Operator Theory
DOI 10.1007/978-3-0348-0692-3_18-1

Discrete Clifford Analysis

Uwe Kaehler*[a] and Frank Sommen[b]
[a]Department of Mathematics, CIDMA - Center for Research and Development in Mathematics and Applications, University of Aveiro, Aveiro, Portugal
[b]Department of Mathematical Analysis, University of Ghent, Gent, The Netherlands

Abstract

This survey is intended as an overview of discrete Clifford analysis and its current developments. Since in the discrete case one has to replace the partial derivative with two difference operators, backward and forward partial difference, one needs to modify the main tools for a development of a discrete function theory, such as the replacement of a real Clifford algebra by a complexified Clifford algebra or of the classic Weyl relations by so-called S-Weyl relations. The main results, like Cauchy integral formula, Fischer decomposition, CK-extension, and Taylor series, will be derived. To give a better idea of the differences between the discrete and continuous case, this chapter contains the problem of discrete Hardy spaces as well as some discrete objects which do not have an equivalent object in continuous Clifford analysis, such as the CK-extension of a discrete Delta function.

Introduction

In the last two decades one can observe an increased interest in the analysis of discrete structures. On the one hand the fact that increased computational power is nowadays available to everybody and that computers can essentially work only with discrete values sparked an increased interest in working with discrete structures. This is true even for persons who are originally unrelated to the field. An outstanding example can be seen in the change of the philosophy of the Finite Element Method. From the classical point of view of being essentially a method for discretization of partial differential equations via a variational formulation the modern approach lifts the problem and, therefore, the finite element modeling directly on to the mesh, resulting in the so-called Finite Element Exterior Calculus. This means that one requires discrete structures which are equivalent to the usual continuous structures. On the other hand, the increased computational power also means that problems in physics which are traditionally modeled by means of continuous analysis are more and more directly studied on the discrete level, the principal example being the Ising model from statistical physics as opposed to the continuous Heisenberg model. But here one can observe also the limitations of this change which is due to a lack of “understanding.” Most of the recent advances on the 2D-Ising model by Smirnov and his collaborators are based on a clever interaction between classic and discrete complex analysis. This is possible since discrete complex analysis is under (more or less) constant development since the forties. Unfortunately, the same cannot be

*E-mail: ukaehler@ua.pt

DOI 10.1007/978-3-0348-0692-3_18-1

said about the higher-dimensional case, the 3D-Ising model being one of the major challenges in modern Mathematics. One of the problems one faces is that most attempts to construct a higher-dimensional analogue of discrete complex analysis are rather recent. For instance, the construction of discrete Dirac operators (as generalizations of discrete Cauchy–Riemann operators) goes back to Becher and Joos in 1982 [1], as earliest reference. Modern constructions using a Hermite-type basis appeared only during the last 10 years, Faustino/Kähler/Sommen [15], Kanamori/Kawamoto [22], Forgy/Schreiber [16], Vaz [27] being the principal references. Discrete Clifford analysis itself started only in the end of the 1980s by combining Rjabenkij's "method of difference potentials" [24] with ideas from complex function theory by Gürlebeck and Sprößig. In their work [18] Gürlebeck and Sprößig construct a function theoretical approach to discrete Clifford analysis for a Dirac operator which only contained forward differences. This is a major drawback since such a Dirac operator does not factorize the Star-Laplacian. Afterwards this work was extended to a version of a "discrete Boundary Element Method" by Güerlebeck and Hommel (c.f. [19, 20]), but also first steps were made in the direction of using a Dirac operator which factorizes the Star-Laplacian by constructing such a Dirac operator in $\mathbb{R}^3$, its fundamental solution, and the corresponding discrete integral operators [3,14,17]. The construction of a discrete Clifford analysis from the classical point of view, i.e. studying polynomial solutions, Fischer decompositions, Taylor series, etc. started only in 2006 with the paper of Faustino and Kähler [12], but again for the case of a Dirac operator with only forward or only backward difference operators. The reason is that while forward and backward difference operators are commuting with each other, their corresponding vector variable operators do not. This could only be overcome by an idea of Sommen [7] of using the so-called S-Weyl relations. Afterwards, this theory was quickly developed from several angles (see, for instance, the Ph.D. thesis of Faustino [10] or the Ph.D. thesis of de Ridder [6] and references [2, 5, 9, 13] as well as references therein). Here, one can find a short overview of this exciting new field. The principal ingredients for a discrete Clifford analysis will be stated, such as Cauchy integral formula, Fischer decomposition, Taylor expansion, and discrete homogeneous monogenic polynomials. Furthermore, a short view on discrete boundary values will be given. All this should give a nice overview for anybody interested in this field and present him with the right tools for its application.

Complex Picture

Discrete Complex Analysis is nowadays a well-established field with a vast literature. For references, one can recommend the expository papers [23, 25]. In fact Discrete Complex Analysis goes back to the work of Kirchhoff on electric circuits in 1845 whose famous rules are just stating that the electric current is a discrete harmonic function at each node. For the first time discrete analytic functions are properly defined by Isaacs [21] in 1941. He introduced two different definitions based on different discretizations of the Cauchy–Riemann equations. His "monodiffric functions of the first kind" satisfy $F(z + i\epsilon) - F(z) = i(F(z + \epsilon) - F(z))$ while his "monodiffric functions of the second kind" fulfill the equation $F(z+i\epsilon)-F(z+\epsilon) = i(F(z+\epsilon(1+i))-F(z))$. Geometrically speaking, in the first definition the differences are taken along the axis of $\mathbb{Z}^n$ while in the second the differences are taken along diagonals of the lattice $\mathbb{Z}^n$, i.e. a kind of cross ratio in each square. Especially, the last definition is used nowadays, for instance, for certain types of Riemann boundary value problems which link with conformal mappings [25]. Later on other definitions of discrete analytic functions appeared, including one based on circle packings [26]

DOI 10.1007/978-3-0348-0692-3_18-1

which is closely linked to conformal mappings. But this close link represents a problem for generalization to higher dimensions. It is a well-known fact that the only conformal mappings in higher dimensions are Möbius transformations which is a too restrictive class for a higher-dimensional function theory. This also means that any type of definition of discrete analytic functions which is linked to conformality will be difficult to generalize. As one can observe from these remarks discrete Clifford analysis as a higher-dimensional function theory is of a quite different nature.

