3

Clifford Analysis

John Ryan

ABSTRACT We introduce the basic concepts of Clifford analysis. This analysis started many years ago as an attempt to generalize one variable complex analysis to higher dimensions. Most of the basic analysis was initially developed over the quaternions which are a division algebra. However, it was soon realized that virtually all of this analysis extends to all dimensions using Clifford algebras. Here we introduce a generalized Cauchy–Riemann operator, often called a Dirac operator, and the analogues of holomorphic functions. These functions are called Clifford holomorphic functions or monogenic functions. We give a generalization of Cauchy's theorem and Cauchy's integral formula. Using Cauchy's theorem, we can establish the Möbius invariance of monogenic functions. We will also introduce the Plemelj formulas and operators, and Hardy spaces.

3.1 Introduction

In this chapter we regard Clifford algebras as natural generalizations of the complex number system. First, note that if z is a complex number, then $\overline{z}z = \|z\|^2$. For a quaternion q, we also have $\overline{q}q = \|q\|^2$. Quaternions in this way may be regarded as a generalization of the complex number system. It seems natural to ask if one can extend basic results of one complex variable analysis on holomorphic function theory to four dimensions using quaternions. The answer is yes. This was developed by the Swiss mathematician Rudolph Fueter in the 1930s and 1940s and also by Moisil and Theodorescu [29]. See for instance [12]. An excellent review of this work is given in the survey article "Quaternionic analysis" by Sudbery, see [47]. There is also earlier work of Dixon [11]. However, in previous lectures we have seen that for a vector $x \in \mathbb{R}^n$, when we consider $\mathbb{R}^n$ embedded in the Clifford algebra $C\ell_n$, then $x^2 = -\|x\|^2$. So it is reasonable to ask if all that is known in the quaternionic setting further extends to the Clifford algebra setting. Again the answer is yes. The earlier aspects of this study were developed

This lecture was presented at "Lecture Series on Clifford Algebras and their Applications", May 18 and 19, 2002, as part of the 6th International Conference on Clifford Algebras and their Applications in Mathematical Physics, Cookeville, TN, May 20–25, 2002.
AMS Subject Classification: 30G35.
Keywords: Cauchy–Riemann operator, monogenic functions, Cauchy's theorem, Cauchy's integral, Plemelj formula, Hardy space.

R. Ablamowicz et al., *Lectures on Clifford (Geometric) Algebras and Applications*

by among others, Richard Delanghe [9], Viorel Iftimie [16] and David Hestenes [15]. The subject that has grown from these works is now called Clifford analysis.

In more recent times, Clifford analysis has found a wealth of unexpected applications in a number of branches of mathematical analysis, particularly classical harmonic analysis. See, for instance, the work of Alan McIntosh and his collaborators [21, 22], Marius Mitrea [27, 28] and papers in [37]. Links to representation theory and several complex variables may be found in [14, 34–36] and elsewhere.

The purpose of this paper is to review the basic aspects of Clifford analysis.

Alternative accounts of much of this work, together with other related results, can be found in [5, 10, 13, 14, 20, 31, 37, 38].

3.2 Foundations of Clifford analysis

We start by replacing the vector $x = x_1e_1+\ldots+x_ne_n$ by the differential operator $D = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}$. One basic, but interesting, property of D is that $D^2 = -\triangle_n$, the Laplacian $\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ in $\mathbb{R}^n$. The differential operator D is called a Dirac operator.

This is because the classical Dirac operator constructed over four-dimensional Minkowski space squares to give the wave operator.

Definition 1. *Suppose that U is a domain in $\mathbb{R}^n$ and f and g are C^1-functions defined on U and taking values in $C\ell_n$. Then f is called a left monogenic function if $Df = 0$ on U, while g is called a right monogenic function on U if $gD = 0$, where $gD = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j} e_j$.*

Left monogenic functions are also called left regular functions and, perhaps most appropriately, left Clifford holomorphic functions. The term Clifford holomorphic functions, or Clifford analytic functions appears to be due to Semmes, see [41] and elsewhere. We shall most often use the term Clifford holomorphic functions.

Examples of such functions include the gradients of real valued harmonic functions on U. If h is harmonic on U, and if it is also real valued, then Dh is a vector valued left monogenic function. It is also a right monogenic function. Such a function is commonly referred to as a conjugate harmonic function, or a harmonic 1-form. See for instance [46]. An example of such a function is $G(x) = \frac{x}{\|x\|^n}$.

It should be noted that if f and g are left monogenic functions then, due to the lack of commutativity of the Clifford algebra, it is not in general true that their product $f(x)g(x)$ is left monogenic.

To introduce other examples of left monogenic functions, suppose that μ is a $C\ell_n$ valued measure with compact support $[\mu]$ in $\mathbb{R}^n$. Then the convolution

$$\int_{[\mu]} G(x-y)d\mu(y)$$

defines a left monogenic function on the maximal domain lying in $\mathbb{R}^n\backslash[\mu]$. The previously defined integral is the Cauchy transform of the measure $[\mu]$.

Another way of constructing examples of left monogenic functions was introduced by Littlewood and Gay in [23], for the case $n = 3$, and independently reintroduced for all n by Sommen [43]. Suppose U' is a domain in $\mathbb{R}^{n-1}$, spanned by $e_2, \ldots, e_n$. Suppose also that $f'(x')$ is a $C\ell_n$-valued function such that at each point $x' \in U'$ there is a multiple series expansion in $x_2, \ldots, x_n$ that converges uniformly on some neighborhood of x' in U' to f'. Such a function is called a real analytic function. The series

$$\sum_{k=0}^{\infty} \frac{1}{k!} x_1^k (-e_1 D')^k f'(x') = \exp(-x_1 e_1 D') f'(x'),$$

where $D' = \sum_{j=2}^{n} e_j \frac{\partial}{\partial x_j}$, defines a left monogenic function f in some neighborhood $U(f')$ in $\mathbb{R}^n$ of U'. The left monogenic function f is the Cauchy–Kowalewska extension of f'.

It should be noted that if f is a left monogenic function, then $\overline{f}$ and $\tilde{f}$ are both right monogenic functions[1].

We now turn to analogues of Cauchy's Theorem and Cauchy's integral formula.

Theorem 1 (Clifford–Cauchy Theorem). *Suppose that f is a left Clifford holomorphic function on U, and g is a right Clifford holomorphic function on U. Suppose also that V is a bounded subdomain of U with piecewise differentiable boundary S lying in U. Then*

$$\int_S g(x) n(x) f(x)\, d\sigma(x) = 0 \tag{2.1}$$

where $n(x)$ is the outward pointing normal vector to S at x and σ is the Lebesgue measure on S.

Proof. The proof follows directly from Stokes' Theorem. One important point to keep in mind is that, since $C\ell_n$ is not a commutative algebra, the order of the quantities g, $n(x)$ and f must be maintained. One then has that

$$\int_S g(x) n(x) f(x)\, d\sigma(x) = \int_V ((g(x)D) f(x) + g(x)(Df(x)))\, dx^n = 0.$$

Suppose that g is the gradient of a real valued harmonic function and $f = 1$. Then the real part of Equation 1 gives the following well-known integral formula:

$$\int < \operatorname{grad} g(x), n(x) > d\sigma(x) = 0.$$

[1]Here, $\overline{f}$ denotes the Clifford conjugate of f while $\tilde{f}$ is the reversion of a $C\ell_n$-valued function f.

We now turn to the analogue of a Cauchy integral formula.

Theorem 2 (Clifford–Cauchy Integral Formula). *Suppose that U, V, S, f and g are all as in Theorem 1 and that $y \in V$. Then*

$$f(y) = \frac{1}{\omega_n} \int_S G(x-y)n(x)f(x)\, d\sigma(x)$$

and

$$g(y) = \frac{1}{\omega_n} \int_S g(x)n(x)G(x-y)\, d\sigma(x)$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

Proof. The proof follows very similar lines to the argument in one variable complex analysis. We shall establish the formula for $f(y)$, the proof being similar for $g(y)$. First, let us take a sphere $S^{n-1}(y,r)$ centered at y and of radius r. The radius r is chosen sufficiently small so that the closed disc with boundary $S^{n-1}(y,r)$ lies in V. Then, by the Clifford–Cauchy theorem,

$$\int_S G(x-y)n(x)f(x)\, d\sigma(x) = \int_{S^{n-1}(y,r)} G(x-y)n(x)f(x)\, d\sigma(x).$$

However, on $S^{n-1}(y,r)$ the vector $n(x) = \frac{y-x}{\|x-y\|}$. So $G(x-y)n(x) = \frac{1}{r^{n-1}}$ and

$$\int_{S^{n-1}(y,r)} G(x-y)n(x)f(x)\, d\sigma(x)$$
$$= \int_{S^{n-1}(y,r)} \frac{1}{r^{n-1}}(f(x)-f(y))\, d\sigma(x) + \int_{S^{n-1}(y,r)} \frac{f(y)}{r^{n-1}}\, d\sigma(x).$$

The right side of the previous expression reduces to

$$\int_{S^{n-1}(y,r)} \frac{(f(x)-f(y))}{r^{n-1}}\, d\sigma(x) + f(y)\int_{S^{n-1}} d\sigma(x).$$

Now $\int_{S^{n-1}} d\sigma(x) = \omega_n$, and by continuity

$$\lim_{r\to 0} \int_{S^{n-1}(y,r)} \frac{(f(x)-f(y)}{r^{n-1}}\, d\sigma(x) = 0.$$

The result follows. □

One important feature to note is that Kelvin inversion, $x^{-1} = \frac{-x}{\|x\|^2}$ whenever x is nonzero, plays a fundamental role in this proof. Moreover, the proof is almost exactly the same as the proof of Cauchy's Integral Formula for piecewise C^1-curves in one variable complex analysis.

Having obtained a Cauchy Integral Formula in $\mathbb{R}^n$, a number of basic results that one might see in a first course in one variable complex analysis carry over more or less automatically to the context described here. This includes Liouville's Theorem and Weierstrass' Convergence Theorem. We leave it to the interested reader to set up and establish the Clifford analysis analogues of these results. Their statements and proofs can be found in [5].

Theorems 1 and 2 show us that the individual components of the equations $Df = 0$ and $gD = 0$ comprise generalized Cauchy–Riemann equations. In the particular case that f is vector valued, $f = \sum_{j=1}^{n} f_j e_j$, the generalized Cauchy–Riemann equations become $\frac{\partial f_j}{\partial x_i} = \frac{\partial f_i}{\partial x_j}$, whenever $i \neq j$, and $\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} = 0$. This system of equations is often referred to as the Riesz system.

Having obtained an analogue of Cauchy's integral formula in Euclidean space, we now exploit this result to show how many consequences of the classical Cauchy integral carry over to the context described here. We begin with the Mean Value Theorem.

Theorem 3 (The Mean Value Theorem). *Suppose that $D(y, R)$ is a closed disc centered at y, of radius R and lying in U. Then, for each left Clifford holomorphic function f on U*

$$f(y) = \frac{1}{R\omega_n} \int_{D(y,R)} \frac{f(x)}{\|x - y\|^{n-1}}\, dx^n.$$

Proof. We have already seen that for each $r \in (0, R)$,

$$f(y) = \frac{1}{\omega_n} \int_{S^{n-1}(y,r)} \frac{f(x)}{\|x - y\|^{n-1}}\, d\sigma(x),$$

where $S^{n-1}(y, r)$ is the $(n-1)$-dimensional sphere centered at y and of radius r. We obtain the result by integrating both sides of this expression with respect to the variable r, and dividing throughout by R. □

Let us now explore the real analyticity properties of Clifford holomorphic functions. First note that when n is even,

$$G(x - y) = (-1)^{\frac{n-2}{2}} (x - y)^{-n+1}.$$

Also

$$(x-y)^{-1} = x^{-1}(1-yx^{-1})^{-1} = (1-x^{-1}y)^{-1}x^{-1}, \ \|x^{-1}y\| = \|yx^{-1}\| = \frac{\|y\|}{\|x\|}.$$

So for, $\|y\| < \|x\|$,

$$\begin{aligned} (x - y)^{-1} &= x^{-1}(1 + yx^{-1} + \ldots + yx^{-1} \ldots yx^{-1} + \ldots) \\ &= (1 + x^{-1}y + \ldots + x^{-1}y \ldots x^{-1}y + \ldots)x^{-1}. \end{aligned}$$

Hence, these two sequences converge uniformly to $(x - y)^{-1}$ provided

$$\|y\| \leq r < \|x\|,$$

and they converge pointwise to $(x-y)^{-1}$ provided $\|y\| < \|x\|$. One now takes $(-1)^{\frac{n-2}{2}}$ times the $(n-1)$-fold product of the series expansions of $(x-y)^{-1}$ with itself to obtain a series expansion for $G(x-y)$. In the process of multiplying series together, in order to maintain the same radius of convergence, one needs to group together all linear combinations of monomials in $y_1, \ldots, y_n$ that are of the same order. Thus, we have deduced that when n is even, the multiple Taylor series expansion

$$\sum_{j=0}^{\infty}\Big(\sum_{\substack{j_1,\ldots,j_n \\ j_1+\ldots+j_n=j}} \frac{y_1^{j_1}\ldots y_n^{j_n}}{j_1!\ldots j_n!}\frac{\partial^j G(x)}{\partial x_1^{j_1}\ldots \partial x_n^{j_n}}\Big)$$

converges uniformly to $G(x-y)$ provided $\|y\| < r < \|x\|$, and converges pointwise to $G(x-y)$ provided $\|y\| < \|x\|$.

