^2 dx$

The Dirac operator is defined as the first order differential operator

$$Df = \sum_{j=1}^{n} \partial_{x_j} e_j f$$

which factorizes the n dimensional Laplacian $\Delta_n = \sum_{j=1}^{n} \partial_{x_j}^2$ in the sense that

$$\Delta_n f = D^2 f.$$

A differentiable function $f : \Omega \subset \mathbb{R}^n \to C\ell_{0,n}$ is said left-monogenic whenever $Df = 0$, on Ω.

This subsection ends with a last remark. When the signal is a para-vector, that is,

$$\mathbb{R}^{n+1} \ni (x_0, x) \mapsto f(x_0, x) = f_0(x_0, x) + f_1(x_0, x)e_1 + \cdots + f_n(x_0, x)e_n,$$

then the Riesz system (11) corresponds to

$$(\partial_{x_0} + D)f = 0,$$

where $\partial_{x_0} + D$ is called the generalized Cauchy–Riemann operator.

The Riemann–Hilbert Problem in the Upper Half Plane

The correspondent Riemann–Hilbert problem for the Dirac operator can be stated in the following form. Denote by $\mathbb{R}^3_+$ the upper half space

$$\mathbb{R}^3_+ := \{x \in \mathbb{R}^3 : x_3 > 0\}.$$

Given $f \in L^2(\mathbb{R}^2)$, determine $F : \mathbb{R}^3_+ \to C\ell_{0,3}$ such that

$$\begin{cases} DF = 0 \quad \text{on } \mathbb{R}^3_+, \text{ (monogenicity)} \\ \mathrm{Re}F = f \text{ in } \mathbb{R}^2. \end{cases} \tag{24}$$

The solution to this problem follows the same lines as before, namely, it relies upon the theory of Calderón, Zygmund, Carleson et al. Let us consider the Cauchy integral operator $C_\Gamma : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, given by

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$$C_\Gamma f(x_1, x_2) = -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x - y}{\|x - y\|^3} (-e_3) f(x_1, x_2) dx_1 dx_2, \quad (x_1, x_2) \in \mathbb{R}^2. \tag{25}$$

By applying the trace operator to (25) one generates the singular integral S_Γ as

$$\begin{aligned} \mathrm{tr} C_\Gamma f(\eta) &= \frac{1}{2}(I + S_\Gamma) f(\eta) \\ &= \frac{1}{2}\left[f(\eta_1, \eta_2) + \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{e_1(x_1 - \eta_1) + e_2(x_2 - \eta_2)}{\|x - \eta\|^3} e_3 f(x_1, x_2) dx_1 dx_2 \right]. \end{aligned} \tag{26}$$

Remark that for an arbitrary $f : \mathbb{R}^2 \to C\ell_{0,3}$ the singular integral operator $S_\Gamma f$ exists for all points in the sense of the Cauchy principal value and that it can be continuously extended to the whole of $L^2(\mathbb{R}^2)$. Moreover, it is a continuous operator in $L_2(\mathbb{R}^2)$ (c.f. [18]).

The connection between (26) and quaternions becomes obvious when one uses the isomorphism between $Cl_{0,3}^+$ and $\mathbb{H}$. In fact, using the identification $\mathbf{i} = e_1 e_2$ and $\mathbf{j} = e_2 e_3$ we get

$$\mathrm{tr} C_\Gamma f(\eta) = \frac{1}{2}\left[f(\eta_1, \eta_2) + \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\mathbf{i}(x_1 - \eta_1) + \mathbf{j}(x_2 - \eta_2)}{\|x - \eta\|^3} f(x_1, x_2) dx_1 dx_2 \right], \tag{27}$$

which corresponds (again, up to the factor $1/2$) to the concept of monogenic signal (c.f. [8, 16]).

Enters the Riesz Transform into the Picture

In [16] one finds a brief historical survey on the development and applications of the analytic signal, as well as an account of some of the early attempts of its extension to higher dimensions. Shortly before, Bülow and Sommer had introduced a *hyper-complex signal representation* (see [3]) in which the authors combined the approach of Hahn (c.f. the previous section) and successfully constructed a hypercomplex signal representing the full symmetry. However, and although that representation clearly identified the Hilbert transform contribution to the signal, it reveals itself to be depending on the directionality. In the work of Larkin et al. the authors presented the concept of a spiral phase spectral (Fourier) operator and of an orientational phase spatial operator both of which combined into a meaningful 2D Hilbert transform. This provided direct solutions to the problem of closed fringe pattern demodulation but, more important, it was recognized to be linked to Riesz transforms in classical harmonic analysis. In the next subsection one recalls properties of the Riesz transforms and their adaptation to Clifford-valued functions.