Discrete Dirac Operators from Finite Difference Operators

There exists a general approach to construct discrete Dirac operators via discrete differential forms. Details can be found in [11, 16, 22]. Since in this survey one considers the case of the grid $\mathbb{Z}^n$ it is enough to follow the approach in [15] where the authors constructed a discrete Dirac operator by using an algebraic splitting $e_k = e_k^+ + e_k^-$.

The standard method of constructing a Dirac operator in classic Clifford analysis is to consider a basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. By introducing a multiplication which satisfies the Euclidean flat metric, i.e. $e_j e_k + e_k e_j = -2\delta_{jk}$ one extends $\mathbb{R}^n$ to its Clifford algebra $C\ell_{0,n}$. Now, due to the anti-commutativity one gets for the operator $D = \sum_{k=1}^n e_k \partial_{x_k}$

$$D^2 = \left(\sum_{k=1}^n e_k \partial_{x_k}\right)\left(\sum_{l=1}^n e_l \partial_{x_l}\right) = \sum_{k=1}^n e_k^2 \partial_{x_k}^2 = -\Delta.$$

If one tries the same idea for a discrete Dirac operator one is faced with one principal problem. While there is only one partial derivative in each space direction there are two (forward and backward) difference operators, i.e. the operators $\partial^{\pm k}$ defined by

$$\partial^{\pm k} f = \mp \left(f(m) - f(m \pm e_k)\right). \tag{1}$$

These forward/backward differences $\partial^{\pm i}$ satisfy the following product rules

$$\partial^{\pm k}(fg) = f(\partial^{\pm k} g) + (\partial^{\pm k} f)(T_{\pm k} g), \tag{2}$$

$$\partial^{\pm k}(fg) = (T_{\pm k} f)(\partial^{\pm k} g) + (\partial^{\pm k} f) g, \tag{3}$$

where $T_{\pm k} u(m) = u(m \pm e_k)$ denotes the translation operator in spatial direction $\pm e_k$. The most common (and natural) discrete analogue to the Laplacian on the grid $\mathbb{Z}^n$ is the so-called star-Laplacian, i.e. the operator given by

$$\Delta_h = \sum_{k=1}^n \partial^{+k} \partial^{-k}$$

which corresponds to the usual form of the Laplacian on a grid $\sum_{v \in N(m)} u(m) - u(v)$, where the sum is taken over all neighbors of m on the grid. In this form the Laplacian already appears in the famous Kirchhoff circuit laws for electrical circuits. Now, if one considers a discrete Dirac operator containing only forward or backward differences one does not get a factorization of

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the star-Laplacian, at most one obtains a factorization of the cross-Laplacian [15]. One idea to overcome this problem is to do an algebraic splitting of the basis elements $e_k = e_k^+ + e_k^-$. This new basis should generate again a Clifford algebra, but of dimension $2m$ with a suitable metric.

That means that the basis should satisfy

1. $e_i^\pm e_j^\pm + e_j^\pm e_i^\pm = -2g_{ij}^\pm$
2. $e_i^+ e_j^- + e_j^- e_i^+ = -2M_{ij}$

with two symmetric matrices $(g_{ij}^+), (g_{ij}^-)$ and one general matrix (M_{ij}). Since $e_k = e_k^+ + e_k^-$ one has the following constraint

$$g_{ij}^+ + g_{ij}^- + M_{ij} + M_{ji} = \delta_{ij} e_0.$$

Furthermore, since no direction of the lattice should be preferred over any other one can assume

$$g_{jj}^+ = \lambda^+, g_{jj}^- = \lambda^-, M_{jj} = \mu.$$

Additionally, the entries should not depend on i and j, i.e. $g_{ij}^+ = g_{ji}^+ = g^+$, $g_{ij}^- = g_{ji}^- = g^-$, and $M_{ij} = M_{ji} = M$. Furthermore, either the $+$ or $-$ directions should be preferred, so that it should hold $g^+ = g^- = g$ and $\lambda^+ = \lambda^- = \lambda$. The non-preference of the cartesian coordinates can be seen as a discrete rotational invariance.

Now, the Dirac operator should have the form

$$D^{+-} f = \sum_{k=1}^{n} e_k^+ \partial_k^+ + e_k^- \partial_k^-,$$

$$D^{-+} f = \sum_{k=1}^{n} e_k^- \partial_k^+ + e_k^+ \partial_k^-,$$

and satisfy $(D^{+-})^2 = -\Delta_h$. Joining all these conditions one gets

$$\begin{aligned} e_i^\pm e_j^\pm + e_j^\pm e_i^\pm &= -2g, \qquad i \neq j, \\ e_i^+ e_j^- + e_j^- e_i^+ &= +2g, \qquad i \neq j, \\ e_i^+ e_i^- + e_i^- e_i^+ &= 2\lambda - 1, \\ (e_i^+)^2 = (e_i^-)^2 &= -\lambda. \end{aligned}$$

As a special example one can use the following basis

$$\begin{aligned} e_i^\pm e_j^\pm + e_j^\pm e_i^\pm &= 0, \qquad i \neq j, \\ e_i^+ e_i^- + e_i^- e_i^+ &= -1, \\ (e_i^\pm)^2 &= 0. \end{aligned}$$

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which corresponds to the Witt basis of a complex Clifford algebra used in Hermitean Clifford analysis [11].

Discrete Cauchy Integral Formula

First of all, a direct calculation leads to the Stokes' formula for the discrete Dirac operator D^{+-}.

Lemma 1. *For any discrete functions f and g one has*

$$\sum_{m\in\mathbb{Z}^n}(fD^{-+}(m))g(m) = -\sum_{m\in\mathbb{Z}^n} f(m)(D^{+-}g)(m)$$

where $D^{-+}f = \sum_{k=1}^{n}(\partial^{-k}f)e_k^+ + (\partial^{+k}f)e_k^-$ provided that the involved series converge.