A similar argument holds when n is odd.

Returning to Cauchy's integral formula, let us suppose that f is a left Clifford holomorphic function defined in a neighborhood of the closure of some ball $B(0,R)$. Then

$$\begin{aligned} f(y) &= \frac{1}{\omega_n}\int_{\partial B(0,R)} G(x-y)n(x)f(x)\,d\sigma(x) \\ &= \frac{1}{\omega_n}\int_{\partial B(0,R)} \sum_{j=0}^{\infty}\Big(\sum_{\substack{j_1\cdots j_n \\ j_1+\ldots+j_n=j}} \frac{y_1^{j_1}\ldots y_n^{j_n}}{j_1!\ldots j_n!}\frac{\partial^j G(x)}{\partial x_1^{j_1}\ldots \partial x_n^{j_n}}\Big)n(x)f(x)\,d\sigma(x) \end{aligned}$$

provided $\|y\| < \|x\|$. Since this series converges uniformly on each ball $B(0,r)$, for each $r < R$, this last integral can be re-written as

$$\frac{1}{\omega_n}\sum_{j=0}^{\infty}\int_{\partial B(0,R)}\Big(\sum_{\substack{j_1\cdots j_n \\ j_1+\ldots+j_n=j}} \frac{y_1^{j_1}\ldots y_n^{j_n}}{j_1!\ldots j_n!}\frac{\partial^j G(x)}{\partial x_1^{j_1}\ldots \partial x_n^{j_n}}n(x)f(x)\Big)\,d\sigma(x).$$

Since the summation within the parentheses is a finite summation, this last expression easily reduces to

$$\frac{1}{\omega_n}\sum_{j=0}^{\infty}\Big(\sum_{\substack{j_1\cdots j_n \\ j_1+\ldots+j_n=j}} \frac{y_1^{j_1}\ldots y_n^{j_n}}{j_1!\ldots j_n!}\int_{\partial B(0,R)}\frac{\partial^j G(x)}{\partial x_1^{j_1}\ldots \partial x_n^{j_n}}n(x)f(x)\Big)\,d\sigma(x).$$

On placing

$$\frac{1}{\omega_n}\int_{\partial B(0,R)}\frac{\partial^j G(x)}{\partial x_1^{j_1}\ldots \partial x_n^{j_n}}n(x)f(x)\,d\sigma(x) = a_{j_1\ldots j_n}$$

it may be seen that on $B(0,R)$ the series

$$\sum_{j=0}^{\infty}\Big(\sum_{\substack{j_1\cdots j_n \\ j_1+\ldots+j_n=j}} \frac{x_1^{j_1}\ldots x_n^{j_n}}{j_1!\ldots j_n!}a_{j_1\ldots j_n}\Big)$$

converges pointwise to $f(y)$. Convergence is uniform on each ball $B(0,r)$, provided $r < R$.

Similarly, if g is a right Clifford holomorphic function defined in a neighborhood of the closure of $B(0,R)$, then the series

$$\sum_{j=0}^{\infty}\Bigg(\sum_{\substack{j_1\cdots j_n \\ j_1+\ldots+j_n=j}} b_{j_1\ldots j_n} \frac{y_1^{j_1}\ldots y_n^{j_n}}{j_1!\ldots j_n!}\Bigg)$$

converges pointwise on $B(0,R)$ to $g(y)$ and converges uniformly on $B(0,r)$ for $r < R$, where

$$b_{j_1\ldots j_n} = \frac{1}{\omega_n}\int_{\partial B(0,R)} g(x)n(x)\frac{\partial^j G(x)}{\partial x_1^{j_1}\ldots\partial x_n^{j_n}}\, d\sigma(x).$$

By translating the ball $B(0,R)$ to the ball $B(w,R)$, where

$$w = w_1e_1 + \ldots + w_ne_n,$$

one may readily observe that for any left Clifford holomorphic function f, defined in a neighborhood of the closure of $B(w,R)$, the series

$$\sum_{j=0}^{\infty}(\sum_{\substack{j_1\cdots j_n \\ j_1+\ldots j_n=j}} \frac{(y_1-w_1)^{j_1}\ldots(y_n-w_n)^{j_n}}{j_1!\ldots j_n!}a'_{j_1\ldots j_n})$$

converges pointwise on $B(w,R)$ to $f(y)$, where

$$a'_{j_1\ldots j_n} = \frac{1}{\omega_n}\int_{\partial B(w,R)} \frac{\partial^j G(x-w)}{\partial x_1^{j_1}\ldots x_n^{j_n}}n(x)f(x)\, d\sigma(x).$$

Again, the series converges uniformly on $B(w,r)$ for each $r < R$. A similar series may be readily obtained for any right Clifford holomorphic function defined in a neighborhood of the closure of $B(w,R)$.

The types of power series that we have developed for left Clifford holomorphic functions are not entirely satisfactory. In particular, unlike their complex analogues, the homogeneous polynomials

$$\sum_{\substack{j_1\cdots j_n \\ j_1+\ldots+j_n=j}} \frac{x_1^{j_1}\ldots x_n^{j_n}}{j_1!\ldots j_n!}a_{j_1\ldots j_n}$$

are not expressed as a linear combination of left Clifford holomorphic polynomials. To rectify this situation, let us first take a closer look at the Taylor expansion for the Cauchy kernel $G(x-y)$ where all the Taylor coefficients are real. Let us first look at the first order terms in the Taylor expansion. This is the expression

$$y_1\frac{\partial G(x)}{\partial x_1} + \ldots + y_n\frac{\partial G(x)}{\partial x_n}.$$

Since G is a Clifford holomorphic function,

$$\frac{\partial G(x)}{\partial x_1} = -\sum_{j=2}^{n} e_1^{-1} e_j \frac{\partial G(x)}{\partial x_j}.$$

Therefore, the first order terms of the Taylor expansion for $G(x-y)$ can be re-expressed as

$$\sum_{j=2}^{n} (y_j - e_1^{-1} e_j y_1) \frac{\partial G(x)}{\partial x_j}.$$

Moreover, for $2 \leq j \leq n$, the first order polynomial $y_j - e_1^{-1} e_j y_1$ is a left Clifford holomorphic polynomial. Let us now go to second order terms. Again, we replace the operator $\frac{\partial}{\partial x_1}$ by the operator

$$-\sum_{j=2}^{n} e_1^{-1} e_j \frac{\partial}{\partial x_j}$$

whenever it arises. Let us consider the term $\frac{\partial^2 G(x)}{\partial x_i \partial x_j}$, where $i \neq j \neq 1$. We end up with the polynomial

$$\begin{aligned} &y_i y_j - y_i y_1 e_1^{-1} e_j - y_j y_1 e_1^{-1} e_i \\ &\quad = \tfrac{1}{2}((y_i - y_1 e_1^{-1} e_i)(y_j - y_1 e_1^{-1} e_j) + (y_j - y_1 e_1^{-1} e_j)(y_i - y_1 e_1^{-1} e_i). \end{aligned}$$

Similarly, the polynomial attached to the term $\frac{\partial^2 G(x)}{\partial x_i^2}$ is $(y_i - y_1 e_1^{-1} e_i)^2$. Using the Clifford algebra anticommutation relations $e_i e_j + e_j e_i = -2\delta_{ij}$, and on replacing the differential operator $\frac{\partial}{\partial x_1}$ by the operator

$$-\sum_{j=2}^{n} e_1^{-1} e_j \frac{\partial}{\partial x_j},$$

the power series we previously obtained for $G(x-y)$ can be replaced by the series

$$\sum_{j=0}^{\infty} \Big(\sum_{\substack{j_2 \cdots j_n \\ j_2 + \cdots + j_n = j}} P_{j_2 \ldots j_n}(y) \frac{\partial^j G(x)}{\partial x_2^{j_2} \ldots \partial x_n^{j_n}} \Big),$$

where $\|y\| < \|x\|$ and

$$P_{j_2 \ldots j_n}(y) = \frac{1}{j!} \Sigma (y_{\sigma(1)} - y_1 e_1^{-1} e_{\sigma(1)}) \ldots (y_{\sigma(j)} - y_1 e_1^{-1} e_{\sigma(j)}).$$

Here, $\sigma(i) \in \{2, \ldots, n\}$ and the previous summation is taken over all permutations of the monomials $(y_{\sigma(i)} - y_1 e_1^{-1} e_{\sigma(i)})$ without repetition. The quaternionic monogenic analogues for these polynomials were introduced by Fueter [12], while the Clifford analogues, $P_{j_2\ldots j_n}$, were introduced by Delanghe in [9]. It should be noted that each polynomial $P_{j_2\ldots j_n}(y)$ takes its values in the space spanned by $\{1, e_1e_2, \ldots, e_1e_n\}$. Also, each such polynomial is homogeneous of degree j. Similar arguments to those just outlined give

$$G(x-y)=\sum_{j=0}^{\infty}\Bigg(\sum_{\substack{j_2\cdots j_n\\ j_2+\ldots+j_n=j}}\frac{\partial^j G(x)}{\partial x_2^{j_2}\ldots x_n^{j_n}}\widetilde{P_{j_2\ldots j_n}(y)}\Bigg)$$

provided $\|y\| < \|x\|$.

Proposition 1. *Each of the polynomials $P_{j_2\ldots j_n}(y)$ is a left Clifford holomorphic polynomial.*

Proof. Calculating

$$DP_{j_2\ldots j_n}(y)=e_1\Big(\frac{\partial}{\partial y_1}+e_1^{-1}\sum_{j=2}^{n}e_j\frac{\partial}{\partial y_j}P_{j_2\ldots j_n}(y)\Big)$$

we then consider the expression

$$(\frac{\partial}{\partial y_1}+\sum_{j=2}^{n}e_1^{-1}e_j\frac{\partial}{\partial y_j})P_{j_2\ldots j_n}(y).$$

This term is equal to

$$\begin{aligned}\Big(\frac{\partial}{\partial y_j}+\sum_{j=2}^{n}e_1^{-1}e_j\frac{\partial}{\partial y_j}\Big)\Sigma(y_{\sigma(1)}-e_1^{-1}e_{\sigma(1)}y_1)\ldots(y_{\sigma(i-1)}-e_1^{-1}e_{\sigma(i-1)}y_1)\\ \times(y_{\sigma(i)}-e_1^{-1}e_{\sigma(i)}y_1)(y_{\sigma(i+1)}-e_1^{-1}e_{\sigma(i+1)}y_1)\ldots(y_{\sigma(j)}-e_1^{-1}e_{\sigma(j)}y_1).\end{aligned}$$

This is equal to

$$\begin{aligned}&\sum(y_{\sigma(1)}-e_1^{-1}e_{\sigma(1)}y_1)\ldots(y_{\sigma(i-1)}-e_1^{-1}e_{\sigma(i-1)}y_1)(-e_1^{-1}e_{\sigma(i)})\\ &\qquad\times(y_{\sigma(i+1)}-e_1^{-1}e_{\sigma(i+1)}y_1)\ldots(y_{\sigma(j)}-e_1^{-1}e_{\sigma(j)}y_1)\\ &\quad+\sum e_1^{-1}e_{\sigma(i)}(y_{\sigma(1)}-e_1^{-1}e_{\sigma(1)}y_1)\ldots(y_{\sigma(i-1)}-e_1^{-1}e_{\sigma(i-1)}y_1)\\ &\qquad\times(y_{\sigma(i+1)}-e_1^{-1}e_{\sigma(i+1)}y_1)\ldots(y_{\sigma(j)}-e_1^{-1}e_{\sigma(j)}y_1).\end{aligned}$$