Clifford-Valued Riesz Transforms

The definition of a Clifford-valued Riesz transform relies on the *Riesz transforms* (acting on real-valued functions).

Definition 5 ([25]). We define the Riesz transforms as $R_j : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, where

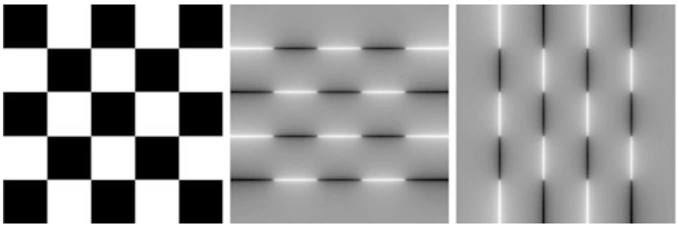

Fig. 2 Checkboard image, together with its first and second components of the Riesz transform

$$u \mapsto R_j u(\eta) = -\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{x_j - \eta_j}{\|x-\eta\|^{n+1}} u(x)dx, \quad j = 1, \dots, n. \tag{28}$$

Then, the Clifford-valued Hilbert transform is given by

$$R : L^2(\mathbb{R}^n, C\ell_{0,n}) \to L^2(\mathbb{R}^n, C\ell_{0,n}), \quad f = \sum_A f_A e_A \mapsto Rf = \sum_{j=1}^n e_j e_A (R_j f_A). \tag{29}$$

The symbol of a Riesz transform is

$$\mathcal{F}(R_j f)(\xi) = i \frac{\xi_j}{\|\xi\|} \mathcal{F} f(\xi), \quad j = 1, \dots, n.$$

This already hints Riesz transforms as an indicator for directionality in the Fourier plane. In other words, Riesz transforms behave similar to partial derivatives. In Fig. 2 one observes the Riesz transforms, that is, the first and second components of the Hilbert transform, acting on the checkerboard image.

Lemma 1. *For $f \in L^2(\mathbb{R}^n)$ it holds*

(i) $\mathcal{F}(\partial_{x_j} f) = \mathcal{F}(R_j(-\Delta)^{1/2})\mathcal{F} f$;
(ii) $\sum_{j=1}^n R_j^2 = -Id$;
(iii) (Plancherel identity) $\| \sum_{j=1}^n R_j^2 f \|_{L^2} = \| f \|_{L^2}$.

Proof.

(i) $\mathcal{F}(\partial_{x_j} f) = i\xi_j \mathcal{F} f = \frac{i\xi_j}{\|\xi\|}\|\xi\| \mathcal{F} f = \mathcal{F}(R_j(-\Delta)^{1/2})\mathcal{F} f$, where $(-\Delta)^{1/2} := \mathcal{F}^{-1}(\|\xi\|)$ denotes the *fractional Laplacian*.
(ii) $\mathcal{F}\left(\sum_{j=1}^n R_j^2 f\right) = \sum_{j=1}^n \left(\frac{i\xi_j}{\|\xi\|}\right)^2 \mathcal{F} f = -\mathcal{F} f$.
(iii) immediate, due to (ii).

□

One of the well-known properties of the one-dimensional Hilbert transform is the so-called Bedrosian identity. In its classic form we have that if $f, g \in L^2(\mathbb{R})$ with $\mathrm{supp}(\hat{f}) \subset [a,b]$ and $\mathrm{supp}(\hat{g}) \subset \mathbb{R} \setminus [a,b]$ then $H(fg) = fHg$. Unfortunately, the Bedrosian identity is something typically one-dimensional and does not hold in case for Riesz transforms. The closest to a

DOI 10.1007/978-3-0348-0692-3_14-1

Bedrosian identity one is aware of was proven in [13] via the consideration of an additional operator which corresponds to integration over hyperplanes passing through the origin.

The Monogenic Signal

In 2001 two new approaches to analytic signals in higher dimensions appeared. Felsberg and Sommer proposed a quaternionic Hilbert transform,

$$f = f(x_1, x_2) \mapsto H_{\mathbb{H}} f(x_1, x_2) = \mathbf{i} R_1 f(x_1, x_2) + \mathbf{j} R_2 f(x_1, x_2),$$

based on the directionality of the Riesz transforms.