Consider a discrete domain $\Omega\subset\mathbb{Z}^n$ and its characteristic function χ_Ω given by

$$\chi_\Omega(m) = \begin{cases} 1 & m\in\Omega \\ 0 & m\notin\Omega \end{cases}.$$

Using the characteristic function one can rewrite the sum over Ω:

$$\sum_{x\in\Omega} f(m)(D^{+-}g)(m) = \sum_{m\in\mathbb{Z}^n} f(m)\chi_\Omega(m)(D^{+-}g)(m).$$

This leads to the corresponding Stokes' formula for the domain Ω by replacing f with the function $f\chi_\Omega$. Therefore, one has to evaluate the sum

$$\sum_{m\in\mathbb{Z}^n}((f\chi_\Omega)D^{-+}(m))g(m)$$

Using Leibniz formula for the operator D^{-+} one gets

$$\begin{aligned}(f\chi_\Omega)D^{-+}(m) &= f(m)(\chi_\Omega D^{-+})(m) + \chi_\Omega(fD^{-+})(m) \\ &\quad + \sum_{j=1}^{n}\left((\partial^{+j}\chi_\Omega(m))(\partial^{+j}f(m))e_j^- - (\partial^{-j}\chi_\Omega(m))(\partial^{-j}f(m))e_j^+\right).\end{aligned}$$

Please notice that the last term can only be non-zero when $\partial^{+j}\chi_\Omega(m)$ or $\partial^{-j}\chi_\Omega(m)$ is non-zero. This defines the boundary of Ω and the boundary terms are given by the first and the last terms in the above formula. Both terms can be joined together as

$$\sum_{j=1}^{n}(\partial^{-j}\chi_\Omega)(T_{-j}f)e_j^+ + (\partial^{+j}\chi_\Omega)(T_{+j}f)e_j^-$$

DOI 10.1007/978-3-0348-0692-3_18-1

where $T_{\pm j}$ denote the shift operators in direction $\pm j$.

Now, this results in the following Stokes' formula for the discrete Dirac operator

Theorem 1. *For any discrete functions f and g one has*

$$-\sum_{m\in\mathbb{Z}^n}\left(\sum_{j=1}^{n}\partial^{-j}\chi_\Omega(m)(T_{-j}f)(m)e_j^{+}+\partial^{+j}\chi_\Omega(m)(T_{+j}f)(m)e_j^{-}\right)g(m)$$
$$=\sum_{m\in\Omega}(fD^{-+})(m)g(m)+f(m)(D^{+-}g)(m)$$

provided that the involved series converge.

To obtain a discrete Cauchy integral formula one still needs the fundamental solution of the adjoint operator to the discrete Dirac operator D^{+-}. There are several methods to obtain such a fundamental solution. The most common way uses the discrete Fourier transform.

The discrete Fourier transform on $l_p(\mathbb{Z}^n,\mathbb{C}_n)$ $\big(1\leq p<+\infty\big)$ is given by

$$\mathcal{F}_h u(\xi)=\sum_{m\in\mathbb{Z}^n}e^{i<m,\xi>}u(m),\quad \xi\in[-\pi,\pi]^n,$$

where $<x,\xi>=\sum_{j=1}^{n}x_j\xi_j$ for arbitrary $x=\sum_{j=1}^{n}e_jx_j,\ \xi=\sum_{j=1}^{n}e_j\xi_j\in\mathbb{R}^n,\ x_j,\xi_j\in\mathbb{R}$ $\big(j=1,\ldots,n\big)$.

Its inverse is given by $\mathcal{F}_h^{-1}=R_h\mathcal{F}$ where R_h denotes the restriction to the lattice $\mathbb{Z}^n$ and $\mathcal{F}$ the (continuous) Fourier transform restricted to the cube $[-\pi,\pi]^n$

$$\mathcal{F}f(x)=\frac{1}{(2\pi)^n}\int_{[-\pi,\pi]^n}e^{-i<x,\xi>}f(\xi)d\xi,\qquad \forall x\in\mathbb{R}^n.$$

Now, using the discrete Fourier transform one has $\mathcal{F}_h(uD^{-+})(\xi)=\mathcal{F}_hu(\xi)\left(\sum_{j=1}^{n}\xi_{-j}^{D}e_j^{+}+\xi_{+j}^{D}e_j^{-}\right)$ with $\xi_{\pm j}^{D}=\mp\big(1-e^{\mp i\xi_j}\big)$ and $\mathcal{F}_h(\Delta u)(\xi)=4\sum_{j=1}^{n}\sin^2\left(\frac{\xi_j}{2}\right)\mathcal{F}_hu(\xi)$. Therefore, one denotes

$$\tilde{\xi}_-=\sum_{j=1}^{n}e_j^{+}\xi_{-j}^{D}+e_j^{-}\xi_{+j}^{D}$$

and

$$d^2=4\sum_{j=1}^{n}\sin^2\left(\frac{\xi_j}{2}\right)$$

as the discrete Fourier symbol of D_h^{-+} and Δ, respectively.

These observations mean that in a way similar to the continuous case the fundamental solution can be given as a Fourier integral

$$E^{-+} = R_h \mathcal{F}\left(\frac{\tilde{\xi}_-}{d^2}\right) = \sum_{j=1}^{n} e_j^+ R_h \mathcal{F}\Big(\frac{\xi_{-j}^D}{d^2}\Big) + e_j^- R_h \mathcal{F}\Big(\frac{\xi_{+j}^D}{d^2}\Big). \tag{4}$$

Lemma 2. *For the fundamental solution E^{-+} one has*

$$(i)\ (E^{-+}D^{-+})(m) = \delta_0(m), \quad \forall m \in \mathbb{Z}^n, \tag{5}$$

$$(ii)\ E^{-+} \in l_p\big(\mathbb{Z}^n, \mathbb{C}_n\big)\big(1 \le p < +\infty\big), \tag{6}$$

where $\delta_k(m)$ denotes the usual Kronecker delta, i.e. $\delta_k(m) = 0, k \neq m$, and $\delta_m(m) = 1$.