If we multiply the previous term by y_1, and add to it the following term, which is equal to zero,

$$\begin{aligned}&\sum(y_{\sigma(1)}-e_1^{-1}e_{\sigma(1)}y_1)\ldots(y_{\sigma(i-1)}-e_1^{-1}e_{\sigma(i-1)}y_1)(y_{\sigma(i)}-y_{\sigma(i)})\\ &\qquad\times(y_{\sigma(i+1)}-e_1^{-1}e_{\sigma(i+1)}y_1)\ldots(y_{\sigma(j)}-e_1^{-1}e_{\sigma(i)}y_1)\end{aligned}$$

we get, after regrouping terms,

$$\sum (y_{\sigma(1)} - e_1^{-1}e_{\sigma(1)}y_1)\dots(y_{\sigma(i-1)} - e_1^{-1}e_{\sigma(i-1)}y_1)(y_{\sigma(i)} - e_1^{-1}e_{\sigma(i)}y_1)$$
$$\times (y_{\sigma(i+1)} - e_1^{-1}e_{\sigma(i+1)}y_1)\dots(y_{\sigma(j)} - e_1^{-1}e_{\sigma(j)}y_1)$$
$$-\sum (y_{\sigma(i)} - e_1^{-1}e_{\sigma(i)}y_1)(y_{\sigma(1)} - e_1^{-1}e_{\sigma(1)}y_1)\dots(y_{\sigma(i-1)} - e_1^{-1}e_{\sigma(i-1)}y_1)$$
$$\times (y_{\sigma(i+1)} - e_1^{-1}e_{\sigma(i+1)}y_1)\dots(y_{\sigma(j)} - e_1^{-1}e_{\sigma(j)}y_j).$$

Since the summation is taken over all possible permutations, without repetition, the last term vanishes. □

Using Proposition 1 and the results we previously obtained on series expansions, we can obtain the following generalization of Taylor expansions from complex analysis.

Theorem 4 (Taylor Series). *Suppose that f is a left Clifford holomorphic function defined in an open neighborhood of the closure of the ball $B(w,R)$. Then*

$$f(y) = \sum_{j=0}^{\infty}\Big(\sum_{\substack{j_2\cdots j_n \\ j_2+\dots+j_n=j}} P_{j_2\dots j_n}(y-w)a_{j_2\dots j_n}\Big)$$

where

$$a_{j_2\dots j_n} = \frac{1}{\omega_n}\int_{\partial B(w,R)} \frac{\partial^j G(x-w)}{\partial x_2^{j_2}\dots\partial x_n^{j_n}} n(x)f(x)\,d\sigma(x)$$

and $\|y-w\| < R$. Convergence is uniform provided $\|x-w\| < r < R$.

A simple application of Cauchy's theorem tells us that the Taylor series that we obtained for f in the previous theorem remains valid on the largest open ball on which f is defined, and on the largest open ball on which g is defined. Also, the previous identities immediately yield the mutual linear independence of the collection of the left Clifford holomorphic polynomials

$$\{P_{j_2\dots j_n} : j_2 + \dots j_n = j,\ 0 \le j < \infty\}.$$

3.3 Other types of Clifford holomorphic functions

Unlike the classical Cauchy–Riemann operator $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$, the generalized Cauchy–Riemann operator D that we have introduced here does not have an identity component. Instead, we could have considered the differential operator

$$D' = \frac{\partial}{\partial x_0} + \sum_{j+1}^{n-1} e_j \frac{\partial}{\partial x_j}.$$

Also for U' a domain in $\mathbb{R} \oplus \mathbb{R}^{n-1}$, spanned by $\{1, e_1, \ldots, e_{n-1}\}$, one can consider $C\ell_{n-1}$-valued differentiable functions f' and g' defined on U' such that $D'f' = 0$ and $g'D' = 0$, where

$$g'D' = \frac{\partial g'}{\partial x_0} + \sum_{j=1}^{n-1} \frac{\partial g'}{\partial x_j} e_j.$$

Traditionally, such functions are also called left monogenic and right monogenic functions. To avoid confusion, we shall call such functions unital left monogenic and unital right monogenic, respectively. In the case where $n = 2$, the operator D' corresponds to the usual Cauchy–Riemann operator, and unital monogenic functions are the usual holomorphic functions studied in one variable complex analysis. The function

$$G'(\underline{x}) = \frac{\overline{\underline{x}}}{\|\underline{x}\|^n} = \underline{x}^{-1}\|\underline{x}\|^{-n+2}$$

is an example of a function which is both unital left monogenic and unital right monogenic. It is a simple matter to observe that f' is unital left monogenic if and only if $\tilde{f}'$ is unital right monogenic. However, $\overline{f}'$ is not unital right monogenic whenever f' is unital left monogenic. Instead, $\overline{f}'$ satisfies the equation $\overline{f}'D' = 0$.

The function theory for unital left monogenic functions is much the same as for left monogenic functions. For instance, if f' is unital left monogenic on U', and g' is unital right monogenic on the same domain, and S' is a piecewise smooth, compact surface lying in U' and bounding a subdomain V', then

$$\int_{S'} g'(\underline{x}) n(\underline{x}) f'(\underline{x}) \, d\sigma(\underline{x}) = 0,$$

where $n(\underline{x})$ is the outward pointing normal vector to S' at $\underline{x}$. Also, for each $\underline{y} \in V'$ there is the following version of Cauchy's integral formula:

$$f'(\underline{y}) = \frac{1}{\omega_n} \int_{S'} G'(\underline{x} - \underline{y}) n(\underline{x}) f'(\underline{x}) \, d\sigma(\underline{x}).$$

To get from the operator D to the operator D', one first rewrites D as

$$e_n \Big(\frac{\partial}{\partial x_n} + \sum_{j=1}^{n-1} e_n^{-1} e_j \frac{\partial}{\partial x_j} \Big).$$

On multiplying on the left by e_n, and changing the variable x_n to x_0, we get the operator

$$D" = \frac{\partial}{\partial x_0} + \sum_{j=1}^{n-1} e_n^{-1} e_j \frac{\partial}{\partial x_j}.$$

This operator takes its values in the even subalgebra $C\ell_n^+$ of $C\ell_n$. Applying the isomorphism

$$\theta : C\ell_{n-1} \to C\ell_n^+, \quad \theta(e_{j_1} \dots e_{j_r}) = e_n^{-1} e_{j_1} \dots e_n^{-1} e_{j_r},$$

it immediately follows that $\theta(D') = D"$. So if f' is unital left monogenic, then $D"\theta(f) = 0$. If we change the variable x_0 of the function $\theta(f(\underline{x}))$ to x_n, we get a left monogenic function, which we denote by $\theta'(f)(x)$, where

$$x = x_1 e_1 + \dots + x_n e_n \in U \subset \mathbb{R}^n$$

if and only if

$$\underline{x} = x_n + x_1 e_1 + \dots + x_{n-1} e_{n-1} \in U' \subset \mathbb{R} \oplus \mathbb{R}^{n-1}.$$

It should be noted that $D'\overline{D'} = \overline{D'}D' = \Delta_n$.

When $n = 3$, the algebra $C\ell_3$ is split by the two projection operators

$$E_\pm = \tfrac{1}{2}(1 \pm e_1 e_2 e_3)$$

into the direct sum

$$C\ell_3 = E_+ C\ell_3 \oplus E_- C\ell_3,$$

and each of these subalgebras is isomorphic to the quaternion algebra $\mathbb{H}$. In this setting, the differential operator $E_\pm D'$ can best be written as

$$E_\pm D' = \frac{\partial}{\partial t} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z},$$

and the operator $E_\pm D$ can best be written as

$$E_\pm D = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}.$$

We shall denote the first of these two operators by $D'_{\mathbb{H}}$ and the second by $D_{\mathbb{H}}$. The operator $D'_{\mathbb{H}}$ is sometimes referred to as the Cauchy–Riemann–Fueter operator. The function theory associated to the differential operators $D_{\mathbb{H}}$ and $D'_{\mathbb{H}}$ is much the same as that associated to the operators D and D'. In fact, historically the starting point for Clifford analysis was to study the function theoretic aspects of the operators $D'_{\mathbb{H}}$ and $D_{\mathbb{H}}$, see for instance [9, 12] and the excellent review article of Sudbery [47].

It is a simple enough matter to set up analogues of Cauchy's theorem and Cauchy's integral formula for the quaternionic valued differentiable functions that are either annihilated by $D'_{\mathbb{H}}$ or $D_{\mathbb{H}}$, either acting on the left or on the right. When dealing with the operator $D'_{\mathbb{H}}$, such functions are called quaternionic monogenic. The quaternionic monogenic Cauchy kernel is the function $q^{-1}\|q\|^{-2}$. Consequently, for each quaternionic left monogenic function $f(q)$ defined on a domain

$U" \subset \mathbb{H}$, and each q_0 lying in a bounded subdomain with piecewise C^1-boundary $S"$,

$$f(q_0) = \frac{1}{\omega_3} \int_{S"} (q - q_0)^{-1} \|q - q_0\|^{-2} n(q) f(q)\, d\sigma(q).$$

Similarly, if g is right quaternionic monogenic on $U"$, then

$$g(q_0) = \frac{1}{\omega_3} \int_{S"} g(q) n(q) (q - q_0)^{-1} \|q - q_0\|^{-2}\, d\sigma(q). \tag{3.2}$$

3.4 The equation $D^k f = 0$

It is reasonably well known that if h is a real valued harmonic function defined on a domain $U \subset \mathbb{R}^n$, then for each $y \in U$ and each compact, piecewise C^1-surface S lying in U such that S bounds a subdomain V of S and $y \in V$,

$$h(y) = \frac{1}{\omega_n} \int_S (H(x-y) < n(x), \text{grad}\, h(x) > - < G(x-y), n(x) > h(x))\, d\sigma(x),$$

where

$$H(x-y) = \frac{1}{(n-2)\|x-y\|^{n-2}}.$$

This is Green's formula for a harmonic function, and it heavily relies on the standard inner product on $\mathbb{R}^n$. Introducing the Clifford algebra $C\ell_n$, the right side of Green's formula is the real part of

$$\frac{1}{\omega_n} \int_S (G(x-y) n(x) h(x) - H(x-y) n(x) Dh(x))\, d\sigma(x).$$

Assuming that the function h is C^2, then on applying Stokes' theorem, the previous integral becomes

$$\frac{1}{\omega_n} \int_{S^{n-1}(y,r(y))} \Big(G(x-y) n(x) h(x) - H(x-y) n(x) Dh(x)\Big)\, d\sigma(x),$$

where $S^{n-1}(y, r(y))$ is a sphere centered at y, of radius $r(y)$ and lying in V. On letting the radius $r(y)$ tend to zero, the first term of the integral tends to $h(y)$, while the second term tends to zero. Consequently, the Clifford analysis version of Green's formula is

$$h(y) = \frac{1}{\omega_n} \int_S \Big(G(x-y) n(x) h(x) - H(x-y) n(x) Dh(x)\Big)\, d\sigma(x).$$

We obtained this formula under the assumption that h is real valued and C^2. The fact that h is real valued can easily be seen to be irrelevant, and so we can assume that h is $C\ell_n$ valued. From now on, we shall assume that all harmonic functions take their values in $C\ell_n$. If h is also a left monogenic function, then the Clifford analysis version of Green's formula becomes Cauchy's integral formula.

Proposition 2. *Suppose that f is a Clifford holomorphic function on some domain U. Then $xf(x)$ is harmonic.*

Proof.

$$Dxf(x) = -nf(x) - \sum_{j=1}^{n} x_j \frac{\partial f(x)}{\partial x_j} - \sum_{\substack{j,k \\ j \neq k}} x_k e_k e_j \frac{\partial f(x)}{\partial x_j}.$$

Now

$$\sum_{\substack{j,k \\ j \neq k}} x_k e_k e_j \frac{\partial f(x)}{\partial x_j} = \sum_{k=1}^{n} \sum_{j \neq k} x_k e_k e_j \frac{\partial f(x)}{\partial x_j}.$$

As f is left monogenic, this last expression simplifies to $\sum_{k=1}^{n} x_k \frac{\partial f(x)}{\partial x_k}$. Moreover, $D(\sum_{j=1}^{n} x_j \frac{\partial f(x)}{\partial x_j}) = 0$. Consequently, $D^2 xf(x) = 0$. □

The previous proof is a generalization of the statement: "if $h(x)$ is a real valued harmonic function, then so is $< x, \text{grad } h(x) >$".