Definition 6 ([8]). Let $f \in L^2(\mathbb{R}^2)$. Then the monogenic signal f_M is defined as

$$f_M = f + H_{\mathbb{H}} f = (Id + \mathbf{i} R_1 + \mathbf{j} R_2) f \in L^2(\mathbb{R}^2).$$

Note that this embeds the 2D signal f into $\mathbb{R}^3$. Moreover, remark that under the action of the 2D Fourier transform (the complex imaginary unit i being scalar and, hence, commuting with quaternions) we get

$$\begin{aligned} \hat{f}_M &= [(Id + \mathbf{i} R_1 + \mathbf{j} R_2) f]\hat{} \\ &= \left(1 + i \frac{\mathbf{i}\xi_1 + \mathbf{j}\xi_2}{\|\xi\|}\right) \hat{f}, \quad \xi = (\xi_1, \xi_2), \end{aligned} \tag{30}$$

thus generating the *transfer function*

$$\hat{H}_{\mathbb{H}}(\xi) = i \frac{\mathbf{i}\xi_1 + \mathbf{j}\xi_2}{\|\xi\|}, \tag{31}$$

as the symbol of the quaternionic Hilbert transform in the Fourier domain. As a final remark, note that Fourier symbol of the monogenic signal (viewed as a transform) coincides with the *spectral idempotents*

$$\chi_{\pm}(\xi) := \frac{1}{2}\left(e_0 \pm i \frac{\xi}{\|\xi\|}\right),$$

introduced by McIntosh in [17]. These elements are projectors in the Clifford algebra satisfying to

$$[\chi_+(\xi)]^2 = \chi_+(\xi), \quad [\chi_-(\xi)]^2 = \chi_-(\xi), \quad \chi_+(\xi)\chi_-(\xi) = \chi_-(\xi)\chi_+(\xi) = 0.$$

As a consequence, the monogenic signal, viewed as an operator, is a projector.

Almost in parallel, Larkin, Bone, and Oldfield proposed a *spiral phase quadrature transform*, based on a complex Riesz transform in each real variable in order to obtain a steerable, or directional, Hilbert transform. In other words, the complex Riesz transform is given by

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$$\mathbf{R}f(x,y) := R_1 f(x,y) + iR_2 f(x,y)$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x'-x)+i(y'-y)}{\|(x,y)-(x',y')\|^3} f(x',y')dx'dy'.$$

The resulting complex Riesz transform is a unitary mapping from $L^2(\mathbb{R}^2)$ into itself, with symbol

$$\mathcal{F}(\mathbf{R}f)(\xi_1,\xi_2) = \frac{i\xi_1 - \xi_2}{\|(\xi_1,\xi_2)\|}\mathcal{F}f$$

while the monogenic signal is defined as the vector $f_M = (f, \mathbf{R}f) = (f, R_1 f, R_2 f)$. Currently, this version is used by M. Unser and his co-authors in a series of papers under the name of *Riesz–Laplace transform*.

However, one must remark that both approaches generate a para-vector (scalar plus vector)-valued function and are equivalent. If one writes the monogenic signal as

$$f_M = (Id + \mathbf{i}R_1 + \mathbf{j}R_2)f = f + \mathbf{i}(R_1 - \mathbf{ij}R_2 f)$$

and identify the imaginary unit i with the unit $-\mathbf{ij}$, we obtain the definition of Larkin, Bone, and Oldfield. The same can be done the other way around by introducing a new imaginary unit $\mathbf{i}$ such that

$$(f, \mathbf{R}f) = f + \mathbf{iR}f$$

and identifying $\mathbf{i}i$ with the imaginary unit $\mathbf{j}$.

Properties of the Monogenic Signal

The main idea is to present the monogenic signal as the boundary value of a certain "analytic" function. For that, let us recall the trace operator of the Cauchy integral operator (27) applied to a signal $f \in L^2(\mathbb{R}^2)$,

$$\mathrm{tr}C_\Gamma f(x_1,x_2) = \frac{1}{2}\left[f(x_1,x_2) + \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\mathbf{i}(\eta_1-x_1)+\mathbf{j}(\eta_2-x_2)}{\|\eta-x\|^3}f(\eta_1,\eta_2)d\eta_1 d\eta_2\right].$$

Theorem 2. *A function $u = u_0 + \mathbf{i}u_1 + \mathbf{j}u_2 \in L^2(\mathbb{R}^2)$ belongs to the Hardy space $H^2(\mathbb{R}^3_+)$ iff*

$$\mathrm{tr}C_\Gamma u = u,$$

with $\mathrm{tr}C_\Gamma$ being the operator (27).