Proof. Statement (i) is obvious, while (ii) follows directly from [17] where it was shown that for each component one has

$$\left|\int_{[-\pi,\pi]^n} \frac{\xi_{\pm j}^D}{d^2} e^{-i<x,\xi>} d\xi\right| \le \frac{M}{(|x|+1)^n} + \frac{M}{(|x|+1)^{n-1}},$$

with $M > 0$ being independent of $x \in \mathbb{Z}^n$. □

Substituting the discrete fundamental solution $E^{-+}(m-l)$ into the Stokes' formula for the discrete Dirac operator one obtains the corresponding Borel–Pompeiu formula.

Theorem 2. *For any discrete function g one has*

$$-\sum_{m\in\mathbb{Z}^n}\left(\sum_{j=1}^{n} \partial^{-j}\chi_\Omega(m)(T_{-j}E^{-+})(m-l)e_j^+ + \partial^{+j}\chi_\Omega(m)(T_{+j}E)^{-+}(m-l)e_j^-\right) g(m)$$
$$= g(l) + \sum_{m\in\Omega} E^{-+}(m-l)(D^{+-}g)(m)$$

provided that the involved series converge.

As a consequence one gets the corresponding Cauchy integral formula.

Theorem 3. *Let $g \in \ker D^{+-}(\Omega)$ then one has*

$$g(l) = -\sum_{m\in\mathbb{Z}^n}\left(\sum_{j=1}^{n} \partial^{-j}\chi_\Omega(m)(T_{-j}E^{-+})(m-l)e_j^+ + \partial^{+j}\chi_\Omega(m)(T_{+j}E^{-+})(m-l)e_j^-\right) g(m)$$

provided that the involved series converge.

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S-Weyl Relations and Fischer Decomposition

Nowadays, in the continuous setting there exist several approaches for the construction of function theories, which clearly state what basic ingredients one has to look for and in which way one has to proceed. The most common one is known under different names, in a more restricted form it is usually called Howe dual pair technique in Clifford analysis. One starts with the so-called Weyl relations for the basic operators, i.e.

$$\partial_{x_k}(x_l f(x)) - x_l \partial_{x_k} f(x) = \delta_{kl} f(x),$$

or in short form, $[\partial_{x_k}, x_l] = \delta_{kl} I$, where $[\cdot, \cdot]$ denotes the commutator and I the identity operator. Starting from these Weyl relations one establishes a super-Lie algebra generated by the Dirac operator, the vector variable operator, and the Euler operator. The last operator is being reduced to a multiple of the identity operator on the spaces of its eigenfunctions which reduces the super-Lie algebra to the standard Heisenberg algebra.

Such a direct approach breaks down almost immediately in the discrete setting. The problem resides in the fact that one has to work with forward and backward differences where the standard Weyl relations take the form

$$\partial^{+j} x T_{-j} - x T_{-j} \partial^{+j} = I,$$
$$\partial^{-j} x T_{+j} - x T_{+j} \partial^{-j} = I.$$

Here, a problem arises from the fact that while $[\partial^{+j}, \partial^{-j}] = 0$ the same does not hold for $[xT_{-j}, xT_{+j}] \neq 0$.

One way to proceed is by modifying the Weyl relations to the so-called S-Weyl relations [7]:

$$\partial^{+j} X_j^{+} - X_j^{-} \partial^{-j} = 1, \tag{7}$$
$$\partial^{-j} X_j^{-} - X_j^{+} \partial^{+j} = 1. \tag{8}$$

While there is no explicit expression for X_j^+ and X_j^- they can be expressed by their action on classic monomials:

Theorem 4. *[7] The polynomials $(P_{k+1}^j)^{\pm}$, resulting from the action of $X_j^{\pm}$ on the classical homogeneous powers x_j^k, $k \in \mathbb{N}$, can be written in terms of the Euler polynomials of even degree. More precisely, for k odd, one has*

$$(P_{k+1}^j)^+ = X_j^+(x_j^k) = E_{k+1}(x_j), \tag{9}$$
$$(P_{k+1}^j)^- = X_j^-(x_j^k) = E_{k+1}(-x_j) \tag{10}$$

while for k even, one has

$$(P_{k+1}^j)^+ = X_j^+(x_j^k) = x_j E_k(x_j), \tag{11}$$
$$(P_{k+1}^j)^- = X_j^-(x_j^k) = x_j E_k(-x_j). \tag{12}$$

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If one would follow the usual path in classic continuous Clifford analysis the next step would be to establish a duality between the coordinate operators X_j^+, X_j^- and the finite difference operators $\partial^{+j}, \partial^{-j}$ via the so-called Fischer inner product. But that would work only if the algebras generated by $\{X_j^+, X_j^-, j = 1\ldots,n\}$ and $\{\partial^{+j}, \partial^{-j}, j = 1\ldots,n\}$ are algebraically isomorphic, which is clearly not possible since $\partial^{+j}, \partial^{-j}$ commute, but X_j^+, X_j^- do not commute. Since one is principally interested in the Dirac operator and the corresponding vector variable operator one can circumvent this problem by considering the operators

$$\delta_j = e_j^+ \partial^{+j} + e_j^- \partial^{-j}$$

and

$$\xi_j = e_j^+ X_j^- + e_j^- X_j^+$$

For these operators it holds

$$\delta_j \xi_j - \xi_j \delta_j = 1, \tag{13}$$

$$\delta_j \xi_k + \xi_k \delta_j = 0. \tag{14}$$

These relations can be written in a shorter form

$$\begin{aligned} &[\delta_j, \xi_j] = 1 \quad [\delta_j, \delta_j] = 0, \quad [\xi_j, \xi_j] = 0, \\ &\{\delta_j, \xi_k\} = 0, \{\delta_j, \delta_k\} = 0, \{\xi_j, \xi_k\} = 0, \quad k \neq j, \end{aligned}$$

where $\{\cdot,\cdot\}$ denotes the anti-commutator.