In fact, in the previous proof, we determine that

$$Dxf(x) = -nf(x) - 2\sum_{j=1}^{n} x_j \frac{\partial f(x)}{\partial x_j}.$$

In the special case where $f(x) = P_k(x)$, a left Clifford holomorphic polynomial of order k, this equation simplifies to

$$DxP_k(x) = -(n + 2k)P_k(x).$$

Suppose now that $h(x)$ is a harmonic function defined in a neighborhood of the ball $B(0, R)$. Then Dh is a left Clifford holomorphic function, and there is a series $\sum_{l=0}^{\infty} P_l(x)$ of left Clifford holomorphic polynomials with each P_l homogeneous of degree l, such that the series converges locally uniformly on $B(0, R)$ to $Dh(x)$. Now consider the series

$$\sum_{l=0}^{\infty} \frac{-1}{n + 2l} P_l(x).$$

Since

$$\frac{1}{n + 2l} \|P_l(x)\| < \|P_l(x)\|,$$

this new series converges locally uniformly on $B(0, R)$ to a left Clifford holomorphic function $f_1(x)$. Moreover, $Dxf_1(x) = Dh(x)$ on $B(0, R)$. Consequently, $h(x) - xf_1(x)$ is equal to a left Clifford holomorphic function $f_2(x)$ on $B(0, R)$. We have established:

Proposition 3. *Suppose that h is a harmonic function defined in a neighborhood of $B(0, R)$. Then there are left Clifford holomorphic functions f_1 and f_2 defined on $B(0, R)$ such that $h(x) = xf_1(x) + f_2(x)$ for each $x \in B(0, R)$.*

This result remains invariant under translation. As a consequence, it shows us that all harmonic functions are real analytic functions. So there is no need to specify whether or not a harmonic function is C^2. The result also provides an Almansi type decomposition of harmonic functions in terms of Clifford holomorphic functions over any ball in $\mathbb{R}^n$.

It should be noted that Proposition 3 remains true only if h is real valued.

Proposition 3 gives rise to an alternative proof of the Mean Value Theorem for harmonic functions.

Theorem 5. *For any harmonic function h defined in a neighborhood of a ball $B(a, R)$ and any $r < R$,*

$$h(a) = \frac{1}{\omega_n} \int_{\partial B(a,r)} h(x)\, d\sigma(x)\,.$$

Proof. Proposition 3 tells us that there is a pair of left Clifford holomorphic functions f_1 and f_2 such that

$$h(x) = (x - a) f_1(x) + f_2(x)$$

on $B(a, R)$. So $h(a) = f_2(a)$, and we have previously shown that

$$\frac{1}{\omega_n} \int_{\partial B(a,r)} f_2(x)\, d\sigma(x) = f_2(a).$$

Now

$$\int_{\partial B(a,r)} (x - a) f_1(x)\, d\sigma(x) = r \int_{\partial B(a,r)} n(x) f_1(x)\, d\sigma(x) = 0.$$

The following is an immediate consequence of Proposition 3.

Proposition 4. *If $h_l(x)$ is a harmonic polynomial homogeneous of degree l, then*

$$h_l(x) = p_l(x) + x p_{l-1}(x)$$

where p_l is a left Clifford holomorphic polynomial homogeneous of degree l while p_{l-1} is a left monogenic polynomial which is homogeneous of degree $l - 1$.

It is well known that pairs of homogeneous harmonic polynomials of differing degrees of homogeneity are orthogonal with respect to the usual inner product over the unit sphere. Proposition 4 offers a further refinement to this. Suppose that f and g are $C\ell_n$-valued functions defined on S^{n-1}, and each component of f and g is square integrable. If we define the $C\ell_n$ inner product of f and g to be

$$< f, g >= \frac{1}{\omega_n} \int_{S^{n-1}} \overline{f(x)} g(x)\, d\sigma(x),$$

then if f and g are both real valued, this inner product is equal to

$$\frac{1}{\omega_n}\int_{S^{n-1}} f(x)g(x)\,d\sigma(x),$$

which is the usual inner product for real-valued square integrable functions defined on S^{n-1}. Now

$$\begin{aligned}< xp_{l-1}(x), p_l(x) > &= -\frac{1}{\omega_n}\int_{S^{n-1}} \overline{p}_{l-1}(x)xp_l(x)\,d\sigma(x)\\ &= -\frac{1}{\omega_n}\int_{S^{n-1}} \overline{p}_{l-1}(x)n(x)p_l(x)\,d\sigma(x) = 0.\end{aligned}$$

The evaluation of the last integral is an application of Cauchy's theorem.

Let us denote the space of $C\ell_n$-valued functions defined on S^{n-1}, and such that each component is square integrable, by $L^2(S^{n-1}, C\ell_n)$. Clearly, the space of real valued square integrable functions defined on S^{n-1} is a subset of $L^2(S^{n-1}, C\ell_{n-1})$. The space $L^2(S^{n-1}, C\ell_n)$ is a $C\ell_n$-module.

We have shown that by introducing the module $L^2(S^{n-1}, C\ell_n)$, Proposition 4 provides a further orthogonal decomposition of harmonic polynomials, using left Clifford holomorphic polynomials. We shall return to this theme later. This decomposition was introduced for the case $n = 4$ by Sudbery [47], and independently extended for all n by Sommen [43].

Let us now consider higher order iterates of the Dirac operator D. In the same way that $DH(x) = G(x)$, there is a function $G_3(x)$ defined on $\mathbb{R}^n\backslash\{0\}$ such that $DG_3(x) = H(x)$. Specifically,

$$G_3(x) = C(n,3)\frac{x}{\|x\|^{n-2}},$$

for some dimensional constant $C(n,3)$. Continuing inductively, we may find a function $G_k(x)$ on $\mathbb{R}^n\backslash\{0\}$ such that $DG_k(x) = G_{k-1}(x)$. Specifically,

$$G_k(x) = C(n,k)\frac{x}{\|x\|^{n-k+1}}$$

where when n is odd, so is k.

$$G_k(x) = C(n,k)\frac{1}{\|x\|^{n-k}}$$

where when n is odd, k is even.

$$G_k(x) = C(n,k)\frac{x}{\|x\|^{n-k+1}}$$

where when n is even, k is odd and $k < n$.

$$G_k(x) = C(n,k)\frac{1}{\|x\|^{n-k}}$$

where when n is even, k is even and $k < n$.

$$G_k(x) = C(n,k)(x^{k-n}\log\|x\| + A(n,k)x^{k-n})$$

where when n is even, $k \geq n$. In the last expression, $A(n,k)$ is a real constant dependent on n and k, and $C(n,k)$ is a constant dependent on n and k throughout.

It should be noted that $G_1(x) = G(x)$ and $G_2(x) = H(x)$. It should also be noted that $D^k G_k(x) = 0$.

Here is a simple technique for constructing solutions to the equation $D^k g = 0$ from left Clifford holomorphic functions. The special case $k = 2$ was illustrated in Proposition 2.

Proposition 5. *Suppose that f is a left Clifford holomorphic function on U. Then $D^k x^{k-1} f(x) = 0$.*

Proof. The proof is by induction. We have already seen the result to be true in the case $k = 2$ in Proposition 2. If k is odd, then

$$Dx^{k-1}f(x) = (k-1)x^{k-2}f(x).$$

If k is even, then

$$Dx^{k-1}f(x) = -n(k-1)x^{k-2}f(x) + x^{k-2}\sum_{j=1}^{n} e_j x \frac{\partial f(x)}{\partial x_j}.$$

By arguments presented in Proposition 5, this expression is equal to

$$-n(k-1)x^{k-2}f(x) + x^{k-2}\sum_{j=1}^{n} x_j \frac{\partial f(x)}{\partial x_j}.$$

The induction hypothesis tells us that the only term we need consider is

$$x^{k-2}\sum_{j=1}^{n} x_j \frac{\partial f(x)}{\partial x_j}.$$

However, $\sum_{j=1}^{n} x_j \frac{\partial f(x)}{\partial x_j}$ is a left Clifford holomorphic function. So the proof by induction is now complete. □

We shall refer to a function $g : U \to C\ell_n$, which satisfies the equation $D^k g = 0$, as a left k-monogenic function. Similarly, if $h : U \to C\ell_n$ satisfies the equation $hD^k = 0$, then h is a right k-monogenic function. In the case where $k = 1$, we return to the setting of left, or right, Clifford holomorphic functions, and when $k = 2$ we return to the setting of harmonic functions. When $k = 4$, the equations $D^4 g = 0$ and $gD^4 = 0$ correspond to the equations $\triangle_n^2 g = 0$ and $\triangle_n^2 h = 0$. So left or right 4-monogenic functions are in fact biharmonic functions.

More generally, if k is even, then a left or right k-monogenic function f automatically satisfies the equation $\triangle_n^{\frac{k}{2}} f = 0$.

Proposition 6. *Suppose that p is a left k-monogenic polynomial homogeneous of degree q. Then there are left Clifford holomorphic polynomials* $f_0, \ldots, f_{k-1}$ *such that*

$$p(x) = f_0(x) + \ldots + x^{k-1} f_{k-1}(x),$$

and each polynomial f_j *is homogeneous of degree* $q - j$, *whenever* $q - j \geq 0$, *and identically zero otherwise.*

Proof. The proof is via induction on k. The case $k = 2$ is established immediately after the proof of Proposition 2. Let us now consider $Dp(x)$. This is a left $(k-1)$-monogenic polynomial homogeneous of degree $q-1$. So by the induction hypothesis,

$$Dp(x) = g_1(x) + \ldots + x^{k-2} g_{k-1}(x),$$

where each g_j is a left Clifford holomorphic polynomial homogeneous of degree $q - j$, whenever $q - j \geq 0$, and is equal to zero otherwise. Using Euler's lemma, and the observations made after the proof of Proposition 5, one may now find left Clifford holomorphic polynomials $f_1(x), \ldots, f_{k-1}(x)$ such that

$$D(xf_1(x) + \ldots + x^{k-1} f_{k-1}(x)) = Dp(x),$$

and

$$f_j(x) = c_j g_j(x)$$

for some $c_j \in \mathbb{R}$, and where $1 \leq j \leq k - 1$. It follows that

$$p(x) - \sum_{j=1}^{k-1} x^j f_j(x)$$

is a left Clifford holomorphic polynomial f_0, homogeneous of degree q. □

One may now use Proposition 6, and the arguments used to establish Proposition 3, to deduce:

Theorem 6. *Suppose that* f *is a left k-monogenic function defined in a neighborhood of the ball* $B(0, R)$. *Then there are left monogenic functions* $f_0, \ldots, f_{k-1}$ *defined on* $B(0, R)$ *such that* $f(x) = f_0(x) + \ldots + x^{k-1} f_{k-1}(x)$ *on* $B(0, R)$.

Theorem 6 establishes an Almansi decomposition for left k-monogenic functions in terms of left Clifford holomorphic functions over any open ball. It also follows from this theorem that each left k-monogenic function is a real analytic function. It is also reasonably well known that if h is a biharmonic function defined in a neighborhood of $B(0, R)$, then there are harmonic functions h_1 and h_2 defined on $B(0, R)$ such that

$$h(x) = h_1(x) + \|x\|^2 h_2(x).$$

In the special case where $k = 4$, Theorem 6 both establishes this result and refines it.

Since each left k-monogenic function is a real analytic function, we can immediately use Stokes' theorem to deduce the following Cauchy–Green type formula.