Moreover, $C_\Gamma u$ is then monogenic in the upper half space $\mathbb{R}^3_+$, that is, it satisfies there

$$(\partial_{x_0} + D)C_\Gamma u = 0,$$

the 3D Riesz system (11).

DOI 10.1007/978-3-0348-0692-3_14-1

It should be remarked that under the previous identification of $\mathbb{H}$ with $C\ell_{0,3}^{+}$ the monogenic signal is a spinor-valued function. Furthermore, one knows that the Poisson kernel

$$P_{x_0} := \frac{x_0}{\|(x_0, x_1, x_2)\|^3}$$

is an approximate identity, i.e. $F = P_{x_0} * f$ implies $\Delta F = 0$ and $\mathrm{tr} F = f$. Since P_{x_0} is the real part of the Cauchy kernel, its harmonic conjugate

$$Q := \frac{\mathbf{i}x_1 + \mathbf{j}x_2}{\|(x_0, x_1, x_2)\|^3} = \mathbf{i}P_{x_1} + \mathbf{j}P_{x_2}$$

gives a new characterization for the kernel of the quaternionic Hilbert transform. In fact, one has that

$$\mathrm{tr}Cf = \mathrm{tr}P_{x_0} * f + \mathbf{i}\mathrm{tr}P_{x_1} * f + \mathbf{j}\mathrm{tr}P_{x_2} * f = f_M,$$

for all $f \in L^2(\mathbb{R}^2)$.

The main goal of the monogenic signal is to obtain its decomposition into a phase and an amplitude.

Theorem 3. *Let $f \in L^2(\mathbb{R}^2)$, and let f_M be its associated monogenic signal*

$$f_M = f + H_{\mathbb{H}} f = f + \mathbf{i}R_1 f + \mathbf{j}R_2 f.$$

Then the monogenic signal admits the following polar representation

$$\begin{aligned} f_M &= \|f_M\| \left(\frac{f}{\|f_M\|} + \frac{H_{\mathbb{H}} f}{\|H_{\mathbb{H}} f\|} \frac{\|H_{\mathbb{H}} f\|}{\|f_M\|} \right), \quad f \neq 0, \\ &= A(\cos\theta + \omega \sin\theta) \\ &= A e^{\omega\theta}, \end{aligned}$$

*where $A = \|f_M\| \geq 0$ denotes the **local amplitude** of the monogenic signal, $\theta = \arccos \frac{f}{\|f_M\|} \in [0, \pi]$ denotes its **local phase**, and $\omega = \frac{H_{\mathbb{H}} f}{\|H_{\mathbb{H}} f\|} \in S^1$ denotes its **local phase direction**.*

Let us remind that there exists a link between the gradient of the signal and its quaternionic Hilbert transform. This can easily be seen via the symbol of the quaternionic Hilbert transform.

Lemma 2. *Let $f \in L^2(\mathbb{R}^2)$. Then*

$$\mathrm{grad} f = H_{\mathbb{H}}(-\Delta)^{1/2} f.$$

Proof. In fact, by Lemma 1(i) we have

$$\mathrm{grad} f = (\mathbf{i}\partial_{x_1} + \mathbf{j}\partial_{x_2}) f = (\mathbf{i}R_1 + \mathbf{j}R_2)(-\Delta)^{1/2} f = H_{\mathbb{H}}(-\Delta)^{1/2} f,$$

DOI 10.1007/978-3-0348-0692-3_14-1

where the first equality is to be understood as an equality between a vector in $(x, y) \in \mathbb{R}^2$ and a quaternion of type $\mathbf{i}x + \mathbf{j}y$. □

Since $(-\Delta)^{1/2}$ is a scalar operator this means that the gradient of f and the quaternionic Hilbert transform $H_{\mathbb{H}}$ point in the same direction.

Another important point of the monogenic signal is its rotational invariance. The rotational covariance of monogenic functions translates to the following property of our quaternionic Hilbert transform

$$H_{\mathbb{H}}[\rho_s f](\overline{s} y s) = s H_{\mathbb{H}}[f]\overline{s}, \quad s \in \text{Spin}(2),$$

where ρ_s denotes the rotation induced by $s \in \text{Spin}(2)$, that is, $\rho_s f(x) = f(sx\overline{s})$. This property is usually known as *steerability* of the quaternionic Hilbert transform in applications.