Using these operators one can write the discrete Dirac operator as $D^{+-} = \sum_{j=1}^n \delta_j$ as well as introduce the following operators

$$X = \sum_{j=1}^n \xi_j \quad E = \sum_{j=1}^n \xi_j \delta_j.$$

All three operators satisfy the following relations

$$\begin{aligned} \{D^{+-}, X\} &= \frac{n}{2} + E, \\ [E, X] &= \xi, \\ \left[D^{+-}, E\right] &= D^{+-}. \end{aligned}$$

Moreover, one has

$$E\xi_j = \xi_j(E+1).$$

The above formulae mean that D^{+-}, ξ, and E form an algebra which is algebraically isomorphic to $osp(1|2)$. In classic Clifford analysis the standard approach to construct monogenic polynomials

DOI 10.1007/978-3-0348-0692-3_18-1

consists in using the Fischer inner product. This is possible since the space of homogeneous polynomials forms a Hilbert space under the Fischer inner product which is widely used in Umbral calculus by considering polynomials of operators. Here, one can follow the same ideas with homogeneity for a polynomial being replaced by being an eigenfunction of the Euler operator E, i.e. one says P_k is a discrete homogeneous polynomial of degree k if it is a polynomial of order k and $EP_k = kP_k$.

Since the basic polynomials are given by the action of ξ_j on the ground state 1

$$\xi^\alpha[1] = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}[1]$$

with the multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ represents a discrete homogeneous polynomial of degree $|\alpha|$ while the set $\{\xi^\alpha[1] : |\alpha| = k\}$ forms a basis for the space of discrete homogeneous polynomials of degree k. The last one is easy to see since $E\xi^\alpha = |\alpha|\xi^\alpha$.

As usual, the Fischer inner product of two polynomials P and Q, being discrete homogeneous of the respective degrees k and m, is given by

$$\langle P, Q\rangle = \mathrm{Sc}\left[P(\xi)^\dagger\, Q(\delta)[1](0)\right] \tag{15}$$

where $Q(\delta)$ denotes the operator obtained by substituting in the polynomial Q the variable x_j by ξ_j, and $P(\xi)$ denotes the difference operator obtained by substituting in the polynomial P the variable x_j by δ_j. † denotes the hermitean conjugate, i.e. $(e_j^{\pm})^\dagger = e_j^{\mp}$. Both P and Q are then acting as operators on the ground state 1, the result of which is evaluated at the point zero.

From the S-Weyl relation one obtains

$$\delta_i \xi_i^m[1] = (\xi_i^{m-1} + \xi_i \delta_i \xi_i^{m-1})[1]$$

for the calculation of the Fischer inner product. A recursive application of this formula results in the next lemma.

Lemma 3. *For all $m \in \mathbb{N}$ one has*

$$\delta_i \xi_i^m[1] = m\xi_i^{m-1}[1].$$

Moreover,

$$\delta_i^m \xi_i^m[1] = m!.$$

This also results in the following statement.

Lemma 4. *For any two multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$, with $|\alpha| = |\beta|$, it holds that*

$$\delta^\alpha \xi^\beta[1] = \begin{cases} \alpha! \text{ if } & \alpha = \beta \\ 0 \text{ if } & \alpha \neq \beta \end{cases}$$

DOI 10.1007/978-3-0348-0692-3_18-1

where one uses the abbreviation $\alpha! = \alpha_1!\alpha_2!\ldots\alpha_n!$.

The above results eventually lead to the following important property.

Proposition 1. *For two discrete homogeneous polynomials* $P_k = \sum_{|\alpha|=k} \xi^\alpha[1]\, p_\alpha$ *and* $Q_k = \sum_{|\alpha|=k} \xi^\alpha[1]\, q_\alpha$, *both homogeneous of degree* k, *one obtains*

$$\langle P_k, Q_k\rangle = \sum_{|\alpha|=k} \alpha!\ \mathrm{Sc}\left[p_\alpha{}^\dagger\, q_\alpha\right]$$

where $\cdot^\dagger$ *stands for the Hermitean conjugate.*

This property obviously implies that, on the space of discrete homogeneous polynomials of given homogeneity k, the Fischer inner product is positive definite, i.e. it indeed represents an inner product. Furthermore, the following corollary holds.

Corollary 1. *For any polynomial* P_{k-1} *of homogeneity* $k-1$ *and any polynomial* Q_k *of homogeneity* k, *one has*

$$\langle X P_{k-1}, Q_k\rangle = \langle P_{k-1}, D^{+-} Q_k\rangle.$$

This property leads to the following theorem.

Theorem 5. *For each* $k \in \mathbb{N}$ *one has*

$$\Pi_k = \mathcal{M}_k + X\,\Pi_{k-1}$$

where Π_k *denotes the space of discrete homogeneous polynomials of degree* k *and* $\mathcal{M}_k$ *denotes the space of discrete monogenic homogeneous polynomials of degree* k. *Furthermore, the subspaces* $\mathcal{M}_k$ *and* $X\,\Pi_{k-1}$ *are orthogonal with respect to the Fischer inner product (15).*

Since it holds that

$$\Pi_k = X\,\Pi_{k-1} + (X\,\Pi_{k-1})^\perp$$

it suffices to prove that $(X\,\Pi_{k-1})^\perp = \mathcal{M}_{k-1}$. To this end, assume that, for some $P_k \in \Pi_k$ one has

$$\langle X P_{k-1}, P_k\rangle = 0, \quad \text{for all } P_{k-1} \in \Pi_{k-1}.$$

On account of Corollary 1 one then has that

$$\langle P_{k-1}, D^{+-} P_k\rangle = 0, \quad \text{for all } P_{k-1} \in \Pi_{k-1}.$$

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As $D^{+-}P_k \in \Pi_{k-1}$ one obtains that $D^{+-}P_k = 0$, or that $P_k \in \mathcal{M}_k$. This means that $(X\,\Pi_{k-1})^{\perp} \subset \mathcal{M}_k$. Conversely, take $P_k \in \mathcal{M}_k$. Then one has, for any $P_{k-1} \in \Pi_{k-1}$, that

$$\langle X\,P_{k-1}, P_k\rangle = \langle P_{k-1}, D^{+-}P_k\rangle = \langle P_{k-1}, 0\rangle = 0$$

from which it follows that $\mathcal{M}_k \subset (X\,\Pi_{k-1})^{\perp}$, and, therefore, $\mathcal{M}_k = (X\,\Pi_{k-1})^{\perp}$.
As a result one arrives at the Fischer decomposition with respect to the discrete Dirac operator D^{+-}.