Theorem 7. *Suppose that f is a left k-monogenic function defined on some domain U, and suppose that S is a piecewise C^1 compact surface lying in U and bounding a bounded subdomain V of U. Then for each $y \in V$,*

$$f(y) = \frac{1}{\omega_n} \int_S (\sum_{j=1}^{k} (-1)^{j-1} G_j(x-y) n(x) D^{j-1} f(x))\, d\sigma(x).$$

3.5 Conformal groups and Clifford analysis

Here we examine the role played by the conformal group within parts of Clifford analysis. Our starting point is to ask what type of diffeomorphisms acting on subdomains of $\mathbb{R}^n$ preserve Clifford holomorphic functions. If a diffeomorphism ϕ can transform the class of left Clifford holomorphic functions on one domain U to a class of left Clifford holomorphic functions on the domain $\phi(U)$ and do the same for the class of right Clifford holomorphic functions on U, then it must preserve Cauchy's theorem. If f and g are left and right Clifford holomorphic on U, respectively, and these functions are transformed to f' and g', left and right Clifford holomorphic functions on $\phi(U)$, then

$$\int_S g(x) n(x) f(x)\, d\sigma(x) = 0 = \int_{\phi(S)} g'(y) n(y) f'(y)\, d\sigma(y)$$

where S is a piecewise C^1-compact surface lying in U and $y = \phi(x)$. An important point to note here is that we need to assume that ϕ preserves vectors orthogonal to the tangent spaces at x and $\phi(x)$. As the choice of x and S is arbitrary, it follows that the diffeomorphism ϕ is angle preserving. In other words, ϕ is a conformal transformation. A theorem of Liouville [24] tells us that for dimensions 3 and greater the only conformal transformations on domains are Möbius transformations.

In order to deal with Möbius transformations using Clifford algebras, we have seen in a previous chapter that one can use Vahlen matrices. We now proceed to show that each Möbius transformation preserves monogenicity. Sudbery [47], and also Bojarski [3], have established this fact. We will need the following two lemmas.

Lemma 1. *Suppose that $\phi(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation; then*

$$G(u-v) = J(\phi, x)^{-1} G(x-y) \tilde{J}(\phi, y)^{-1},$$

where $u = \phi(x)$, $v = \phi(y)$ and

$$J(\phi, x) = \frac{\widetilde{(cx+d)}}{\|cx+d\|^n}.$$

Proof. The proof essentially follows from the fact that

$$(x^{-1} - y^{-1}) = x^{-1}(y - x)y^{-1}.$$

Consequently,

$$\|x^{-1} - y^{-1}\| = \|x\|^{-1}\|x - y\|\|y\|^{-1}, \quad \text{and} \quad ax\tilde{a} - ay\tilde{a} = a(x - y)\tilde{a}.$$

If one breaks the transformation down into terms arising from the generators of the Möbius group, using the previous set of equations, then one readily arrives at the result. □

Lemma 2. *Suppose that* $y = \phi(x) = (ax + b)(cx + d)^{-1}$ *is a Möbius transformation, and for domains* U *and* V *we have* $\phi(U) = V$. *Then*

$$\int_S f(u)n(u)g(u)\,d\sigma(u) = \int_{\psi^{-1}(S)} f(\psi(x))\tilde{J}(\psi, x)n(x)J(\psi, x)g(\psi(x))\,d\sigma(x)$$

where $u = \psi(x)$, S *is a orientable hypersurface lying in* U, *and*

$$J(\psi, x) = \frac{\widetilde{cx + d}}{\|cx + d\|^n}.$$

Proof Outline. On breaking ψ up into the generators of the Möbius group, the result follows by noting that

$$\frac{\partial x^{-1}}{\partial x_j} = -x^{-1}e_j x^{-1}.$$

It follows from Cauchy's Theorem that if $g(u)$ is a left Clifford holomorphic function in the variable u, then $J(\psi, x)f(\psi(x))$ is left Clifford holomorphic in the variable x.

When $\phi(x)$ is the Cayley transformation

$$y = (e_n x + 1)(x + e_n)^{-1},$$

we can use this transformation to establish a Cauchy–Kowalewska extension in a neighborhood of the sphere. If $f(x)$ is a real analytic function defined on an open subset U of $S^{n-1}\backslash\{e_n\}$, then

$$l(y) = J(\phi^{-1}, y)^{-1}f(\phi(y))$$

is a real analytic function on the open set $V = \phi^{-1}(U)$. This function has a Cauchy–Kowalewska extension to a left Clifford holomorphic function $L(y)$ defined on an open neighborhood $V(g) \subset \mathbb{R}^n$ of V. Consequently,

$$F(x) = J(\phi^{-1}, x)L(\phi^{-1}(x))$$

is a left Clifford holomorphic defined on an open neighborhood

$$U(f)=\phi^{-1}(V(g))$$

of U. Moreover $F_{|U} = f$. Combining with similar arguments for the other Cayley transformation

$$y = (-e_n x + 1)(x - e_n)^{-1},$$

one can deduce:

Theorem 8 (Cauchy–Kowalewska Theorem). *Suppose that f is a $C\ell_n$-valued real analytic function defined on S^{n-1}. Then there is a unique left Clifford holomorphic function F defined on an open neighborhood $U(f)$ of S^{n-1} such that $F_{|S^{n-1}} = f$.*

In fact, if $f(u)$ is defined on some domain and satisfies the equation $D^k f = 0$, then the function $J_k(\psi, x) f(\psi(x))$ satisfies the same equation, where

$$J_k(\psi, x) = \frac{\widetilde{cx + d}}{\|cx + d\|^{n-k+1}}.$$

Theorem 9 (Fueter–Sce Theorem). *Suppose that $f = u + iv$ is a holomorphic function on a domain $\Omega \subset \mathbb{C}$ and that $\Omega = \overline{\Omega}$ and $f(\overline{z}) = \overline{f(z)}$. Then the function*

$$F(\underline{x}) = u(x_1, \|x'\|) + e_1^{-1} \frac{x'}{\|x'\|} v(x_1, \|x'\|)$$

is a unital left$(n-1)$-monogenic function on the domain $\{\underline{x} : x_1 + i\|x'\| \in \Omega\}$ whenever n is even. Here $x' = x_2 e_2 + \ldots + x_n e_n$.

Proof. First let us note that $x^{-1}e_1$ is left $n-1$ monogenic whenever n is even. It follows that

$$\frac{\partial^k}{\partial x_1^k} x^{-1} e_1 = c_k x^{-k-1} e_1$$

is $n-1$ left monogenic for each positive integer k. Here c_k is some nonzero real number. Using Kelvin inversion, it follows that $x^k e_1$ is left $n-1$ monogenic for each positive integer k. By taking translations and Taylor series expansions for the function f, the result follows. □

This result was first established for the case $n = 4$ by Fueter [12], see also Sudbery [47]. It was extended to all even dimensions by Sce [40], though the methods used do not make use of the conformal group. This result has been applied in [32, 33] to study various types of singular integral operators acting on L^p spaces of Lipschitz perturbations of the sphere.

3.6 Conformally flat spin manifolds

The invariance of monogenic functions under Möbius transformations, described in the previous section, makes use of a conformal weight factor $J(\psi, x)$. This invariance can be seen as an automorphic form invariance, which leads to a natural generalization of the concept of a Riemann surface to the Euclidean setting. A manifold M is said to be conformally flat if there is an atlas $\mathcal{A}$ of M whose transition functions are Möbius transformations. For instance, via the Cayley transformations

$$(e_{n+1}x+1)(x+e_{n+1})^{-1} \quad \text{and} \quad (-e_{n+1}x+1)(x-e_{n+1})^{-1},$$

one can see that the sphere $S^n \subset \mathbb{R}^{n+1}$ is an example of a conformally flat manifold. Another way of constructing conformally flat manifolds is to take a simply connected domain U of $\mathbb{R}^n$, and consider a Kleinian subgroup Γ of the Möbius group at acts discontinuously on U. Then the factorization $U\backslash\Gamma$ is a conformally flat manifold. For instance, let $U = \mathbb{R}^n$ and let Γ be the integer lattice

$$Z^k = Ze_1 + \ldots + Ze_k$$

for some positive integer $k \leq n$. In this case, $\mathbb{R}^n\backslash Z^k$ gives the cylinder C_k, and when $k = n$ we get the n-torus. Also, if we let

$$U = \mathbb{R}^n\backslash\{0\} \quad \text{and} \quad \Gamma = \{2^k : k \in Z\},$$

the resulting manifold is $S^1 \times S^{n-1}$.

We locally construct a spinor bundle over M by making the identification (u, X) with either $(x, \pm J(\psi, x)X)$, where

$$u = \psi(x) = (ax+b)(cx+d)^{-1} = (-ax-b)(-cx-d)^{-1}.$$

If we can compatibly choose the signs, then we have created a spinor bundle over the conformally flat manifold. Note, it might be possible to create more than one spinor bundle over M. For instance, consider the cylinder C_k. If we make the identification (x, X) with $(x + \underline{m}, (-1)^{m_1+\ldots+m_l}X)$, where l is a fixed integer with $l \leq k$, and $\underline{m} = m_1e_1 + \ldots + m_le_l + \ldots + m_ke_k$, then we have created k different spinor bundles $E^1, \ldots, E^k$ over C_k.

We have used the conformal weight function $J(\psi, x)$ to construct the spinor bundle E. It is easy to see that a section $f : M \to E$ could be called a left monogenic section if it is locally a left monogenic function. It is now natural to ask if one can construct Cauchy integral formulas for such sections. To do this, we need to construct a kernel over the Euclidean domain U that is periodic with respect to Γ, and then use the projection map $p : U \to M$ to construct from this kernel a Cauchy kernel for U. In [19], we show that the Cauchy kernel for C_k, with spinor bundle E^l, is constructed from the kernel

$$\cot_{k,l}(x,y) = \sum_{\underline{m}\in Z^l, \underline{n}\in Z^{k-l}} (-1)^{m_1+\ldots+m_l} G(x-y+\underline{m}+\underline{n}),$$

where $\underline{n} = n_{l+1}e_{l+1} + \ldots + n_n e_n$. While for the conformally flat spin manifold $S^1 \times S^{n-1}$ with trivial bundle $C\ell_n$, the Cauchy kernel is constructed from the kernel

$$\sum_{k=0}^{\infty} G(2^k x - 2^k y) + 2^{2-2n} G(x) \left(\sum_{k=-1}^{-\infty} G(2^{-k} x^{-1} - 2^{-k} y^{-1}) \right) G(y).$$

See [17–19] for more details and related results.

It should be noted that one may set up a Dirac operator over arbitrary Riemannian manifolds, see for instance [4], and one may set up Cauchy integral formulas for functions annihilated by these Dirac operators [6, 28].

3.7 Boundary behavior and Hardy spaces

Possibly the main topic that unites all that has been previously discussed here on Clifford analysis is its applications to boundary value problems. This, in turn, leads to a study of boundary behavior of classes of Clifford holomorphic functions and Hardy spaces. Let us look first at one of the simplest cases. Previously, we noted that if θ is a square integrable function defined on the sphere S^{n-1}, then there is a harmonic function h defined on the unit ball in $\mathbb{R}^n$ with boundary value θ almost everywhere. Also, we have seen that $h(x) = f_1(x) + x f_2(x)$ where f_1 and f_2 are left Clifford holomorphic. However, on S^{n-1} the function $G(x) = x$. One can see that on S^{n-1} we have $\theta(x) = f_1(x) + g(x)$ almost everywhere. Here, f_1 is left monogenic on the unit ball $B(0,1)$ and g is left Clifford holomorphic on $\mathbb{R}^n \backslash \overline{B(0,1)}$, where $\overline{B(0,1)}$ is the closure of the open unit ball. Let $H^2(B(0,1))$ denote the space of Clifford holomorphic functions defined on $B(0,1)$, with extensions to a square integrable functions on S^{n-1}, and let $H^2(\mathbb{R}^n \backslash \overline{B(0,1)})$ denote the class of left Clifford holomorphic functions defined on $\mathbb{R}^n \backslash \overline{B(0,1)}$, with square integrable extensions to S^{n-1}. What we have so far outlined is that

$$L^2(S^{n-1}) = H^2(B(0,1)) \oplus H^2(\mathbb{R}^n \backslash \overline{B(0,1)}),$$

where $L^2(S^{n-1})$ is the space of $C\ell_n$ valued Lebesgue square integrable functions defined on S^{n-1}. This is the Hardy 2-space decomposition of $L^2(S^{n-1})$. It is also true if we replace 2 by p where $1 < p < \infty$. We will not go into more details here, since it is beyond the scope of the material presented here.