A principal practical problem is how to use the monogenic signal on an image. Usually, one considers first a decomposition of the image in terms of a frame. Just let us recall the definition of a quaternionic frame:

Definition 7. $\{g_k\}$ is a frame for $L^2(\mathbb{R}^2; \mathbb{H})$ if there exist real constants $A, B > 0$ such that

$$A\|f\|_{L^2}^2 \leq \sum_{k \in \mathbb{Z}^2} \|\langle f, g_k \rangle\|^2 \leq B\|f\|_{L^2}^2, \quad \forall f \in L^2(\mathbb{R}^2; \mathbb{H}). \tag{32}$$

Frames are a generalization of the concept of a basis in the sense that although their elements span the space in general they are not linearly independent, but allow a larger redundancy. In the case of the monogenic signal we have the following observation.

Theorem 4 ([13]). *Let $\{u_k\}$ be a frame for $L^2(\mathbb{R}^2, \mathbb{R})$ with frame bounds $0 < A \leq B < \infty$. Then the following holds*

- *$\{u_k\}$ is a frame for $L^2(\mathbb{R}^2, \mathbb{H})$ with the same frame bounds*
- *$\{H_{\mathbb{H}} u_k\}$ is a frame for $L^2(\mathbb{R}^2, \mathbb{H})$ with the same frame bounds*

In particular, one can start with real-valued frames for the original signal which are rotationally invariant. From the above theorem one knows that the application of the quaternionic Hilbert transform will yield rotationally invariant frames for the monogenic signal. This idea was applied in [13, 14] to construct monogenic wavelets and in [26] to obtain monogenic curvelets. In both cases the term monogenic is applied to further clarify that the resulting frame will be a frame for the monogenic Hardy space $H^2(\mathbb{R}^2, \mathbb{H})$. In the case of curvelets it allows an easier detection of edges as elements of the wave front set [26]. The monogenic wavelet transform as the wavelet transform over $H^2(\mathbb{R}^2, \mathbb{H})$ was also explored in papers by Unser and his co-authors [28, 29] as *Riesz–Laplace Wavelet Transform*.

The principal application of the monogenic signal is the detection of intrinsically one-dimensional signals. One of the principal examples of such a signal is given by the real part of the so-called plane wave function

$$u(x) = e^{i(<x,\omega>+t)}$$

DOI 10.1007/978-3-0348-0692-3_14-1

which is constant along lines perpendicular to ω, and has its phase function given by

$$i(< x, \omega > +t).$$

There is a close connection between plane wave functions and monogenic functions [21]. It also leads to the connection with the Radon transform. Consider the classic Radon transform

$$\mathcal{R}f(\omega,t) = \int_{\mathbb{R}^n} f(\delta(\langle\omega,x\rangle + t))dx$$

Since an intrinsically 1D-signal is in fact the real part of a plane wave function one has

$$\mathcal{R}u(\omega,t) = c, c \in \mathbb{R}$$

for all directions $\omega = (\cos\theta, \sin\theta)$ different from the orientation of the plane wave. Furthermore, one can obtain the Hilbert transform in the case of the monogenic signal via the following well-known formula

$$(H_{\mathbb{H}}f)(x) = \mathcal{R}^{-1}(\omega H(\mathcal{R}f(\omega,\cdot))(t))(x),$$

where H_1 denotes the one-dimensional Hilbert transform. Using this link instead of the usual study via the phase of a monogenic signal in time domain one can also discuss its properties in Radon space [33].

Further Developments

Higher-Order Riesz Transforms

In [27] the authors proposed to use higher-order Riesz transforms. The basic idea behind is that Riesz transforms R_k form an algebra of linear bounded operators, invariant under translation and dilation over $L^p(R^2)$. This algebra is isomorphic to the algebra of polynomials of two variables. A higher-order Riesz transform is then defined as any homogeneous polynomial of degree l, i.e.

$$P_\alpha(R_k) = \sum_{|\alpha|=l} R_1^{\alpha_1} R_2^{\alpha_2} \lambda_{\alpha_1,\alpha_2}.$$

Specific examples are R_1R_2, R_1^2, R_2^2 which can be linked to the Hessian of f, i.e.