Theorem 6 (Fischer Decomposition). *Let P_k be a discrete homogeneous polynomial of degree k. Then*

$$P_k = M_k + XM_{k-1} + X^2M_{k-2} + \ldots + X^kM_0 \tag{16}$$

where each M_j denotes a homogeneous discrete monogenic polynomial of degree j.

A simple combinatorial argument shows that the dimension on $\mathcal{M}_k$ is equal to

$$\dim\mathcal{M}_k = \frac{(k+m-1)!}{k!(m-1)!}$$

For determining the monogenic projection $\mathrm{proj}_{\mathcal{M}_k}$, i.e. the projection of a discrete homogeneous polynomial P_k onto the space of discrete monogenic homogeneous polynomials $\mathcal{M}_k$ one usually makes the ansatz

$$r = P_k + a_1XD^{+-} + \ldots + a_kX^k(D^{+-})^kP_k.$$

Since it is required that $D^{+-}r = 0$ one can evaluate the right-hand side and get the following theorem.

Theorem 7. *The monogenic projection of a homogeneous polynomial P_k of degree k is given by*

$$\mathrm{proj}_{\mathcal{M}_k}P_k = P_k + a_1XD^{+-} + \ldots + a_kX^k(D^{+-})^kP_k$$

with $a_1 = -\frac{1}{2k-2+n}, a_2 = \frac{a_1}{2}, a_3 = -\frac{a_2}{2k-4+n}, \ldots$ and

$$\begin{cases} a_k = \frac{a_{k-1}}{k} & k \text{ even} \\ a_k = -\frac{a_{k-1}}{k-1+n} & k \text{ odd} \end{cases}$$

Amazing Action of $SO(n)$ on the Space of Discrete Spherical Harmonics

As one could observe in the previous section, D^{+-}, X, and E form a super-algebra isomorphic to $osp(1|2)$. Since on the space of discrete homogeneous polynomials of degree k it holds $EP_k =$

DOI 10.1007/978-3-0348-0692-3_18-1

kP_k one can reduce that algebra to the classic Heisenberg algebra H_1 as the operator algebra over this space, E basically acting as a multiple of the identity on this space.

Cauchy–Kovalevskaya Extension and Discrete Taylor Series

One of the principal tools to construct monogenic functions in classic Clifford analysis is the so-called Cauchy–Kovalevskaya (CK-) extension. It is based on the fact that one can consider a Cauchy problem for the Dirac operator with respect to one variable, i.e.

$$\begin{cases} (\partial_{x_0} + \sum_{i=1}^{n} e_j \partial_{x_j})u = (\partial_{x_0} + \tilde{D})u = 0 \\ u(0, x_1, \ldots, x_n) = u_0 \end{cases}$$

for which the solution is given in form of the usual operator power series

$$u(x) = e^{-x_0 \tilde{D}} u_0.$$

To get the corresponding method in discrete Clifford analysis the authors of [8] proposed the following CK-extension:

$$\mathrm{CK}[f](x_1, x_2, \ldots, x_n) = \sum_{k=0}^{\infty} \frac{\xi_1^k[1](x_1)}{k!} f_k(x_2, \ldots, x_n)$$

A direct evaluation allowed the authors to obtain the following definition:

Definition 1. The CK-extension $\mathrm{CK}[f]$ of a discrete function $f \in \mathbb{Z}^{n-1}$ is given by

$$\mathrm{CK}[f](x_1, x_2, \ldots, x_n) = \sum_{k=0}^{\infty} \frac{\xi_1^k[1](x_1)}{k!} f_k(x_2, \ldots, x_n)$$

where $f_0 = f$ and $f_{k+1} = (-1)^{k+1} \partial' f_k$.

That this extension is unique can be deduced from the following theorem.

Theorem 8. *Consider a discrete function $f : \mathbb{Z}^n \to \mathbb{C}_n$, monogenic over $\mathbb{Z}^n$. If $f|_{x_1} = 0$ then $f = 0$ everywhere.*

Like in the continuous case the CK-extension establishes an isomorphism between the space $\mathcal{M}_k$ of discrete homogeneous monogenic polynomials of degree k and the space Π_{k-1} of discrete homogeneous polynomials of degree k in dimension $n-1$. Since a basis for Π_{k-1} is given by

$$\xi_2^{\alpha_2} \ldots \xi_n^{\alpha_n}[1], \quad |\alpha_2| + \cdots + |\alpha_n| = k - 1$$

one gets that the system

$$\{V_\alpha : V_\alpha = \mathrm{CK}[\xi_2^{\alpha_2} \ldots \xi_n^{\alpha_n}[1]]\}$$

DOI 10.1007/978-3-0348-0692-3_18-1

forms a basis of the space $\mathcal{M}_k$ of discrete homogeneous monogenic polynomials of degree k. The elements of this basis are called discrete Fueter polynomials of degree k.

The discrete Fueter polynomials can also be obtained in a different way [6]. Consider $z_i = \xi_i - \xi_1$ and $\hat{z}_i = \xi_i + \xi_1$ as well as the product $(z_{l_1} \ldots z_{l_k})^{E_{r_1, r_2, \ldots}}$ where every second, fourth, sixth, ... occurrence of z_{r_1} is replaced by $\hat{z_{r_1}}$, then the same with r_2, and so on. Then one has

$$V_{\alpha_1,\ldots,\alpha_k} = \frac{1}{k!} \sum_{\pi(\alpha_1,\ldots,\alpha_k)} \operatorname{sgn}(\imath)(z_{\pi(\alpha_1)} \ldots z_{\pi(\alpha_k)}))^{E_{2,\ldots,n}}.$$

This basis allows to obtain a power series expansion for a discrete monogenic function. For simplicity one can restrict oneself to the case of a Taylor series with respect to the origin. The general case can be seen in [6]. Using the basis $\{\xi^\alpha[1] : |\alpha| \in \mathbb{N} \cup \{0\}\}$ one can write the discrete Taylor series of a discrete function f defined on $\mathbb{Z}^n$ in the form

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{\alpha!} \sum_{|\alpha|=k} \xi^\alpha[1] \delta^\alpha f(0)$$

For the convergence of this series one can observe that

$$\xi_j^k[1](x_j) = 0, \qquad \forall k \geq 2|x_j| + 1.$$

What looks strange at first view simply means that the points inside a certain rectangle are zeros of the corresponding coordinate polynomials starting from a certain degree. A similar fact can be observed in classic Newton interpolation where the coefficients are given by the divided differences and the basic polynomials are products of terms like $(x - x_j)$ which naturally means that x_j is a zero of all polynomials starting from a certain degree. In fact, in the discrete setting Newton interpolation can be seen as a kind of Taylor series. The consequence of this observation is that the above discrete Taylor series is in fact a finite series in each point and for a function f defined on a bounded cube centered at the origin the discrete Taylor series of f is finite in each point of the cube with a maximum degree N_k for all points. Furthermore, a discrete Taylor series converges normally over any bounded domain.