Let us now take an alternative look at a way of obtaining this decomposition. This method will generalize to all reasonable surfaces. We will clarify what we mean by a reasonable surface later. Instead of considering an arbitrary square integrable function on S^{n-1}, let us instead assume that θ is a continuously differentiable function. Let us now consider the integral

$$\frac{1}{\omega_n} \int_{S^{n-1}} G(x-y) n(x) \theta(x) \, d\sigma(x)$$

where $y \in B(0,1)$. This defines a left Clifford holomorphic function on $B(0,1)$. Now let the point y approach a point $z \in S^{n-1}$ along a differentiable path $y(t)$. Let us also assume that $\frac{dy(t)}{dt}$ is evaluated at $t = 1$, so that $y(t) = z$ is not tangential to S^{n-1} at z. We can essentially ignore this last point at a first read. We want to evaluate

$$\lim_{t\to 1} \frac{1}{\omega_n} \int_{S^{n-1}} G(x - y(t))n(x)\theta(x)\, d\sigma(x).$$

We do this by removing a small ball on $B(0,1)$ from S^{n-1}. The ball is centered at z and is of radius ϵ. We denote this ball by $b(z,\epsilon)$. The previous integral now splits into an integral over $b(z,\epsilon)$ and an integral over $S^{n-1}\backslash b(z,\epsilon)$. On $b(z,\epsilon)$, we can express $\theta(x)$ as $(\theta(x) - \theta(z)) + \theta(z)$. As θ is continuously differentiable,

$$\|\theta(x) - \theta(z)\| < C\|x - z\|$$

for some $C \in \mathbb{R}^+$. It follows that

$$\lim_{\epsilon\to 0}\lim_{t\to 1} \int_{b(z,\epsilon)} \|G(x - y(t))n(x)(\theta(x) - \theta(z))\|\, d\sigma(x) = 0.$$

Moreover, the term

$$\lim_{\epsilon\to 0}\lim_{t\to 1} \frac{1}{\omega_n} \int_{b(z,\epsilon} G(x - y(t))n(x)\theta(z)\, d\sigma(x)$$

can be replaced by the term

$$\lim_{\epsilon\to 0}\lim_{t\to 1} \int_{B(0,1)\cap\partial B(z,\epsilon)} G(x - y(t))n(x)\theta(z)\, d\sigma(x),$$

since $\theta(z)$ is a Clifford holomorphic function. By the residue theorem the limit of this integral evaluates to $\frac{1}{2}\theta(z)$.

We leave it to the interested reader to note that the singular integral or principal valued integral

$$\lim_{\epsilon\to 0}\lim_{t\to 1} \frac{1}{\omega_n} \int_{S^{n-1}\backslash b(z,\epsilon)} G(x - y(t))n(x)\theta(x)\, d\sigma(x)$$
$$= P.V. \frac{1}{\omega_n} \int_{S^{n-1}} G(x - z)n(x)\theta(x)\, d\sigma(x)$$

is bounded.

We have established that

$$\lim_{t\to 1} \int_{S^{n-1}} G(x - y(t))n(x)\theta(x)\, d\sigma(x)$$
$$= \tfrac{1}{2}\theta(z) + P.V. \frac{1}{\omega_n} \int_{S^{n-1}} G(x - z)n(x)\theta(x)\, d\sigma(x).$$

If we now assumed that $y(t)$ is a path tending to z on the complement of $\overline{B(0,1)}$, then similar arguments give

$$\lim_{t\to 1}\int_{S^{n-1}} G(x-y(t))n(x)\theta(x)\,d\sigma(x)$$
$$= -\tfrac{1}{2}\theta(z) + P.V.\frac{1}{\omega_n}\int_{S^{n-1}} G(x-z)n(x)\theta(x)\,d\sigma(x).$$

We will write these expressions as

$$(\pm\tfrac{1}{2}I + C_{S^{n-1}})\theta.$$

If we consider the limit

$$\lim_{t\to 1}\frac{1}{\omega_n}\int_{S^{n-1}} G(x-y(t))n(x)(\tfrac{1}{2}I + C_{S^{n-1}})\theta(x)\,d\sigma(x),$$

we may determine that

$$(\tfrac{1}{2}I + C_{S^{n-1}})^2 = \tfrac{1}{2}I + C_{S^{n-1}}.$$

Furthermore,

$$(\tfrac{1}{2}I + C_{S^{n-1}})(-\tfrac{1}{2}I + C_{S^{n-1}}) = 0 \quad\text{and}\quad (-\tfrac{1}{2}I + C_{S^{n-1}})^2 = -\tfrac{1}{2}I + C_{S^{n-1}}.$$

It is known that each function $\psi \in L^2(S^{n-1})$ can be approximated by a sequence of functions, each with the same properties as θ. This tells us that the previous formulas can be repeated, but this time simply for $\theta \in L^2(S^{n-1})$. It follows that for such a θ we have

$$\theta = (\tfrac{1}{2}I + C_{S^{n-1}})\theta + (\tfrac{1}{2}I - C_{S^{n-1}})\theta.$$

This formula gives the Hardy space decomposition of $L^2(S^{n-1})$. In fact, if one looks more carefully at the previous calculations used to obtain these formulas we see that it is not so significant that the surface used is a sphere, and we can redo the calculations for any "reasonable" hypersurface S. In this case, we get

$$\theta = (\tfrac{1}{2}I + C_S)\theta + (\tfrac{1}{2}I - C_S)\theta$$

where θ now belongs to $L^2(S)$, and

$$C_S\theta = P.V.\frac{1}{\omega_n}\int_S G(x-y)n(x)\theta(x)\,d\sigma(x).$$

This gives rise to the Hardy space decomposition

$$L^2(S) = H^2(S^+) \oplus H^2(S^-),$$

where $S^{\pm}$ are the two domains that complement the surface S (we are assuming that S divides $\mathbb{R}^n$ into two complementary domains).

Lastly, one should address the smoothness of S. In some parts of the literature, one simply assumes that S is compact and C^2. More recently, one assumes that S has rougher conditions, usually that the surface is Lipschitz continuous, see for instance [21, 22, 27]. The formulas given above, involving the singular integral operator C_S, are called Plemelj formulas. It is a simple exercise to show that these formulas are conformally invariant. Using Kelvin inversion, or even a Cayley transformation, one can show that these formulas and the Hardy space decompositions are also valid on unbounded surfaces and domains. A great deal of recent Clifford analysis has been devoted to the study of such Hardy spaces and singular integral operators. This is due to an idea of R. Coifman, that various hard problems in classical harmonic analysis, studied in Euclidean space, might be more readily handled using tools from Clifford analysis, particularly the singular Cauchy transform and associated Hardy spaces.

In particular, Coifman speculated that a more direct proof of the celebrated Coifman–McIntosh–Meyer Theorem [7], could be derived using Clifford analysis. The Coifman–McIntosh–Meyer Theorem establishes the L^2 boundedness of the double layer potential operator for Lipschitz graphs in $\mathbb{R}^n$. Coifman's observation was that the double layer potential operator is the real or scalar part of the singular Cauchy transform arising in Clifford analysis and discussed earlier. If one can establish the L^2 boundedness of the singular Cauchy transform for a Lipschitz graph in $\mathbb{R}^n$, then one automatically has the L^2 boundedness for the double layer potential operator for the same graph. The L^2 boundedness of the singular Cauchy transform was first established for Lipschitz graphs with small constant by Murray [30], and extended to the general case by McIntosh, see [26, 27]. One very important reason for needing to know that the double layer potential operator is L^2 bounded for Lipschitz graphs is to be able to solve boundary value problems for domains with Lipschitz graphs as boundaries. Such boundary value problems would include the Dirichlet problem and Neuman problem for the Laplacian. See [26, 27] for more details. In [49] Clifford analysis, and more precisely the Hardy space decomposition mentioned here, is specifically used to solve the water wave problem in three dimensions.

3.8 More on Clifford analysis on the sphere

In the previous section, we saw that $L^2(S^{n-1})$ splits into a direct sum of Hardy spaces for the corresponding complementary domains $B(0,1)$ and $\mathbb{R}^n\backslash\overline{B(0,1)}$. In an earlier section, we saw that any left Clifford holomorphic function $f(x)$ can be expressed as a locally uniformly convergent series $\sum_{j=0}^{\infty} f_j(x)$, where each $f_j(x)$ is left Clifford holomorphic and homogeneous of degree j. Now following [47], consider the operator

$$D = x^{-1}xD = x^{-1}(\sum_{i<k} e_i e_k (x_i \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_i} - \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j})).$$

By letting the last term in this expression act on homogeneous polynomials, one may determine from Euler's lemma that

$$\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}$$

is the radial operator $r\frac{\partial}{\partial r}$. So $r\frac{\partial}{\partial r}f_j(x) = jf_j(x)$. Since each polynomial f_k is Clifford holomorphic, it follows that each f_j is an eigenvector of the spherical Dirac operator

$$x\Lambda_{n-1} = x\sum_{i<k} e_i e_k \big(x_i \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_i}\big)$$

with eigenvalue k. Now using Kelvin inversion, we know that f_k is homogeneous of degree k and left Clifford holomorphic if and only if $G(x)f_k(x^{-1})$ is homogeneous of degree $-n+1-k$ and is left Clifford holomorphic. On restricting $G(x)f_k(x^{-1})$ to the unit sphere, this function becomes $xf_k(x^{-1})$ and this function is an eigenvector for the spherical Dirac operator $x\Lambda_{n-1}$. Since each $f \in H^2(\mathbb{R}^n \backslash \overline{B(0,1)})$ can be written as

$$\sum_{k=0}^{\infty} G(x)f_k(x^{-1}),$$

where each f_k is homogeneous of degree k and is left Clifford holomorphic, it follows that if $h \in L^2(S^{n-1})$, then

$$\Lambda_{n-1}xh(x) = (1-n)xh(x) - x\Lambda_{n-1}h(x).$$

Similarly, if we replace S^{n-1} by the n-sphere S^n embedded in $\mathbb{R}^{n+1}$, then we have the identity

$$\Lambda_n xh(x) = -nxh(x) - x\Lambda_n h(x)$$

for each $h \in L^2(S^n)$. As all C^∞ functions defined on S^n belong to $L^2(S^n)$, this identity holds for all such functions too.

It should be noted that for each $x \in S^n$, if we restrict the operator $x\Lambda_n$ to the tangent bundle TS^n_x, then we obtain the Euclidean Dirac operator acting on this tangent space.

By using the Cayley transformation

$$x = \psi(y)(e_{n+1}y + 1)(y + e_{n+1})^{-1}$$

from $\mathbb{R}^n$ to $S^n \backslash \{e_{n+1}\}$, one can transform left Clifford holomorphic functions from domains in $\mathbb{R}^n$ to functions defined on domains lying on the sphere. If $f(y)$ is left Clifford holomorphic on the domain U lying in $\mathbb{R}^n$, then we obtain a function $f'(x) = J(\psi^{-1}, x)f(\psi^{-1}(x)$ defined on the domain $U' = \psi(U)$ lying on S^n. Here

$$J(\psi^{-1}, x) = \frac{x+1}{\|x+1\|^n}.$$

Similarly, if $g(y)$ is right Clifford holomorphic on U, then

$$g'(x) = g(\psi^{-1}(x)J(\psi^{-1}, x)$$

is a well-defined function on U'. Moreover for any smooth, compact hypersurface S bounding a subdomain V of U, we have from the conformal invariance of Cauchy's Theorem,

$$\int_{S'} g'(x)n(x)f'(x)\, d\sigma'(x) = 0$$

where $S' = \psi(S)$, and $n(x)$ is the unit vector lying in the tangent space TS^n_x of S^n at x and outer normal to S' at x. Furthermore σ' is the Lebesgue measure on S'.

From Lemma 1, it now follows that for each point $y' \in V' = \psi(V)$, we have the following version of Cauchy's Integral Formula:

$$f'(y') = \frac{1}{\omega_n}\int_{S'} G(x-y')n(x)f'(x)\, d\sigma(x),$$

where, as before,

$$G(x-y') = \frac{x-y'}{\|x-y'\|^n},$$

but now x and $y' \in S^n$. It would now appear that the functions f' and g' are solutions to some spherical Dirac equations. We need to isolate this Dirac operator. We shall achieve this by applying the operator $x\Lambda_n$ to the kernel $G_s(x, y') = G(x-y')$. Since x and $y' \in S^n$,

$$\|x-y'\|^2 = 2 - 2 < x, y' >,$$

where $< x, y' >$ is the inner product of x and y'. So

$$G_s(x, y') = 2^{\frac{-n}{2}} \frac{x-y'}{(1- < x, y' >)^{\frac{n}{2}}}.$$

In calculating $x\Lambda_n G_s(x, y')$, we need to know what $\Lambda_n < x, y' >$ evaluates to. It is a simple exercise to determine that

$$\Lambda_n < x, y' >= xy' + < x, y' >,$$

which is the wedge product $x \wedge y'$ of x with y'.