$$Hf = \begin{pmatrix} \partial_1^2 f & \partial_1\partial_2 f \\ \partial_1\partial_2 f & \partial_1^2 f \end{pmatrix} = \begin{pmatrix} R_1^2\Delta f & R_1R_2\Delta f \\ R_1R_2\Delta f & R_1^2\Delta f \end{pmatrix}.$$

One can observe that the Hessian corresponds to a matrix operator given by second-order Riesz transforms up to the action of the Laplacian, which is a symmetric negative definite scalar operator. Considering the Clifford algebra $C\ell_{0,3}$ (keep in mind that $\mathbb{H} \sim C\ell_{0,3}^+$ by disregarding the scalar operator the above Hessian also resembles the even part of the so-called monogenic curvature tensor [34]

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$$T_{\text{even}}(f) = \begin{pmatrix} R_1^2 f & -R_1 R_2 f e_{12} \\ R_1 R_2 f e_{12} & R_2^2 f \end{pmatrix}$$

while the odd part is given by

$$T_{\text{odd}}(f) = T_{\text{even}}(R_1 f + R_2 f e_{12}) = \begin{pmatrix} (R_1^3 + R_1^2 R_2 e_{12}) f & -R_1(R_2 R_1 + R_2^2 e_{12}) f e_{12} \\ R_1(R_2 R_1 + R_2^2 e_{12}) f e_{12} & (R_2^2 R_1 + R_2^3 e_{12}) f \end{pmatrix}.$$

For the application and interpretation of the monogenic curvature tensor we refer to [33, 34].

One of the drawbacks is that in general higher-order Riesz transforms are not invariant under rotations. This leads in [13] to the consideration of higher-order Riesz transforms based on spherical harmonics Y_k^l. Taking into account that $\{Y_k^l\}$ spans the space of homogeneous polynomials of degree k which are invariant under rotations we can define the so-called higher-order hypercomplex Riesz transform

$$R_k f = \sum_{l=0}^{d} e_l Y_k^l(R_1, R_2),$$

where e_l is an orthonormal basis of R^d which generates the Clifford algebra $C\ell_{0,d}$ [13].

The function $f_M = f + R_k f$ is a boundary value of a function F which belongs to the kernel of the higher-order Dirac operator

$$\mathcal{D} = \partial_0^k + \sum_{l=0}^{d} e_l Y_k^l(\partial_1, \partial_2).$$

For an interpretation of the resulting phase we refer to [13]

Conformal Monogenic Signals

In 2008 Wietzke and Sommer [31] introduced the concept of the so-called conformal monogenic signal to introduce curvature terms into the monogenic signal. One of the key properties is the link between the classic Radon transform and the Riesz transforms which allows the detection of intrinsically 1D-signal like plane waves. To detect other features, such as curvature or intrinsically 2D-features, one can also start from a modification of the Radon transform, such as the circular Radon transform. The basic idea of the conformal monogenic signal is to map the signal to the sphere S^2 conformally:

$$c(x, y, z) = \begin{cases} f(C^{-1}(x, y, z)), & x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4} \\ 0, & \text{else} \end{cases},$$

where $C^{-1}(x, y, z) := \frac{1}{1-z}(x, y)$, and to apply the 3D-Riesz transforms to c. This results in a quaternion-valued signal with energy $||c(0, s)||$, phase $\phi(0, s) = \text{atan2}(\sqrt{(R_1 c)^2 + (R_2 c)^2 + (R_3 c)^2}, c)$, and direction $\theta(0, s) = \text{atan2}(R_2 c, R_1 c)$, where

DOI 10.1007/978-3-0348-0692-3_14-1

$c(\cdot, s)$ denotes the evaluation of c restricted to the plane orthogonal to $s \in S^2$ and passing through the origin. Moreover, it allows the consideration of a monogenic curvature

$$\kappa(0, s) = \frac{R_3 c}{\sqrt{(R_1 c)^2 + (R_2 c)^2}}.$$

More details can be found in [31, 32].

Conclusion

Although rather recent, the theory of monogenic signals is steadily imposing itself as a major tool for analysis and mathematical manipulation of signals. Their range of applications is vast, ranging from image processing to demodulation of AM-FM signals. Their success also stems from its close connection with monogenic functions from which they inherit several important properties such as rotational invariance. Furthermore, being para-vector-valued the monogenic signals allow for an easy interpretation in terms of a phase decomposition in terms of a direction and a phase value.

Acknowledgements This work was supported by Portuguese funds through the CIDMA-Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology (FCT-Fundação para a Ciência e a Tecnologia), within project PEst-OE/MAT/UI4106/2014.

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