If one considers now the Taylor series of a discrete monogenic function f, then one can take the monogenic projection of each discrete homogeneous polynomial and arrive at the following theorem.

Theorem 9 ([6]). *Let $\Omega \subset \mathbb{Z}^n$ be a discrete bounded set containing the origin such that for all $x \in \Omega$ the rectangle $\{y : |y_j| \leq |x_j|\} \subset \Omega$ and f be defined on the set*

$$\bigcup_{x \in \Omega} \{y : |y_j| \leq |x_j| + 1\}.$$

If f is discrete monogenic in Ω, then f can be developed into a convergent series of discrete homogeneous monogenic polynomials as follows:

$$f(x) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} V_\alpha \delta^\alpha f(0).$$

There is also the converse theorem.

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Theorem 10 ([6]). *Let* $\Omega \subset \mathbb{Z}^n$ *be a discrete bounded set containing the origin such that for all* $x \in \Omega$ *the rectangle* $\{y : |y_j| \leq |x_j|\} \subset \Omega$ f *be defined on the set*

$$\bigcup_{x\in\Omega}\{y : |y_j| \leq |x_j| + 1\}$$

given by

$$f(x) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} V_\alpha \lambda_\alpha, \quad \lambda_\alpha \in \mathbb{C}_n.$$

Then f *represents a discrete monogenic function on the set* Ω *and* $\lambda_\alpha = \delta^\alpha f(0)$.

Discrete Boundary Values and Hardy Spaces

In [4] the authors consider the question of Hardy spaces (for spacial dimension 3). To this end they start with the following discrete Cauchy integral formulae for the upper and lower discrete half space:

Theorem 11. *Let* f *be a discrete left monogenic function with respect to operator* D^{+-}*, then the upper discrete Cauchy formula*

$$\sum_{\underline{n}\in\mathbb{Z}^2} \left[E^{-+}((\underline{n}-\underline{m}, -m_3))e_3^+ f((\underline{n}, 1)) + E^{-+}((\underline{n}-\underline{m}, 1-m_3))e_3^- f(\underline{n}, 0)\right]$$
$$= \begin{cases} 0, & \text{if } m_3 \leq 0, \\ -f(m), & \text{if } m_3 > 0. \end{cases} \tag{17}$$

holds under the condition that the involved series converge.

In the same way the lower discrete Cauchy formula can also be given by

$$\sum_{\underline{n}\in\mathbb{Z}^2} \left[E^{-+}((\underline{n}-\underline{m}, -1-m_3))e_3^+ f(\underline{n}, 0) + E^{-+}((\underline{n}-\underline{m}, -m_3))e_3^- f((\underline{n}, -1))\right]$$
$$= \begin{cases} 0, & \text{if } m_3 \geq 0, \\ f(m), & \text{if } m_3 < 0. \end{cases} \tag{18}$$

A sufficient condition for the convergence of the series is $f \in l_p(\mathbb{Z}^3, \mathbb{C}_3)$, $1 \leq p < \infty$. From these Cauchy formulae they obtain the following discrete Cauchy transforms.

Definition 2. For a discrete l_p-function f, $1 \leq p < +\infty$, defined on the boundary layers $(\underline{n}, 0)$, $(\underline{n}, 1)$ with $\underline{n} \in \mathbb{Z}^2$, one defines the upper Cauchy transform for $m = (\underline{m}, m_3) \in \mathbb{Z}^3_+$ as

$$C^+[f](m) = -\sum_{\underline{n}\in\mathbb{Z}^2} \left[E^{-+}((\underline{n}-\underline{m}, -m_3))e_3^+ f((\underline{n}, 1)) + E^{-+}((\underline{n}-\underline{m}, 1-m_3))e_3^- f(\underline{n}, 0)\right], \tag{19}$$

DOI 10.1007/978-3-0348-0692-3_18-1

and for a discrete l_p-function f, $1 \leq p < \infty$, defined on the boundary layers $(\underline{n}, -1), (\underline{n}, 0)$ with $\underline{n} \in \mathbb{Z}^2$, one defines the lower Cauchy transform at $m = (\underline{m}, m_3) \in \mathbb{Z}^3_-$ by

$$C^-[f](m) = \sum_{\underline{n} \in \mathbb{Z}^2} \left[E^{-+}((\underline{n} - \underline{m}, -1 - m_3)) e_3^+ f(\underline{n}, 0) + E^{-+}((\underline{n} - \underline{m}, -m_3)) e_3^- f((\underline{n}, -1)) \right]. \tag{20}$$

From the discrete Cauchy formulae (17) and (18) and the definition of the discrete Cauchy transforms one can observe quite clearly the dependence of discrete monogenic functions on the boundary values where the boundary consists of three different layers (two for each). The discrete Cauchy transforms have the following properties:

Theorem 12. *Consider the upper and lower Cauchy transforms (19) and (20), respectively. Here, one has*

$$(i)\ C^+[f] \in \mathfrak{l}_p\left(\mathbb{Z}^3_+, \mathbb{C}_3\right), \quad C^-[f] \in \mathfrak{l}_p\left(\mathbb{Z}^3_-, \mathbb{C}_3\right), \quad 1 \leq p < +\infty, \tag{21}$$

$$(ii)\ D^{+-} C^+[f](m) = 0, \forall m = (\underline{m}, m_3) \in \mathbb{Z}^3 \textit{ with } m_3 > 1, \tag{22}$$

$$(iii)\ D^{+-} C^-[f](m) = 0, \forall m = (\underline{m}, m_3) \in \mathbb{Z}^3 \textit{ with } m_3 < -1. \tag{23}$$