Now let us calculate $x\Lambda_n G_s(x, y')$:

$$x\Lambda_n G_s(x, y') = 2^{\frac{-n}{2}}(x\Lambda_n \frac{x}{(1- < x, y' >)^{\frac{n}{2}}} - x\Lambda_n \frac{y'}{(1- < x, y' >)^{\frac{n}{2}}}) =$$

$$2^{\frac{n}{2}}(-x\frac{nx}{(1- < x, y' >)^{\frac{n}{2}}} + \Lambda_n \frac{1}{(1- < x, y' >)^{\frac{-n}{2}}} - x\Lambda_n \frac{1}{(1- < x, y' >)^{\frac{n}{2}}} y').$$

First,

$$\Lambda_n \frac{1}{(1- < x, y' >)^{\frac{n}{2}}} = \frac{n}{2} \frac{x \wedge y'}{(1- < x, y' >)^{\frac{n+2}{2}}},$$

so

$$x\Lambda_n G_s(x, y')$$
$$= 2^{\frac{-n}{2}} \frac{n}{2(1- < x, y >)^{\frac{n+2}{2}}} (2(1- < x, y' >) + x \wedge y' - x(x \wedge y')y').$$

The expression

$$2(1- < x, y' >) + x \wedge y' - x(x \wedge y')y'$$

is equal to

$$2 - 2 < x, y' > +xy' + < x, y' > -x(xy + < x, y' >)y'.$$

This expression simplifies to

$$1- < x, y' > +xy' - xy' < x, y' >,$$

which in turn simplifies to

$$(1- < x, y' >)(1 + xy') = -x(1- < x, y' >)(x - y').$$

So

$$x\Lambda_n G_s(x, y') = \frac{-n}{2} x G_s(x, y').$$

Hence

$$x(\Lambda_n + \frac{n}{2}) G_s(x, y') = 0,$$

so the Dirac operator, D_s, over the sphere is $x(\Lambda_n + \frac{n}{2})$. It follows from our Cauchy integral formula for the sphere that $D_s f'(x) = 0$. For more details on this operator, related operators and their properties see [8, 25, 38, 39, 48].

Besides the operator D_s, we also need a Laplacian $\triangle_s$ acting on functions defined on domains on S^n. To do this, we will work backwards and look for a fundamental solution to $\triangle_n$. A strong candidate for such a fundamental solution is the kernel

$$H_s(x, y') = \frac{1}{n-2} \frac{1}{\|x - y'\|^{n-2}}.$$

By similar considerations to those made in the previous calculation, we find that

$$D_s H_s(x, y') = -x H_s(x, y') + G_s(x, y').$$

So

$$(D_s + x) H_s(x, y') = G_s(x, y').$$

Therefore we may define our Laplacian $\triangle_s$ to be $D_s(D_s + x)$.

Definition 2. *Suppose h is a $C\ell_n$ valued function defined on a domain U' of S^n. Then h is called a harmonic function on U' if $\Delta_s h = 0$.*

In much the same way as one would derive Green's Theorem in $\mathbb{R}^n$, one now has

Theorem 10. *Suppose U' is a domain on S^n and $h : U' \to C\ell_n$ is a harmonic function on U'. Suppose also that S' is a smooth hypersurface lying in U' and that S' bounds a subdomain V' of U' and that $y' \in V'$. Then*

$$h(y') = \frac{1}{\omega_n}\int_{S'}(G_s(x,y')n(x)h(x) + H_s(x,y')n(x)D_s h(x))\,d\sigma'(x).$$

See [25] for more details.

3.9 The Fourier transform and Clifford analysis

Closely related to Hardy spaces is the Fourier transform. Here we will consider $\mathbb{R}^n$ as divided into the upper and lower half spaces $\mathbb{R}^{n+}$ and $\mathbb{R}^{n-}$, where

$$\mathbb{R}^{n+} = \{x = x_1e_1 + \ldots + x_ne_n : x_n > 0\}$$

and

$$\mathbb{R}^{n-} = \{x = x_1e_1 + \ldots + x_ne_n : x_n < 0\}.$$

These two domains have $\mathbb{R}^{n-1} = \text{span} < e_1, \ldots, e_{n-1} >$ as a common boundary. As before,

$$L^2(\mathbb{R}^{n-1}) = H^2(\mathbb{R}^{n+}) \oplus H^2(\mathbb{R}^{n-}).$$

Let us now consider a function $\psi \in L^2(\mathbb{R}^{n-1})$. Then

$$\psi(y) = \left(\tfrac{1}{2}\psi(y) + \frac{1}{\omega_n}P.V.\int_{\mathbb{R}^{n-1}} G(x'-y)e_n\psi(x')\,dx^{n-1}\right) + \left(\tfrac{1}{2}\psi(y) - \frac{1}{\omega_n}P.V.\int_{\mathbb{R}^{n-1}} G(x'-y)e_n\psi(x')\,dx^{n-1}\right)$$

almost everywhere. Here

$$\tfrac{1}{2}\psi(y) + \frac{1}{\omega_n}P.V.\int_{\mathbb{R}^{n-1}} G(x'-y)e_n\psi(x')\,dx^{n-1}$$

is the nontangential limit of

$$\frac{1}{\omega_n}\int_{\mathbb{R}^{n-1}} G(x'-y(t))e_n\psi(x')\,dx^{n-1},$$

as $y(t)$ tends to y nontangentially through a smooth path in the upper half space, and

$$\tfrac{1}{2}\psi(y) - \frac{1}{\omega_n}P.V.\int_{\mathbb{R}^{n-1}} G(x'-y)e_n\psi(x')\,dx^{n-1}$$

is the nontangential limit of

$$\frac{-1}{\omega_n}\int_{\mathbb{R}^{n-1}} G(x'-y(t))e_n\psi(x')\,dx^{n-1}$$

as $y(t)$ tends nontangentially to y through a smooth path in lower half space.

Consider now the Fourier transform, $\mathcal{F}(\psi)=\hat{\psi}$, of ψ. In order to proceed, we need to calculate

$$\mathcal{F}(\tfrac{1}{2}\psi \pm \frac{1}{\omega_n} P.V. \int_{\mathbb{R}^{n-1}} G(x'-y)e_n\psi(x')\,dx^{n-1}).$$

In particular, we need to determine

$$\mathcal{F}(\frac{1}{\omega_n} P.V. \int_{\mathbb{R}^{n-1}} G(x'-y)e_n\psi(x')\,dx^{n-1}).$$

Following [26], it may be determined that this is

$$\tfrac{1}{2} i \frac{\zeta}{\|\zeta\|} e_n \hat{\psi}(\zeta),$$

where $\zeta=\zeta_1 e_1+\ldots+\zeta_{n-1}e_{n-1}$. So

$$\mathcal{F}(\tfrac{1}{2}\psi \pm \frac{1}{\omega_n}\int_{\mathbb{R}^{n-1}} G(x'-y)e_n\psi(x')\,d\sigma(x')) = \tfrac{1}{2}(1\pm i\frac{\zeta}{\|\zeta\|}e_n).$$

Now, as observed in [26],

$$(\tfrac{1}{2}(1\pm i\frac{\zeta}{\|\zeta\|}e_n))^2 = \tfrac{1}{2}(1\pm i\frac{\zeta}{\|\zeta\|}e_n), \quad \tfrac{1}{2}(1\pm i\frac{\zeta}{\|\zeta\|}e_n)\tfrac{1}{2}(1\mp i\frac{\zeta}{\|\zeta\|}e_n)=0$$

and

$$i\zeta e_n \tfrac{1}{2}(1\pm i\frac{\zeta}{\|\zeta\|}e_n) = \|\zeta\|\tfrac{1}{2}(1\pm i\frac{\zeta}{\|\zeta\|}e_n).$$

Taking the Cauchy–Kowalewska extension of $e^{i<x',\zeta>}$, we get

$$\exp(i<x',\zeta> -ix_n e_n\zeta)$$

defined on some neighborhood in $\mathbb{R}^n$ of $\mathbb{R}^{n-1}$.

Now consider

$$\exp(i<x',\zeta> -ix_n e_n\zeta)\tfrac{1}{2}(1\pm i\frac{\zeta}{\|\zeta\|}e_n),$$

which simplifies to

$$e^{i<x',\zeta>-x_n\|\zeta\|}\tfrac{1}{2}(1+i\frac{\zeta}{\|\zeta\|}e_n)$$

if $x_n > 0$, and to

$$e^{i<x',\zeta>+x_n\|\zeta\|}\tfrac{1}{2}(1 - i\frac{\zeta}{\|\zeta\|}e_n)$$

if $x_n < 0$. The first of these series converges locally uniformly on the upper half space while the second series converges locally uniformly on the lower half space. We denote these two functions by $e_\pm(x,\zeta)$, respectively. The integrals

$$\frac{1}{\omega_n}\int_{\mathbb{R}^{n-1}} e_\pm(x,\zeta)\hat{\psi}(\zeta)\,d\sigma(\zeta)$$

define the left monogenic functions $\Psi_\pm(x)$ on the upper and lower half spaces, respectively. Moreover,

$$\Psi_\pm \in H^2(\mathbb{R}^{n\pm})$$

explicitly gives the Hardy space decomposition of $\psi \in L^2(\mathbb{R}^{n-1})$. The links between the Fourier transform and Clifford analysis were first found in [44], and later independently rediscovered and applied in [22].

The integrals

$$\frac{1}{\omega_n}\int_{\mathbb{R}^{n-1}} e_\pm(x,\zeta)\,d\zeta^{n-1}$$

can be expressed in polar coordinates as

$$\frac{1}{\omega_n}\int_{S^{n-2}}\int_0^\infty e^{ir<x',\zeta'>\pm x_n r}r^{n-2}\tfrac{1}{2}(1\pm\frac{\zeta}{\|\zeta\|}e_n)\,drdS^{n-2},$$

where $\zeta' = \frac{\zeta}{\|\zeta\|}$. In [22], Chun Li observed that the integrals

$$\int_0^\infty e^{ir<x',\zeta'>\pm x_n r}\tfrac{1}{2}(1\pm\zeta' e_n)r^{n-2}\,dr$$

are Laplace transforms of the function $f(R) = R^{n-2}$. So the integral

$$\int_0^\infty e^{ir<x',\zeta'>-x_n r}\tfrac{1}{2}(1 - i\zeta' e_n)\,dr$$

evaluates to

$$(-i)^n(n-2)!(<x',\zeta'> + ix_n)^{-n+1}.$$

Hence the integral

$$\frac{1}{\omega_n}\int_{\mathbb{R}^{n-1}} e_+(x,\zeta)\,d\zeta^{n-1}$$

becomes

$$\frac{1}{\omega_n}\int_{S^{n-2}}(-i)^n(n-2)!(<x',\zeta'> + ix_n)^{-n+1}\tfrac{1}{2}(1+ie_n\zeta')\,dS^{n-2}.$$

For any complex number $a+ib$, the product $(a+ib)\frac{1}{2}(1+ie_n\zeta')$ is equal to $(a-be_n)\frac{1}{2}(1+ie_n\zeta')$. Thus, the previous integral becomes

$$\frac{1}{\omega_n}\int_{S^{n-2}}(-i)^n(<x',\zeta'>-x_ne_n\zeta')^{-n+1}\tfrac{1}{2}(1+ie_n\zeta')\,dS^{n-2}.$$

The imaginary, or $iC\ell_n$ part of this integral is the integral of an odd function, so when n is even, the integral becomes

$$\frac{1}{\omega_n}\int_{S^{n-2}}\frac{(n-2)!e_n\zeta'}{2(<x',\zeta'>-x_ne_n\zeta')^{n-1}}\,dS^{n-2},$$

and when n is odd, the integral becomes

$$\frac{-1}{\omega_n}\int_{S^{n-2}}\frac{(n-2)!}{(<x',\zeta'>-x_ne_n\zeta')^{n-1}}\,dS^{n-2}.$$

These integrals are the plane wave decompositions of the Cauchy kernel for the upper half space, described by Sommen in [45]. It should be noted that while introducing the Fourier transform and exploring some of its links with Clifford analysis, we have also been forced to complexify the Clifford algebra $C\ell_n$ to the complex Clifford algebra $C\ell_n(\mathbb{C})$. Furthermore, the functions $\frac{1}{2}(1\pm i\frac{\zeta}{\|\zeta\|e_n})$ are defined on spheres lying in the null cone

$$\{x_ne_n+iw' : w'\in\mathbb{R}^{n-1},\ x_n^2-\|w'\|^2=0\}.$$

This leads naturally to the question: What domains in $\mathbb{C}^n$ do the functions $e_\pm(x,\zeta)$ extend to?

Here we are replacing the real vector variable $x\in\mathbb{R}^n$ by a complex vector variable $\underline{z}=z_1e_1+\ldots+z_ne_n\in\mathbb{C}^n$, where $z_1,\ldots,z_n\in\mathbb{C}$. Letting $\underline{z}=x+iy$ where x and y are real vector variables, the term

$$e^{-<\zeta,y'>-x_n\|\zeta\|}$$

is well defined for $x_n\|\zeta\|>|<\zeta,y'>|$. In this case, $iy'+x_ne_n$ varies over the interior of the forward null cone

$$\{iy'+x_ne_n : x_n>0 \quad\text{and}\quad x_n>\|y'\|\},$$

so $e_+(\underline{z},\zeta)$ is well defined for each

$$\underline{z}=x+iy=x'+iy'+(x_n+iy_n)e_n\in\mathbb{C}^n,$$

where $x'\in\mathbb{R}^{n-1}$, $y_n\in\mathbb{R}$, $x_n>0$ and $\|y'\|<x_n$. Similarly, $e_-(x,\zeta)$ holomorphically extends to

$$\{\underline{z}=x'+iy'+(x_n+iy_n)e_n : x'\in\mathbb{R}^{n-1},\ x_n<0,\ y_n\in\mathbb{R},\ \|y'\|<|x_n|\}.$$

We denote these domains by $C^\pm$, respectively. It should be noted that $\Psi^\pm$ holomorphically continue to $C^\pm$, respectively. We denote these holomorphic continuations of $\Psi^\pm$ by $\Psi'^\pm$. The domains $C^\pm$ are examples of tube domains.

3.10 Complex Clifford analysis

In the previous section, we showed that any left Clifford holomorphic function $f \in H^2(\mathbb{R}^{n,+})$ can be holomorphically continued to a function $f^\dagger$ defined on a tube domain C^+ in $\mathbb{C}^n$. In this section, we will briefly show how this type of holomorphic continuation happens for all Clifford holomorphic functions defined on a domain $U \subset \mathbb{R}^n$.

Let S be a compact smooth hypersurface lying in U, and suppose that S bounds a subdomain V of U. Cauchy's integral formula gives

$$f(y) = \frac{1}{\omega_n} \int_S G(x-y)n(x)f(x)\, d\sigma(x)$$

for each $y \in V$. Let us now complexify the Cauchy kernel. The function $G(x)$ holomorphically continues to

$$\frac{\underline{z}}{(\underline{z}^2)^{\frac{n}{2}}}.$$

In even dimensions this is a well-defined function on $\mathbb{C}^n \backslash N(0)$, where $N(0) = \{\underline{z} \in \mathbb{C}^n : \underline{z}^2 = 0\}$. In odd dimensions this lifts to a well-defined function on a complex n-dimensional Riemann surface double covering $\mathbb{C}^n \backslash N(0)$. Though things work out well in odd dimensions, for simplicity we will work with the cases where n is even. In holomorphically extending $G(x-y)$ in the variable y, we obtain a function

$$G^\dagger(x - \underline{z}) = \frac{x - \underline{z}}{((x-\underline{z})^2)^{\frac{n}{2}}}.$$

This function is well defined on $\mathbb{C}^n \backslash N(x)$, where

$$N(x) = \{\underline{z} \in \mathbb{C}^n : (x - \underline{z})^2 = 0\}.$$

It follows that the integral

$$\frac{1}{\omega_n} \int_S G^\dagger(x - \underline{z})n(x)f(x)\, d\sigma(x)$$

is well defined provided $\underline{z}$ is not in $N(x)$ for any $x \in S$. The set

$$\mathbb{C}^n \backslash \cup_{x \in S} N(x)$$

is an open set in $\mathbb{C}^n$. We shall take the component of this open set which contains V and denote it by $V^\dagger$. It follows that the left Clifford holomorphic function $f(y)$ has a holomorphic extension $f^\dagger(\underline{z})$ to $V^\dagger$. Furthermore, this function is given by the integral formula

$$f^\dagger(\underline{z}) = \frac{1}{\omega_n} \int_S G^\dagger(x - \underline{z})n(x)f(x)\, d\sigma(x).$$

The holomorphic function $f^\dagger$ is a solution to the complex Dirac equation $D^\dagger f^\dagger = 0$, where

$$D^\dagger = \sum_{j=1}^{n} e_j \frac{\partial}{\partial z_j}.$$

By allowing the hypersurface to deform and move out to include more of U in its interior, we see that $f^\dagger$ is a well-defined holomorphic function on $U^\dagger$, where $U^\dagger$ is the component of

$$\mathbb{C}^n \setminus \cup_{x \in \overline{U} \setminus U} N(x)$$

which contains U. In the special cases where U is either of $\mathbb{R}^{n,\pm}$, then $U^\dagger = C^\pm$. See [34–36] for more details.

3.11 References

[1] L. V. Ahlfors, Möbius transformations in $\mathbb{R}^n$ expressed through 2×2 matrices of Clifford numbers, *Complex Variables* **5** (1986), 215–224.

[2] M. F. Atiyah, R. Bott and A. Shapiro, Clifford modules, *Topology* **3** (1965), 3–38.

[3] B. Bojarski, Conformally covariant differential operators, *Proceedings, XXth Iranian Math. Congress*, Tehran, 1989.

[4] B. Booss-Bavnbek and K. Wojciechowski, *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser, Boston, 1993.

[5] F. Brackx, R. Delanghe and F. Sommen, *Clifford Analysis*, Pitman, London, 1982.

[6] D. Calderbank, *Clifford analysis for Dirac operators on manifolds with boundary*, Max-Plank-Institut für Mathematik preprint, Bonn, 1996.

[7] R. Coifman, A. McIntosh and Y. Meyer, L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes, *Annals of Mathematics* **116** (1982), 361–387.

[8] J. Cnops and H. Malonek, *An introduction to Clifford analysis*, Univ. Coimbra, Coimbra, 1995.

[9] R. Delanghe, On regular-analytic functions with values in a Clifford algebra, *Math. Ann.* **185** (1970), 91–111.

[10] R. Delanghe, F. Sommen and V. Soucek, *Clifford Algebra and Spinor-Valued Functions*, Kluwer, Dordrecht, 1992.

[11] A. C. Dixon, On the Newtonian potential, *Quarterly Journal of Mathematics*, 35, 1904, 283–296.

[12] R. Fueter, Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit Vier Reallen Variablen, *Commentarii Mathematici Helvetici* **7** (1934–1935), 307–330.

[13] K. Guerlebeck and W. Sproessig, *Quaternionic and Clifford Calculus for Physicists and Engineers*, Wiley, Chichester, 1998.

[14] J. Gilbert and M. A. M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, CUP, Cambridge, 1991.

[15] D. Hestenes, Multivector functions, *J. Math. Anal. Appl.* **24** (1968), 467–473.

[16] V. Iftimie, Fonctions hypercomplexes, *Bull. Math. de la Soc. Sci. Math. de la R. S. de Roumanie* **9** (1965), 279–332.

[17] R. S. Krausshar, Automorphic forms in Clifford analysis, *Complex Variables* **47** (2002), 417–440.

[18] R. S. Krausshar and J. Ryan, Clifford and harmonic analysis on cylinders and tori, to appear.

[19] R. S. Krausshar and J. Ryan, Some conformally flat spin manifolds, Dirac operators and automorphic forms, to appear.

[20] V. Kravchenko and M. Shapiro, *Integral Representations for Spatial Models of Mathematical Physics*, Pitman Research Notes in Mathematics, London, No. 351, 1996.

[21] C. Li, A. McIntosh and S. Semmes, Convolution singular integrals on Lipschitz surfaces, *JAMS* **5** (1992), 455–481.

[22] C. Li, A. McIntosh and T. Qian, Clifford algebras, Fourier transforms, and singular convolution operators on Lipschitz surfaces, *Revista Mathematica Iberoamericana* **10** (1994), 665–721.

[23] T. E. Littlewood and C. D. Gay, Analytic spinor fields, *Proc. Roy. Soc., A313* (1969), 491–507.

[24] J. Liouville, *Extension au cas des trois dimensions de la question du trace géographique. Applications de l'analyse à la géométrie*, G. Monger, Paris, 1850, pp. 609–616.

[25] H. Liu and J. Ryan, Clifford analysis techniques for spherical pde, *Journal of Fourier Analysis and its Applications* **8** (2002), 535–564.

[26] A. McIntosh, Clifford algebras, Fourier theory, singular integrals, and harmonic functions on Lipschitz domains, *Clifford Algebras in Analysis and Related Topics*, ed. J. Ryan, CRC Press, Boca Raton, 1996, pp. 33–88.

[27] M. Mitrea, *Clifford Wavelets, Singular Integrals and Hardy Spaces*, Lecture Notes in Mathematics, Springer-Verlag, Heidelberg, No. 1575, 1994.

[28] M. Mitrea, Generalized Dirac operators on non-smooth manifolds and Maxwell's equations, *Journal of Fourier Analysis and its Applications* **7** (2001), 207–256.

[29] Gr. C. Moisil and N. Theodorescu, Fonctions holomorphes dans l'espace, *Mathematica (Cluj)* **5** (1931), 142–159.

[30] M. Murray, The Cauchy integral, Calderon commutation, and conjugation of singular integrals in $\mathbb{R}^n$, *Trans. of the AMS* **298** (1985), 497–518.

[31] E. Obolashvili, *Partial Differential Equations in Clifford Analysis*, Pitman Monographs and Surveys in Pure and Applied Mathematics No. 96, Harlow, 1998.

[32] T. Qian, Singular integrals on star shaped Lipschitz surfaces in the quaternionic space, *Math Ann.* **310** (1998), 601–630.

[33] T. Qian, Fourier analysis on starlike Lipschitz surfaces, *Journal of Functional Analysis* **183** (2001), 370–412.

[34] J. Ryan, Cells of Harmonicity and generalized Cauchy integral formulae, *Proc. London Math. Soc* **60** (1990), 295–318.

[35] J. Ryan, Intrinsic Dirac operators in C^n, *Adv. in Math.* **118** (1996), 99–133.

[36] J. Ryan, Complex Clifford analysis and domains of holomorphy, *Journal of the Australian Mathematical Society, Series A* **48** (1990), 413–433.

[37] J. Ryan, Ed., *Clifford Algebras in Analysis and Related Topics*, Studies in Advanced Mathematics, CRC Press, Boca Raton, 1995.

[38] J. Ryan, Basic Clifford analysis, *Cubo Matematica Educacional* **2** (2000), 226–256.

[39] J. Ryan, Dirac operators on spheres and hyperbolae, *Bolletin de la Sociedad Matematica a Mexicana* **3** (1996), 255–270.

[40] M. Sce, Osservaziono sulle series di potenzi nei moduli quadratici, *Atti. Acad. Nax. Lincei Rend Sci Fis Mat. Nat*, **23** (1957), 220–225.

[41] S. Semmes, Some remarks concerning integrals of curvature on curves and surfaces, to appear.

[42] F. Sommen, Spherical monogenics and analytic functionals on the unit sphere, *Tokyo J. Math.* **4** (1981), 427–456.

[43] F. Sommen, A product and an exponential function in hypercomplex function theory, *Appl. Anal.* **12** (1981), 13–26.

[44] F. Sommen, Microfunctions with values in a Clifford algebra II, *Scientific Papers of the College of Arts and Sciences, The University of Tokyo*, 36, 1986, pp. 15–37.

[45] F. Sommen, Plane wave decomposition of monogenic functions, *Ann. Polon. Math.* **49** (1988), 101–114.

[46] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Space*, Princeton University Press, Princeton, 1971.

[47] A. Sudbery, Quaternionic analysis, *Math. Proc. of the Cambridge Philosophical Soc.* **85** (1979), 199–225.

[48] P. Van Lancker, Clifford analysis on the sphere, *Clifford Algebras and their Applications in Mathematical Physics*, V. Diedrich et al., editors, Kluwer, Dordrecht, 1998, pp. 201–215.

[49] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 3-D, *JAMS* **12** (1999), 445–495.

John Ryan
Department of Mathematics
University of Arkansas
Fayetteville, AR 72701, U. S. A.
E-mail: jryan@uark.edu

Received: May 12, 2002; Revised: March 15, 2003.