Furthermore, one obtains a discrete equivalent to the boundary behavior of a monogenic function. Formula (17) means that for the boundary values (at the layer $m_3 = 1$) of a function which is discrete monogenic in the upper half plane it holds

$$-\sum_{\underline{n} \in \mathbb{Z}^2} \left[E^{-+}((\underline{n} - \underline{m}, -1)) e_3^+ f((\underline{n}, 1)) + E^{-+}((\underline{n} - \underline{m}, 0)) e_3^- f((\underline{n}, 0)) \right] = f((\underline{m}, 1)), \tag{24}$$

while formula (18) states that for the boundary values ($m_3 = -1$) of a function which is discrete monogenic in the lower half plane it holds

$$\sum_{\underline{n} \in \mathbb{Z}^2} \left[E^{-+}((\underline{n} - \underline{m}, 0)) e_3^+ f((\underline{n}, 0)) + E^{-+}((\underline{n} - \underline{m}, 1)) e_3^- f((\underline{n}, -1)) \right] = f((\underline{m}, -1)). \tag{25}$$

Calculating and evaluating the Fourier symbols over the boundary one obtains the following expression for the upper and lower Hilbert transforms

$$H_+ f = \mathcal{F}_h^{-1} \left[\frac{\underline{\tilde{\xi}}}{\underline{d}} \left(e_3^+ \frac{\underline{d} - \sqrt{4 + \underline{d}^2}}{2} - e_3^- \frac{\underline{d} + \sqrt{4 + \underline{d}^2}}{2} \right) \right] \mathcal{F}_h f, \tag{26}$$

$$H_- f = \mathcal{F}_h^{-1} \left[\frac{\underline{\tilde{\xi}}}{\underline{d}} \left(e_3^+ \frac{\underline{d} + \sqrt{4 + \underline{d}^2}}{2} - e_3^- \frac{\underline{d} - \sqrt{4 - \underline{d}^2}}{2} \right) \right] \mathcal{F}_h f, \tag{27}$$

DOI 10.1007/978-3-0348-0692-3_18-1

where $\mathcal{F}_h$ denotes the two-dimensional discrete Fourier transform. These operators satisfy $H_+^2 = H_-^2 = I$. This means that one can introduce the discrete Hardy spaces $h_p^{\pm}$ as the space of discrete functions $f \in l_p(h\mathbb{Z}^2, \mathbb{C}_3)$ which satisfy $P_{\pm} f = \frac{1}{2}(1 + H_{\pm}) f = f$. The last conditions can be thought of as the discrete equivalent to the Plemelj–Sokhotzki formulae. Please note that in the discrete case one has $P_- \neq I - P_+$, unlike the continuous case.

Some Genuine Discrete Objects and Properties from Discrete Function Theory

Let us summarize some more facts about discrete function theory which are unique for the discrete case and do not have an analogue in the continuous case.

First of all, any discrete function $f : \mathbb{Z}^n \mapsto \mathbb{C}_n$ can be expressed in terms of discrete delta functions, i.e.

$$f(m) = \sum_{k \in \mathbb{Z}^n} \delta_k(m) \lambda_k = \sum_{k \in \mathbb{Z}^n} \delta(k - m) \lambda_k$$

with $\delta_k(m) = \delta(k - m) = 0, k \neq m$, and $\delta_m(m) = \delta(0) = 1$. One consequence of this rather unique property of the discrete case is that the study of discrete functions can be reduced to the study to discrete delta functions in several cases. In particular, δ_k has a Taylor series expansion as well as a Cauchy–Kovalevskaya extension. The CK-extension in the case of $n = 2$ is given by (c.f. [6])

$$CK[\delta_0](m_1, m_2) = \sum_{k \in \mathbb{Z}^2} \frac{\xi_1^k[1](m_1)}{k!} f_k(x_2)$$

with $f_0 = \delta_0$ and

$$f_{2l} = \sum_{j=0}^{2l} (-1)^{j+l} \binom{2l}{j} \delta_{l-j},$$

$$f_{2l+1} = \sum_{j=0}^{2l+1} (-1)^{j+l+1} \binom{2l+1}{j} (e_2^+ \delta_{j-l-1} - e_2^- \delta_{l+1-j}).$$

The Taylor series expansion of δ_0 also provides another method for the calculation of discrete fundamental solutions [6]. In section "Discrete Cauchy Integral Formula" one had an expression for the fundamental solution in terms of a Fourier integral. Here, starting the Taylor expansion of the delta function δ_0 and determining the Fischer decomposition of each discrete homogeneous term one gets an expansion of the form

$$\delta_0(m_1, \ldots, m_n) = \sum_{s \geq 0} X^s M_k^{(s)},$$

DOI 10.1007/978-3-0348-0692-3_18-1

where $M_k^{(s)}$ is a discrete homogeneous monogenic polynomial. Now, the discrete fundamental solution will be given by

$$E(m) = \sum_{l,k\geq 0} \frac{1}{2l + 2k + n} X^{2l+1} M_k^{(2l)} + \sum_{l,k\geq 0} \frac{1}{2l + 2} X^{2l+2} M_k^{(2l+1)}.$$

Specific examples of such fundamental solutions can be found in [6].

Conclusion

Discrete Clifford analysis is a quite recent research field. Even in its simplest case as the function theory of a discrete Dirac operator over the standard lattice in several dimensions it already provides all ingredients for an exciting theory. Main tools from standard Clifford analysis, like Cauchy integral formula, Fischer decomposition, CK-extension, and Taylor series are available. Some of these tools require modifications in their construction due to the nature of discrete analysis, such as the replacement of classic Weyl relations by the so-called S-Weyl relations. Other results like discrete boundary values and Hardy spaces are asymptotically equivalent to its continuous counterpart, i.e. one obtains the continuous case when the lattice constants converge to zero. There are also genuinely discrete objects, like a CK-extension of a discrete delta function, which have no equivalent in the continuous case. Altogether, it shows that discrete Clifford analysis is a very interesting research field with a lot of potential for future development.

Acknowledgements This work was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology (FCTFundao para a Cincia e a Tecnologia), within project PEst-OE/MAT/UI4106/2014.

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